Universidade Federal da Bahia - UFBA
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Tese de Doutorado

# Some topics about singular hyperbolicity and invariant measures

Vinicius Coelho dos Santos

**Salvador-BA**
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Tese de Doutorado apresentada ao Colegiado da Pós-Graduação em Matemática da Universidade Federal da Bahia como requisito parcial para obtenção do título de Doutor em Matemática, aprovada em 18 de abril de 2019.

**Orientadora:** Profa. Dra. Luciana Silva Salgado.

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Aos meus pais e amigos

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*“Para Tales... a questão primordial não era o que sabemos, mas como sabemos.”*

(Aristóteles)

# Resumo

Mostramos a existência de métricas adaptadas para conjuntos singulares hiperbólicos de codimensão um com respeito a um campo $C^1$ em uma variedade compacta de dimensão finita sem uso de formas quadráticas. Analisando as medidas de um sistema, provamos um teorema tipo-Kingman para medidas finitas arbitrárias assumindo algumas condições em um espaço métrico qualquer, e fornecemos condições necessárias que garatem a existência de medidas invariantes em espaços separáveis e localmente compactos para funções próprias contínuas. Além disso, usamos o operador de Perron-Frobenius e as técnicas desenvolvidas aqui para obter um outro critério que garante a existência de medidas invariantes para funções contínuas (não necessariamente funções próprias) em espaços métricos localmente compactos e separáveis.

**Palavras-chave:** Conjunto singular hipérbolico, métricas adaptadas, Teorema tipo-Kingman, localmente compacto, separável, medidas invariantes, operador de Perron-Frobenius.

# Abstract

We show the existence of singular adapted metrics for any codimension one singular hyperbolic set with respect to a $C^1$ vector field on finite dimensional compact manifolds without using quadradic forms. Considering the measures of a system, we provide a Kingman-like Theorem for an arbitrary finite measure assuming some conditions in any metric space, and we give necessary conditions to guarantee the existence of invariant measures in locally compact and separable metric spaces for continuous proper maps. Moreover, we use the Perron-Frobenius operator and the techniques developed here to obtain other criteria to guarantee the existence of invariant measures for continuous maps (not necessarily a proper maps) in locally compact separable metric spaces.

**Keywords:** Singular hyperbolic set, adapted metrics, Kingman-like Theorem, locally compact, separable, invariant measures, Perron-Frobenius operator.

# Contents

# Introduction

Let $M$ be a connected compact finite $m$-dimensional manifold, $m \geq 3$, with or without boundary. We consider a vector field $X$, such that $X$ is inwardly transverse to the boundary $\partial M$, if $\partial M \neq \emptyset$. The flow generated by $X$ is denoted by $X_t$.

A hyperbolic set for a flow $X_t$ on a finite dimensional Riemannian manifold $M$ is a compact invariant set $\Gamma$ with a continuous splitting of the tangent bundle, $T_\Gamma M = E^s \oplus E^X \oplus E^u$, where $E^X$ is the direction of the vector field, for which the subbundles are invariant under the derivative $DX_t$ of the flow $X_t$

$$DX_t \cdot E^*_x = E^*_{X_t(x)}, \quad x \in \Gamma, \quad t \in \mathbb{R}, \quad * = s, X, u; \tag{1}$$

and $E^s$ is uniformly contracted by $DX_t$ and $E^u$ is likewise expanded: there are $K, \lambda > 0$ so that

$$\|DX_t \mid_{E^s_x} \| \leq K e^{-\lambda t}, \quad \|(DX_t \mid_{E^u_x})^{-1}\| \leq K e^{-\lambda t}, \quad x \in \Gamma, \quad t \in \mathbb{R}. \tag{2}$$

Very strong properties can be deduced from the existence of such hyperbolic structure; see for instance [19, 20, 67, 43, 60].

An important feature of hyperbolic structures is that it does not depends on the metric on the ambient manifold (see [36]). We recall that a metric is said to be *adapted* to the hyperbolic structure if we can take $K = 1$ in equation (2).

Weaker notions of hyperbolicity (e.g. dominated splitting, partial hyperbolicity, volume hyperbolicity, sectional hyperbolicity, singular hyperbolicity) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see [18] and specifically [72, 6, 11] for singular hyperbolicity and Lorenz-like attractors.

In the same work [36], Hirsch, Pugh and Shub asked about adapted metrics for dominated splittings. The positive answer was given by Gourmelon [34] in 2007, where it is given adapted metrics to dominated splittings for both diffeomorphisms and flows, and he also gives an adapted metric for partially hyperbolic splittings as well.

Proving the existence of some hyperbolic structure is, in general, a non-trivial matter, even in its weaker forms.

In [48], Lewowicz proved that a diffeomorphism on a compact riemannian manifold is Anosov if and only if its derivative admits a nondegenerate Lyapunov quadratic function.

An example of application of the adapted metric from [34] is contained in [7], where L. Salgado and V. Araújo, following the spirit of Lewowicz's result, construct quadratic forms which characterize partially hyperbolic and singular hyperbolic structures on a trapping region for flows.

In [8], L. Salgado and V. Araújo provided an alternative way to obtain singular hyperbolicity for three-dimensional flows using the same expression as in Proposition 1.11 applied to the infinitesimal generator of the exterior square $\wedge^2 DX_t$ of the cocycle $DX_t$. This infinitesimal generator can be explicitly calculated through the infinitesimal generator $DX$ of the linear multiplicative cocycle $DX_t$ associated to the vector field $X$.

Here, in Chapter 1, the author and L. Salgado provide a similar result as above for $m$-dimensional flows if this admits a partially hyperbolic splitting for which one of the invariant subbundles is one-dimensional.

In [8], V. Araújo and L. Salgado noted that the existence of an adapted metric could be considered for singular hyperbolic splittings, and they proved it for a three-dimensional vector field by using quadratic forms.

In [62, Theorem B], the author and L. Salgado showed the existence of adapted metrics for any singular hyperbolic set $\Gamma$ of a $C^1$ vector fields in the particular setting where $\Gamma$ has a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ with $F$ volume expanding and $E$ an one-dimensional uniformly contracting bundle, extending the result from [8] for any codimension one singular hyperbolic set. This is also done under the point of view of $\mathfrak{J}$-algebras of Potapov [77], confirming the very interesting feature of the quadratic forms technique from which we can get adapted metrics.

Here, in Chapter 1, in a joint work with V. Araújo and L. Salgado, we also proved this result but this is made in a certain different way from [8, 62]. Now, we make this without using quadratic forms, we only use multilinear algebra and the dynamics.

In Chapter 2, our purpose is to investigate Kingman-like Theorems for arbitrary finite measures.

Let $(M, \mathcal{A}, \mu)$ be a measure space equipped with a $\sigma$-finite measure, and $T : M \to M$ be a measurable map.

If $\mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{A}$ then $\mu$ is said to be *invariant* under $T$ or, equivalently, $T$ is *measure-preserving.*

The most important results of invariant measures theory are Kingman's Theorem (see [13]) and Birkhoff's Theorem (see [16]).

The basic idea to proof Kingman's Theorem is to apply Fekete's Subadditive

Lemma. This Lemma yields information about subadditive sequences $(a_n)_n$ in $\mathbb{R}$ proving that the limit $\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n} = a$ and satisfies $-\infty \leq a < \infty$. This sequence occurs naturaly when we work with invariant measures and a subbaditive sequence of functions for a transformation in a manifold.

Derriennic [29] generalized Fekete's Lemma as follows. Let $(a_n)_n$ be a sequence in $\mathbb{R}$ and $(c_n)_n$ be a sequence such that $c_n \geq 0$. If $a_{n+m} \leq a_n + a_m + c_n$ for all $n, m \geq 1$, and $\lim_n \frac{c_n}{n} = 0$ then the limit $\lim_n \frac{a_n}{n} = a$ and satisfies $-\infty \leq a < \infty$. He utilizes this result and others techniques to provide a generalization for Kingman's Theorem.

Other generalisations of Kingman's Theorem were proved by Akcoglu and Sucheston [2] (for superadditive processes), Shurger [65] (a stochastic analogue of generalization of Kingman's Theorem given by Derriennic), and recently by A. Karlsson and Margulis [41] (for ergodic measure preserving transformations).

Here, we will show a Kingman-like Theorem for an arbitrary finite measure assuming some conditions. This theorem was inspired by the proof of Kingman's Theorem given by Avila and Bochi [13].

Generalisations of Birkhoff's Theorem were proved by E.Hopf [37] (for infinite measure preserving transformations), J. Aaronson [1, Theorem 2.4.2] (for conservative ergodic measure transformations), W. Hurewicz [38] (for conservative nonsingular transformations where the observables are defined by means of Radon-Nykodim Theorem and the measure can be finite or infinite), R. Chacon, D.Ornsten [25] (for Markov operators), M. Carvalho and F. Moreira [22] (for half-invariant measures), and recently M. Carvalho and F. Moreira [23] (for ultralimits by means of ultrafilters).

As an application of our Kingman-like Theorem, we formulated a version of Birkhoff's Theorem for bounded observables and finite measures. Our result are not contemplated by previous work:

($a$) in [38], Hurewicz worked in context of conservative transformations and bounded observables defined by means of Radon-Nykodim Theorem;

($b$) in [22, Theorem 1.2], Carvalho and Moreira showed that every finite and half-invariant measure is an invariant measure, and our theorem was proved for a finite arbitraty measure;

($c$) in [23], Carvalho and Moreira showed that the Birkhoff's Theorem holds for each non-principal ultrafilter, so for this Theorem to imply our result it is necessary that the value of integral be the same for each non-principal ultrafilter, however it is not clear how to compute this, because the ultrafilters are obtained by Zorn's Lemma, and therefore we do not have an expression for these ultrafilters.

In Chapter 3, we are interested in finding necessary conditions to ensure the existence of invariant finite measures in locally compact separable metric spaces for continuous proper maps.

We recall that under theses conditions (locally compact and separable metric spaces and continuous proper maps) some authors (see [49, 51, 58]) constructed a variational principle. Recently, Caldas and Patrão [21] dropped the proper condition of the map and extended the result for any continuous map.

Results that guarantee the existence of invariant measures were proved for Groups ([26, 32, 57]), Markov Chains ([46, 47, 66, 75]), and Dynamical Systems (for recent results see [9]). The most celebrated result of this theory was proved by Krylov and Bogolyubov [17] for compact metric space. Precisely, they showed that if $f : M \to M$ is a continuous map then $f$ admits an invariant Borel probability measure where $M$ is a compact metric space.

We observe that our result is obtained by means of Functional Analysis and Measure Theory, and allows us to provide a natural characterization for the existence of invariant measures in this context (locally compact and separable metric spaces for continuous proper maps). Moreover, we use the Perron-Frobenius operator and the techniques developed here to obtain other criteria to guarantee the existence of invariant measures in locally compact and separable metric spaces for continuous functions (not necessarily a proper map).

To facilite acess to the individual topics, the chapters are rendered as self-contained as possible.

Finally, in Chapter 4, we discuss future perspectives of this work, considering some problems and conjectures.

# Chapter 1

# Singular hyperbolic flows

The best known and simplest examples of chaotic dynamical systems are hyperbolic systems. A hyperbolic set is defined to be a compact invariant set $\Gamma$ of a diffeomorphism $f$ in a compact manifold such that there exists a splitting of the tangent bundle $T\Gamma$ into two supplementary, $df$-invariant subbundles, called the stable and the unstable bundles that are uniformly contracted and expanded, by the derivative $df^n$, for some $n > 0$. The hyperbolicity of $\Gamma$ does not depend on the metric on the manifold, but the smallest time $n$ where the contraction/expansion phenomena are seen depends on the metric; a Riemannian metric is said to be adapted to the hyperbolic set $\Gamma$ if one can take $n = 1$. Hirsch, Pugh and Shub obtain that any hyperbolic set admits an adapted Riemannian metric applying Holmes' Theorem (see [36] p.15).

Weaker notions of hyperbolicity (e.g. dominated splitting, partial hyperbolicity, volume hyperbolicity, sectional hyperbolicity, singular hyperbolicity) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see [18] and specifically [72, 6, 11] for singular hyperbolicity and Lorenz-like attractors.

In the same work [36], Hirsch, Pugh and Shub asked about adapted metrics for dominated splittings. The positive answer was given by Gourmelon [34] in 2007, where it is given adapted metrics to dominated splittings for both diffeomorphisms and flows, and he also gives an adapted metric for partially hyperbolic splittings as well. To do this Gourmelon adapted the Holmes' Theorem to the case of dominated behaviours.

For a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ of $\Gamma$, a $C^1$ vector field $X$ on a $m$-manifold, we provided an alternative way to obtain singular-hyperbolicity using only the tangent map $DX$ of $X$ and its derivative $DX_t$ whether $E$ is one-dimensional subspace.

Moreover, we show the existence of singular adapted metrics for any codimension one singular hyperbolic set $\Gamma$ with respect to a $C^1$ vector field on finite dimensional compact manifolds.

This results were published in [62] (in a joint work with L. Salgado) and [5] (in a joint work with V.Araújo and L. Salgado).

### 1.0.1 Statements of main results of Chapter

In the sequel, we write $\tilde{\mathfrak{J}}(v) =< \tilde{J}_x v, v >$, where $\tilde{J}_x$ is given in Proposition 1.11, that is, $\tilde{\mathfrak{J}}(v)$ is the time derivative of a quadratic form $\mathfrak{J}$ under the action of the flow.

The absolute value of the *cross product* (also called *vector product*) on a 3-dimensional vector space $V$, denote by $w = u \times v$, provides the length of the vector $w$. It is very useful to calculate the area expansion of the parallelogram generated by $u, v$, under the action of a linear operator.

Following this way, in [8], L. Salgado and V. Araújo proved the result below.

**Theorem 1.1.** *[8, Theorem B] Suppose that $X$ is 3-dimensional vector field on $M$ which is non-negative strictly $\mathfrak{J}$-separated over a non-trivial subset $\Gamma$, where $\mathfrak{J}$ has index 1. Then*

1. *$\wedge^2 DX_t$ is strictly $(-\mathfrak{J})$-separated;*
2. *$\Gamma$ is a singular hyperbolic set if either one of the following properties is true*
   - *(a) $\widetilde{\Delta}_0^t(x) \xrightarrow[t\to+\infty]{} -\infty$ for all $x \in \Gamma$.*
   - *(b) $\tilde{\mathfrak{J}} - 2\,\mathrm{tr}(DX)\mathfrak{J} > 0$ on $\Gamma$.*

Here, we generalized this result to $m$ and $k = m - 1$, as follows.

If $\wedge^k DX_t$ is strictly separated with respect to some family $\mathfrak{J}$ of quadratic forms, then there exists the function $\delta_k$ as stated in Proposition 1.11 with respect to the cocyle $\wedge^k DX_t$. We set

$$\widetilde{\Delta}_a^b(x) := \int_a^b \delta_k(X_s(x))\, ds$$

the area under the function $\delta_k : U \to \mathbb{R}$ given by Proposition 1.11 with respect to $\wedge^k DX_t$ and its infinitesimal generator.

If $k = m - 1$, it is not difficult to see that this function is related to $X$ and $\delta$ as follows: let $\delta : \Gamma \to \mathbb{R}$ be the function associated to $\mathfrak{J}$ and $DX_t$, as given by Proposition 1.11, then $\delta_k = 2\,\mathrm{tr}(DX) - \delta$, where $\mathrm{tr}(DX)$ represents the trace of the linear operator $DX_x : T_xM \circlearrowleft, x \in M$.

We recall that $\tilde{\mathfrak{J}} = \partial_t \mathfrak{J}$ is the time derivative of $\mathfrak{J}$ along the flow; see Remark 1.12.

Our first main result is the following.

**Theorem A.** *Suppose that $X$ is $m$-dimensional vector field on $M$ which is non-negative strictly $\mathfrak{J}$-separated over a non-trivial subset $\Gamma$, where $\mathfrak{J}$ has index 1. Then*

1. *If* $\Gamma$ *is a singular hyperbolic set then* $\wedge^{(m-1)} DX_t$ *is strictly* $(-\mathfrak{J})$*-separated;*

2. $\Gamma$ *is a singular hyperbolic set if either one of the following properties is true*

   (a) $\widetilde{\Delta}_0^t(x) \xrightarrow[t\to+\infty]{} -\infty$ *for all* $x \in \Gamma$.

   (b) $\tilde{\mathfrak{J}} - 2\,\mathrm{tr}(DX)\mathfrak{J} > 0$ *on* $\Gamma$.

We work here with exterior products of codimension one. See [50] for more details on this subject.

This result provides useful sufficient conditions for a $m$-dimensional vector field to be singular hyperbolic if $k = m-1$, using only one family of quadratic forms $\mathfrak{J}$ and its space derivative $DX$, avoiding the need to check cone invariance and contraction/expansion conditions for the flow $X_t$ generated by $X$ on a neighborhood of $\Gamma$.

Now we recall the definition of adapted metrics in the singular hyperbolic setting.

**Definition 1.2.** *We say a Riemannian metric* $\langle \cdot, \cdot \rangle$ adapted to a singular hyperbolic splitting $T\Gamma = E \oplus F$ *if it induces a norm* $|\cdot|$ *such that there exists* $\lambda > 0$ *satisfying for all* $x \in \Gamma$ *and* $t > 0$ *simultaneously*

$$|DX_t\,|_{E_x}| \cdot \left|(DX_t\,|_{F_x})^{-1}\right| \le e^{-\lambda t}, \; |DX_t\,|_{E_x}| \le e^{-\lambda t} \quad \textit{and} \quad |\det(DX_t\,|_{F_x})| \ge e^{\lambda t}.$$

*We call it* singular adapted metric*, for simplicity.*

This extends the notion of adapted metric for dominated and partially hyperbolic splittings; see e.g. [34].

In [8], L. Salgado and V. Araújo proved the next result.

**Theorem 1.3.** *[8, Theorem C] Let* $\Gamma$ *be a singular-hyperbolic set for a* $C^1$ *three-dimensional vector field* $X$*. Then* $\Gamma$ *admits a singular adapted metric.*

In [62, Theorem B], the author and L. Salgado showed the existence of adapted metrics for any singular hyperbolic set $\Gamma$ of a $C^1$ vector fields in particular setting where $\Gamma$ has a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ with $F$ volume expanding and $E$ a one-dimensional uniformly contracting bundle, extending the result from [8] for any codimension one singular hyperbolic set.

Here, in a joint work with V. Araújo and L. Salgado, we also proved this result but in this work this is made in a certain different way from [8, 62]. Now, we make this without using quadratic forms, we only use multilinear algebra and the dynamics.

Consider a partially hyperbolic splitting $T_\Gamma M = E \oplus F$ where $E$ is uniformly contracted and $F$ is volume expanding. We show that for $C^1$ flows having a singular-hyperbolic set $\Gamma$ such that $E$ is one-dimensional subspace there exists a metric adapted to the partial hyperbolicity and the area expansion, as follows.

**Theorem B.** *Let* $\Gamma$ *be a singular-hyperbolic set of codimension one for a* $C^1$ $m$*-dimensional vector field* $X$*. Then* $\Gamma$ *admits a singular adapted metric.*

We present the relevant definitions and auxiliary results in the next section.

The chapter is organized as follow. In the present Section we provide an introduction and statement of main results. In Section 1.1 we give the main definitions and useful properties of quadratic forms. In Section 1.2 we provide some auxiliary results. In Section 1.3 we give the proofs of our theorems.

## 1.1 Preliminary definitions and results

We now present preliminary definitions and results.

We recall that a *trapping region* $U$ for a flow $X_t$ is an open subset of the manifold $M$ which satisfies: $X_t(U)$ is contained in $U$ for all $t > 0$, and there exists $T > 0$ such that $\overline{X_t(U)}$ is contained in the interior of $U$ for all $t > T$. We define $\Gamma(U) = \Gamma_X(U) := \cap_{t>0}\overline{X_t(U)}$ to be the *maximal positive invariant subset in the trapping region* $U$.

A *singularity* for the vector field $X$ is a point $\sigma \in M$ such that $X(\sigma) = \vec{0}$ or, equivalently, $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. The set formed by singularities is the *singular set of* $X$ denoted $\mathrm{Sing}(X)$. We say that a singularity is hyperbolic if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity $\sigma$ have nonzero real part.

**Definition 1.4.** *A* dominated splitting *over a compact invariant set* $\Lambda$ *of* $X$ *is a continuous* $DX_t$*-invariant splitting* $T_\Lambda M = E \oplus F$ *with* $E_x \neq \{0\}$*,* $F_x \neq \{0\}$ *for every* $x \in \Lambda$ *and such that there are positive constants* $K, \lambda$ *satisfying*

$$\|DX_t|_{E_x}\| \cdot \|DX_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \text{ for all } x \in \Lambda, \text{ and all } t > 0. \tag{1.1}$$

A compact invariant set $\Lambda$ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle $E$ is *uniformly contracted*, i.e., there exists $C > 0$ and $\lambda > 0$ such that $\|DX_t|_{E_x}\| \leq Ce^{-\lambda t}$ for $t \geq 0$. In this case $F$ is the *central subbundle* of $\Lambda$. Or else, we may replace uniform contraction along $E$ by uniform expansion along $F$ (the right hand side condition in (2)).

We say that a $DX_t$-invariant subbundle $F \subset T_\Lambda M$ is a *sectionally expanding* subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants $C, \lambda$ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \text{ for all } t > 0. \tag{1.2}$$

**Definition 1.5.** *[53, Definition 2.7] A* sectional-hyperbolic set *is a partially hyperbolic set whose central subbundle is sectionally expanding.*

This is a particular case of the so called *singular hyperbolicity* whose definition we recall now. A $DX_t$-invariant subbundle $F \subset T_\Lambda M$ is said to be a *volume expanding* if in the above condition 1.2, we may write

$$|\det(DX_t|_{F_x})| > Ce^{\lambda t}, \text{ for all } t > 0. \tag{1.3}$$

**Definition 1.6.** *[54, Definition 1] A* singular hyperbolic set *is a partially hyperbolic set whose central subbundle is volume expanding.*

Clearly, in the three-dimensional case, these notions are equivalent.

This is a feature of the Lorenz attractor as proved in [69] and also a notion that extends hyperbolicity for singular flows, because sectional hyperbolic sets without singularities are hyperbolic; see [55, 6].

### 1.1.1 Linear multiplicative cocycles over flows

Let $A : G \times \mathbb{R} \to G$ be a smooth map given by a collection of linear bijections

$$A_t(x) : G_x \to G_{X_t(x)}, \quad x \in \Gamma, t \in \mathbb{R},$$

where $\Gamma$ is the base space of the finite dimensional vector bundle $G$, satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in \Gamma, t, s \in \mathbb{R},$$

with $\{X_t\}_{t\in\mathbb{R}}$ a complete smooth flow over $M \supset \Gamma$. We note that for each fixed $t > 0$ the map $A_t : G \to G, v_x \in G_x \mapsto A_t(x) \cdot v_x \in G_{X_t(x)}$ is an automorphism of the vector bundle $G$.

The natural example of a linear multiplicative cocycle over a smooth flow $X_t$ on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle $G = TM$ of a finite dimensional compact manifold $M$. Another example is given by the exterior power $A_t(x) = \wedge^k DX_t$ of $DX_t$ acting on $G = \wedge^k TM$, the family of all $k$-vectors on the tangent spaces of $M$, for some fixed $1 \leq k \leq \dim G$.

It is well-known that the exterior power of a inner product space has a naturally induced inner product and thus a norm. Thus $G = \wedge^k TM$ has an induced norm from the Riemannian metric of $M$. For more details see e.g. [12].

In what follows we assume that the vector bundle $G$ has a smoothly defined inner product in each fiber $G_x$ which induces a corresponding norm $\|\cdot\|_x, x \in \Gamma$.

**Definition 1.7.** *A continuous splitting $G = E \oplus F$ of the vector bundle $G$ into a pair of subbundles is* dominated *(with respect to the automorphism $A$ over $\Gamma$) if*

- *the splitting is* invariant*: $A_t(x) \cdot E_x = E_{X_t(x)}$ and $A_t(x) \cdot F_x = F_{X_t(x)}$ for all $x \in \Gamma$ and $t \in \mathbb{R}$; and*
- *there are positive constants $K, \lambda$ satisfying*

$$\|A_t|_{E_x}\| \cdot \|A_{-t}|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \text{ for all } x \in \Gamma, \text{ and all } t > 0. \tag{1.4}$$

We say that the splitting $G = E \oplus F$ is *partially hyperbolic* if it is dominated and the subbundle $E$ is uniformly contracted: $\|A_t \mid E_x\| \le Ce^{-\mu t}$ for all $t > 0$ and suitable constants $C, \mu > 0$.

### 1.1.2 Fields of quadratic forms, positive and negative cones

Let $E_U$ be a finite dimensional vector bundle with inner product $\langle \cdot, \cdot \rangle$ and base given by the trapping region $U \subset M$. Let $\mathcal{J} : E_U \to \mathbb{R}$ be a continuous family of quadratic forms $\mathcal{J}_x : E_x \to \mathbb{R}$ which are non-degenerate and have index $0 < q < \dim(E) = n$. The index $q$ of $\mathcal{J}$ means that the maximal dimension of subspaces of non-positive vectors is $q$. Using the inner product, we can represent $\mathcal{J}$ by a family of self-adjoint operators $J_x : E_x \circlearrowleft$ as $\mathcal{J}_x(v) = \langle J_x(v), v \rangle, v \in E_x, x \in U$.

We also assume that $(\mathcal{J}_x)_{x \in U}$ is continuously differentiable along the flow. The continuity assumption on $\mathcal{J}$ means that for every continuous section $Z$ of $E_U$ the map $U \ni x \mapsto \mathcal{J}(Z(x)) \in \mathbb{R}$ is continuous. The $C^1$ assumption on $\mathcal{J}$ along the flow means that the map $\mathbb{R} \ni t \mapsto \mathcal{J}_{X_t(x)}(Z(X_t(x))) \in \mathbb{R}$ is continuously differentiable for all $x \in U$ and each $C^1$ section $Z$ of $E_U$.

Using Lagrange diagonalization of a quadratic form, it is easy to see that the choice of basis to diagonalize $\mathcal{J}_y$ depends smoothly on $y$ if the family $(\mathcal{J}_x)_{x \in U}$ is smooth, for all $y$ close enough to a given $x$. Therefore, choosing a basis for $T_x$ adapted to $\mathcal{J}_x$ at each $x \in U$, we can assume that locally our forms are given by $\langle J_x(v), v \rangle$ with $J_x$ a diagonal matrix whose entries belong to $\{\pm 1\}$, $J_x^* = J_x$, $J_x^2 = I$ and the basis vectors depend as smooth on $x$ as the family of forms $(\mathcal{J}_x)_x$.

We let $\mathcal{C}_\pm = \{C_\pm(x)\}_{x \in U}$ be the family of positive and negative cones associated to $\mathcal{J}$

$$C_\pm(x) := \{0\} \cup \{v \in E_x : \pm \mathcal{J}_x(v) > 0\}, \quad x \in U,$$

and also let $\mathcal{C}_0 = \{C_0(x)\}_{x\in U}$ be the corresponding family of zero vectors $C_0(x) = \mathcal{J}_x^{-1}(\{0\})$ for all $x \in U$.

### 1.1.3 Strict $\mathcal{J}$-separation for linear multiplicative cocycles

Let $A : E \times \mathbb{R} \to E$ be a linear multiplicative cocycle on the vector bundle $E$ over the flow $X_t$. The following definitions are fundamental to state our results.

**Definition 1.8.** *Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$ with constant index on the positively invariant open subset $U$ for the flow $X_t$, we say that the cocycle $A_t(x)$ over $X_t$ is*

- $\mathcal{J}$-separated *if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (simple cone invariance);*
- strictly $\mathcal{J}$-separated *if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (strict cone invariance).*
- $\mathcal{J}$-monotone *if $\mathcal{J}_{X_t(x)}(DX_t(x)v) \geq \mathcal{J}_x(v)$, for each $v \in T_xM \setminus \{0\}$ and $t > 0$;*
- strictly $\mathcal{J}$-monotone *if $\partial_t\big(\mathcal{J}_{X_t(x)}(DX_t(x)v)\big) \mid_{t=0} > 0$, for all $v \in T_xM \setminus \{0\}$, $t > 0$ and $x \in U$;*
- $\mathcal{J}$-isometry *if $\mathcal{J}_{X_t(x)}(DX_t(x)v) = \mathcal{J}_x(v)$, for each $v \in T_xM$ and $x \in U$.*

*We say that the flow $X_t$ is (strictly)* $\mathcal{J}$-separated *on $U$ if $DX_t(x)$ is (strictly)* $\mathcal{J}$-separated *on $T_UM$. Analogously, the flow of $X$ on $U$ is (strictly)* $\mathcal{J}$-monotone *if $DX_t(x)$ is (strictly)* $\mathcal{J}$-monotone.

**Remark 1.9.** *If a flow is strictly $\mathcal{J}$-separated, then for $v \in T_xM$ such that $\mathcal{J}_x(v) \leq 0$ we have $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) < 0$, for all $t > 0$, and $x$ such that $X_{-s}(x) \in U$ for every $s \in [-t, 0]$. Indeed, otherwise $\mathcal{J}_{X_{-t}(x)}(DX_{-t}(v)) \geq 0$ would imply $\mathcal{J}_x(v) = \mathcal{J}_x\big(DX_t(DX_{-t}(v))\big) > 0$, contradicting the assumption that $v$ was a non-positive vector.*

*This means that a flow $X_t$ is strictly $\mathcal{J}$-separated if, and only if, its time reversal $X_{-t}$ is strictly $(-\mathcal{J})$-separated.*

**Remark 1.10.** *Let $V$ be a real finite dimensional vector space, and $L : V \to V$ be a $\mathcal{J}$-separated linear operator. Then $L$ can be uniquely represented by $L = RT$, where $T$ is a $\mathcal{J}$-isometry (i.e. $\mathcal{J}(T(v)) = \langle J_x(Tv), Tv\rangle = \langle J_x(v), v\rangle = \mathcal{J}(v), v \in V$) and $R$ is $\mathcal{J}$-symmetric (i.e. $\langle J_x(Rv), w\rangle = \langle v, J_xRw\rangle$, for $v, w \in V$) with positive spectrum; the operator $R$ can be diagonalized by a $\mathcal{J}$-isometry, and there exist constants $r_-$ and $r_+$ such that the operator $L$ is (strictly) $\mathcal{J}$-monotonous if, and only, if $r_- \leq (<)\ 1$ and $r_+ \geq (>)\ 1$. For more details see [7, Proposition 2.4] and comments below of the Theorem* 1.2 *in [77].*

A vector field $X$ is $\mathfrak{J}$-*non-negative* on $U$ if $\mathfrak{J}(X(x)) \geq 0$ for all $x \in U$, and $\mathfrak{J}$-*non-positive* on $U$ if $\mathfrak{J}(X(x)) \leq 0$ for all $x \in U$. When the quadratic form used in the context is clear, we will simply say that $X$ is non-negative or non-positive.

We say that a $C^1$ family $\mathfrak{J}$ of indefinite and non-degenerate quadratic forms is *compatible* with a continuous splitting $E_\Gamma \oplus F_\Gamma = E$ of a vector bundle over some compact subset $\Gamma$ if $E_x$ is a $\mathfrak{J}$-negative subspace and $F_x$ is a $\mathfrak{J}$-positive subspace for all $x \in \Gamma$.

**Proposition 1.11.** *[7, Proposition 1.3] A $\mathfrak{J}$-non-negative vector field $X$ on $U$ is strictly $\mathfrak{J}$-separated if, and only if, there exists a compatible family $\mathfrak{J}_0$ of forms and there exists a function $\delta : U \to \mathbb{R}$ such that the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^* \cdot J_0$ satisfies*

$$\tilde{J}_{0,x} - \delta(x) J_0 \quad \text{is positive definite}, \quad x \in U,$$

*where $DX(x)^*$ is the adjoint of $DX(x)$ with respect to the adapted inner product.*

**Remark 1.12.** *The expression for $\tilde{J}_{0,x}$ in terms of $J_0$ and the infinitesimal generator of $DX_t$ is, in fact, the time derivative of $\mathfrak{J}_0$ along the flow direction at the point $x$, which we denote $\partial_t J_0$; see item 1 of Proposition 1.18. We keep this notation in what follows.*

A characterization of dominated splittings, via quadratic forms is given in [7] (see also [77]) as follow.

**Theorem 1.13.** *[7, Theorem 2.13] The cocycle $A_t(x)$ is strictly $\mathfrak{J}$-separated if, and only if, $E_U$ admits a dominated splitting $F_- \oplus F_+$ with respect to $A_t(x)$ on the maximal invariant subset $\Lambda$ of $U$, with constant dimensions* $\dim F_- = q, \dim F_+ = p, \dim M = p + q$.

This is an algebraic/geometrical way to prove the existence of dominated splittings. As we have said in the introduction, proving existence of some hyperbolic structure is not an easy work to do, in general. One of the most habitual way is to use cone field techniques, see for instance [56, 42, 59].

In [8, Example 5], L. Salgado and V. Araújo checked out the singular hyperbolicity of geometric Lorenz attractor, in a most simple way, by using Theorem 1.1. It was proved by Tucker [69], under computer assistance, that the Lorenz attractor exist for the classical parameters.

In fact, we have an analogous result about partial hyperbolic splittings, as follow.

We say that a compact invariant subset $\Lambda$ is *non-trivial* if

- either $\Lambda$ does not contain singularities;
- or $\Lambda$ contains at most finitely many singularities, $\Lambda$ contains some regular orbit and is connected.

**Theorem 1.14.** *[7, Theorem A] A non-trivial compact invariant subset $\Gamma$ is a partially hyperbolic set for a flow $X_t$ if, and only if, there is a $C^1$ field $\mathcal{J}$ of non-degenerate and indefinite quadratic forms with constant index, equal to the dimension of the stable subspace of $\Gamma$, such that $X_t$ is a non-negative strictly $\mathcal{J}$-separated flow on a neighborhood $U$ of $\Gamma$.*

*Moreover $E$ is a negative subspace, $F$ a positive subspace and the splitting can be made almost orthogonal.*

Here strict $\mathcal{J}$-separation corresponds to strict cone invariance under the action of $DX_t$ and $\langle\cdot,\cdot\rangle$ is a Riemannian inner product in the ambient manifold. We recall that the index of a field quadratic forms $\mathcal{J}$ on a set $\Gamma$ is the dimension of the $\mathcal{J}$-negative space at every tangent space $T_xM$ for $x \in U$. Moreover, we say that the splitting $T_\Gamma M = E \oplus F$ is *almost orthogonal* if, given $\varepsilon > 0$, there exists a smooth inner product $\langle\cdot,\cdot\rangle$ on $T_\Gamma M$ so that $|\langle u, v\rangle| < \varepsilon$, for all $u \in E, v \in F$, with $\|u\| = 1 = \|v\|$.

We note that the condition stated in Theorem 1.14 allows us to obtain partial hyperbolicity checking a condition at every point of the compact invariant set that depends only on the tangent map $DX$ to the vector field $X$ together with a family $\mathcal{J}$ of quadratic forms without using the flow $X_t$ or its derivative $DX_t$. This is akin to checking the stability of singularity of a vector field using a Lyapunov function. For example, it is well known by Lyapunov's Stability Theorem that if a singularity $\sigma$ of a $C^1$ vector field $Y : U \subset \mathbb{R}^n \to \mathbb{R}^n$, defined over an open set $U$, admits a strict Lyapunov funtion on $\sigma$, then this is a asymptotically stable singularity. Lewowicz, in [48], used this idea replacing stability of a singularity by topological stability of Anosov diffeomorphisms.

### 1.1.4 Exterior powers

We note that if $E \oplus F$ is a $DX_t$-invariant splitting of $T_\Gamma M$, with $\{e_1, \dots, e_\ell\}$ a family of basis for $E$ and $\{f_1, \dots, f_h\}$ a family of basis for $F$, then $\widetilde{F} = \wedge^k F$ generated by $\{f_{i_1} \wedge \cdots \wedge f_{i_k}\}_{1 \le i_1 < \cdots < i_k \le h}$ is naturally $\wedge^k DX_t$-invariant by construction. In addition, $\tilde{E}$ generated by $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \le i_1 < \cdots < i_k \le \ell}$ together with all the exterior products of $i$ basis elements of $E$ with $j$ basis elements of $F$, where $i + j = k$ and $i, j \ge 1$, is also $\wedge^k DX_t$-invariant and, moreover, $\widetilde{E} \oplus \widetilde{F}$ gives a splitting of the $k$th exterior power $\wedge^k T_\Gamma M$ of the subbundle $T_\Gamma M$. Let $T_\Gamma M = E_\Gamma \oplus F_\Gamma$ be a $DX_t$-invariant splitting over the compact $X_t$-invariant subset $\Gamma$ such that $\dim F = k \ge 2$. Let $\widetilde{F} = \wedge^k F$ be the $\wedge^k DX_t$-invariant subspace generated by the vectors of $F$ and $\tilde{E}$ be the $\wedge^k DX_t$-invariant subspace such that $\widetilde{E} \oplus \widetilde{F}$ is a splitting of the $k$th exterior power $\wedge^k T_\Gamma M$ of the subbundle $T_\Gamma M$.

We consider the action of the cocycle $DX_t(x)$ on $k$-vector that is the $k$-exterior $\wedge^k DX_t$ of the cocycle acting on $\wedge^k T_\Gamma M$.

We denote by $\|\cdot\|$ the standard norm on $k$-vectors induced by the Riemannian norm of $M$, see [12].

**Remark 1.15.** *Let $V$ to be a vector space of dimension $N$.*

*(i) The dimension of space $\wedge^r V$ is $\dim \wedge^r V = \binom{N}{r}$. If $\{e_1, \cdots, e_N\}$ is a basis of $V$, so the set $\{e_{k_1} \wedge \cdots \wedge e_{k_r} : 1 \leq k_1 < \cdots < k_r \leq N\}$ is a basis in $\wedge^r V$ with $\binom{N}{r}$ elements.*

*(ii) If $V$ has the inner product $\langle , \rangle$, then the bilinear extension of*

$$\langle u_1 \wedge \cdots \wedge u_r, v_1 \wedge \cdots \wedge v_r \rangle := \det(\langle u_i, v_j \rangle)_{r \times r}$$

*defines a inner product in $\wedge^r V$. In particular, $||u_1 \wedge \cdots \wedge u_r|| = \sqrt{\det(\langle u_i, u_j \rangle)_{r \times r}}$ is the volume of $r$-dimensional parallelepiped $H$ spanned by $u_1, \cdots, u_r$, we write $\mathrm{vol}(u_1, \cdots, u_r) = \mathrm{vol}(H) = \det(H) = |\det(u_1, \cdots, u_r)|$.*

*(iii) If $A : V \to V$ is a linear operator then the linear extension of $\wedge^r A(u_1 \wedge \cdots \wedge u_r) = A(u_1) \wedge \cdots \wedge A(u_r)$ defines a linear operator $\wedge^r A$ on $\wedge^r V$.*

*(iv) Let $A : V \to V$, and $\wedge^r A : \wedge^r V \to \wedge^r V$ linear operators with $G$ spanned by $v_1, \cdots, v_s \in V$. Define $H := A|_G$, then $H$ is spanned by $A(v_1), \cdots, A(v_s)$ . So $|\det A|_G| = \mathrm{vol}(A|_G) = \mathrm{vol}(H) = \mathrm{vol}(A(v_1), \cdots, A(v_s)) = ||A(v_1) \wedge \cdots \wedge A(v_s)|| = || \wedge^s A(v_1 \wedge \cdots \wedge v_s)||$.*

When $DX_t(u_i) = v_i(t) = v_i$, where G is spanned by $u_1, \cdots, u_r \in T_\Gamma M$, and $H$ is spanned by $v_1, \cdots, v_r$, we have $H = DX_t(G) = DX_t|_G$. Thus,

$$|\det(DX_t|_G)| = \mathrm{vol}(DX_t(u_1), \cdots, DX_t(u_r)) =$$
$$||DX_t(u_1) \wedge \cdots \wedge DX_t(u_r)|| = || \wedge^r DX_t(u_1 \wedge \cdots \wedge u_r)||.$$

It is natural to consider the linear multiplicative cocyle $\wedge^k DX_t$ over the flow $X_t$ of $X$ on $U$, that is, for any $k$ choice, $u_1, u_2, \cdots, u_k$ of vectors in $T_x M, x \in U$ and $t \in \mathbb{R}$ such that $X_t(x) \in U$ we set

$$(\wedge^k DX_t) \cdot (u_1 \wedge u_2 \wedge \cdots \wedge u_k) = (DX_t \cdot u_1) \wedge (DX_t \cdot u_2) \wedge \cdots \wedge (DX_t \cdot u_k)$$

see [12, Chapter 3, Section 2.3] or [74] for more details and standard results on exterior algebra and exterior products of linear operator.

In [8], L. Salgado and V. Araújo proved the following relation between a dominated splitting and its exterior power.

**Theorem 1.16.** *[8, Theorem A] The splitting $T_\Gamma M = E \oplus F$ is dominated for $DX_t$ if, and only if, $\wedge^k T_\Gamma M = \widetilde{E} \oplus \widetilde{F}$ is a dominated splitting for $\wedge^k DX_t$.*

Hence, the existence of a dominated splitting $T_\Gamma M = E_\Gamma \oplus F_\Gamma$ over the compact $X_t$-invariant subset $\Gamma$, is equivalent to the bundle $\wedge^k T_\Gamma M$ admits a dominated splitting with respect to $\wedge^k DX_t : \wedge^k T_\Gamma M \to \wedge^k T_\Gamma M$.

As a consequence, they obtain the next characterization of three-dimensional singular sets.

**Corollary 1.17.** *[8, Corollary 1.5] Assume that $M$ has dimension 3, $E$ is uniformly contracted by $DX_t$, and that $k = 2$. Then $E \oplus F$ is a singular-hyperbolic splitting for $DX_t$ if, and only if, $\widetilde{E} \oplus \widetilde{F}$ is partially hyperbolic splitting for $\wedge^2 DX_t$ such that $\widetilde{F}$ is uniformly expanded by $\wedge^2 DX_t$.*

### 1.1.5 Properties of $\mathfrak{J}$-separated linear multiplicative cocycles

We present some useful properties about $\mathfrak{J}$-separated linear cocycles whose proofs can be found in [7].

Let $A_t(x)$ be a linear multiplicative cocycle over $X_t$. We define the infinitesimal generator of $A_t(x)$ by

$$D(x) := \lim_{t \to 0} \frac{A_t(x) - Id}{t}. \tag{1.5}$$

The following is the basis for arguments given by L. Salgado and V. Araújo in [7] to prove the Theorem 1.14.

**Proposition 1.18.** *[7, Proposition 2.7] Let $A_t(x)$ be a cocycle over $X_t$ defined on an open subset $U$ and $D(x)$ its infinitesimal generator. Then*

1. *$\tilde{\mathfrak{J}}(v) = \partial_t \mathfrak{J}(A_t(x)v) = \langle \tilde{J}_{X_t(x)} A_t(x)v, A_t(x)v \rangle$ for all $v \in E_x$ and $x \in U$, where*

$$\tilde{J}_x := J \cdot D(x) + D(x)^* \cdot J \tag{1.6}$$

*and $D(x)^*$ denotes the adjoint of the linear map $D(x) : E_x \to E_x$ with respect to the adapted inner product at $x$;*

2. *the cocycle $A_t(x)$ is $\mathfrak{J}$-separated if, and only if, there exists a neighborhood $V$ of $\Lambda$, $V \subset U$ and a function $\delta : V \to \mathbb{R}$ such that*

$$\tilde{\mathfrak{J}}_x \geq \delta(x)\mathfrak{J}_x \quad \text{for all} \quad x \in V. \tag{1.7}$$

*In particular we get $\partial_t \log |\mathfrak{J}(A_t(x)v)| \geq \delta(X_t(x))$, $v \in E_x, x \in V, t \geq 0$;*

3. *if the inequalities in the previous item are strict, then the cocycle $A_t(x)$ is strictly $\mathfrak{J}$-separated. Reciprocally, if $A_t(x)$ is strictly $\mathfrak{J}$-separated, then there exists a compatible family $\mathfrak{J}_0$ of forms on $V$ satisfying the strict inequalities of item (2).*

4. *For a $\mathfrak{J}$-separated cocycle $A_t(x)$, we have $\frac{|\mathfrak{J}(A_{t_2}(x)v)|}{|\mathfrak{J}(A_{t_1}(x)v)|} \geq \exp \Delta_{t_1}^{t_2}(x)$ for all $v \in E_x$ and reals $t_1 < t_2$ so that $\mathfrak{J}(A_t(x)v) \neq 0$ for all $t_1 \leq t \leq t_2$, where $\Delta_{t_1}^{t_2}(x)$ was defined in (1.8).*

5. *we can bound $\delta$ at every $x \in \Gamma$ by $\inf_{v \in C_+(x)} \frac{\tilde{\mathfrak{J}}(v)}{\mathfrak{J}(v)} \leq \delta(x) \leq \sup_{v \in C_-(x)} \frac{\tilde{\mathfrak{J}}(v)}{\mathfrak{J}(v)}$.*

**Remark 1.19.** *We stress that the necessary and sufficient condition in items (2-3) of Proposition 1.18, for (strict) $\mathfrak{J}$-separation, shows that a cocycle $A_t(x)$ is (strictly) $\mathfrak{J}$-separated if, and only if, its inverse $A_{-t}(x)$ is (strictly) $(-\mathfrak{J})$-separated.*

**Remark 1.20.** *Item (2) above of Proposition 1.18 shows that $\delta$ is a measure of the "minimal instantaneous expansion rate" of $|\mathfrak{J} \circ A_t(x)|$.*

The area under the function $\delta$ provided by Proposition 1.18 allows us to detect different dominated splittings with respect to linear multiplicative cocycles on vector bundles (Proposition 1.21). For this, define the function

$$\Delta_a^b(x) := \int_a^b \delta(X_s(x))\, ds, \quad x \in \Gamma, a, b \in \mathbb{R}. \tag{1.8}$$

**Proposition 1.21.** *[7, Theorem 2.23] Let $\Gamma$ be a compact invariant set for $X_t$ admitting a dominated splitting $E_\Gamma = F_- \oplus F_+$ for $A_t(x)$, a linear multiplicative cocycle over $\Gamma$ with values in $E$. Let $\mathfrak{J}$ be a $C^1$ family of indefinite quadratic forms such that $A_t(x)$ is strictly $\mathfrak{J}$-separated. Then*

1. *$F_- \oplus F_+$ is partially hyperbolic with $F_+$ uniformly expanding if $\Delta_0^t(x) \xrightarrow[t \to +\infty]{} +\infty$ for all $x \in \Gamma$.*

2. *$F_- \oplus F_+$ is partially hyperbolic with $F_-$ uniformly contracting if $\Delta_0^t(x) \xrightarrow[t \to +\infty]{} -\infty$ for all $x \in \Gamma$.*

3. *$F_- \oplus F_+$ is uniformly hyperbolic if, and only if, there exists a compatible family $\mathfrak{J}_0$ of quadratic forms in a neighborhood of $\Gamma$ such that $\mathfrak{J}_0'(v) > 0$ for all $v \in E_x$ and all $x \in \Gamma$.*

For the proof and more details about the Proposition 1.21, see [7].

## 1.2 Auxiliary results

### 1.2.1 Exterior products and main Lemma

From now, we present some properties about exterior products and the main lemma to prove the Theorem A. Next, we are going to use Proposition 1.21 to obtain sufficient conditions for a flow $X_t$ on a $m$-manifold $M$ to have a $\wedge^{m-1}DX_t$-invariant one-dimensional uniformly expanding direction orthogonal to the $(m-1)$-dimensional center-unstable bundle.

Let $V$ a $m$-dimensional vector space, we denote $V$ by $V^m$, consider $\wedge^k V^m$ where $2 \leq k \leq m$. Let $\mathcal{B} = \{e_1, \cdots, e_m\}$ a basis of $V^m$. So $\{e_{j_1} \wedge \cdots \wedge e_{j_k} : 1 \leq j_1 < \cdots < j_k \leq m\}$ is a basis of $\wedge^k V^m$, and $J := \{(j_1, \cdots, j_k) \in \mathbb{N}^k : 1 \leq j_1 < \cdots < j_k \leq m\}$. Let $l = \binom{m}{k}$, so we have $l$ combination of $k$ vectors in $\{e_1, \cdots, e_m\}$, and $|J| = l$.

Take $u_1, u_2, \cdots, u_k \in V^m$ where $u_j = (u_j^1, u_j^2, \cdots, u_j^m)_{\mathcal{B}}$ for all $j \in \{1, \cdots, k\}$. Define

$$\mathcal{C} := \begin{pmatrix} u_1^1 & ... & u_k^1 \\ ... & ... & ... \\ u_1^m & ... & u_k^m \end{pmatrix}_{m \times k}. \tag{1.9}$$

For $(j_1, \cdots, j_k) \in J$, consider

$$\mathcal{C}^{j_1,\ldots,j_k} := \begin{pmatrix} u_1^{j_1} & ... & u_k^{j_1} \\ ... & ... & ... \\ u_1^{j_k} & ... & u_k^{j_k} \end{pmatrix}_{k \times k} \tag{1.10}$$

The following result holds

$$u_1 \wedge \cdots \wedge u_k = \sum_{(j_1,\ldots,j_k) \in J} \det(\mathcal{C}^{j_1,\ldots,j_k})(e_{j_1} \wedge \cdots \wedge e_{j_k}). \tag{1.11}$$

Let $A : V^m \to V^m$ a linear operator with matrix in basis $\mathcal{B}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}_{(m\times m)} . \tag{1.12}$$

We will denote this matrix by $A$ too.

Consider $\wedge^k A : \wedge^k V^m \to \wedge^k V^m$, note that $A(u_1)\wedge\cdots\wedge A(u_k) = \wedge^k A(u_1\wedge\cdots\wedge u_k)$, by (1.11) and the linearity of $\wedge^k A$, we have that

$$A(u_1) \wedge \cdots \wedge A(u_k) = \sum_{(j_1,\cdots,j_k)\in J} \det(\mathcal{C}^{j_1,\cdots,j_k}) \wedge^k A(e_{j_1} \wedge \cdots \wedge e_{j_k}) \tag{1.13}$$

Define $A_j := A(e_j)$, so $A_j$ is the $j$-th column of $A$, i.e., $A(e_j) = A_j = (a_{1j}, \cdots, a_{mj})^T$, so $A(e_j) = [a_{ij}]_{m\times 1}$. Let $A_{j_1\cdots j_k} := (A_{j_1} \cdots A_{j_k})_{m\times k}$ where $(j_1, \cdots, j_k) \in J$. For each $(i_1 \cdots i_k), (j_1 \cdots j_k) \in J$ consider

$$A^{i_1\cdots i_k}_{j_1\cdots j_k} := \begin{pmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_k} \\ \dots & \dots & \dots \\ a_{i_k j_1} & \dots & a_{i_k j_k} \end{pmatrix}_{k\times k} \tag{1.14}$$

Using that $\wedge^k A(e_{j_1} \wedge \cdots \wedge e_{j_k}) = A(e_{j_1}) \wedge \cdots \wedge A(e_{j_k})$ with matrix

$$A_{j_1\cdots j_k} := (A_{j_1} \cdots A_{j_k})_{m\times k},$$

by (1.11) we obtain that

$$A(e_{j_1}) \wedge \cdots \wedge A(e_{j_k}) = \sum_{(i_1,\cdots,i_k)\in J} \det(A^{i_1\cdots i_k}_{j_1\cdots j_k})(e_{i_1} \wedge \cdots \wedge e_{i_k}). \tag{1.15}$$

**Lemma 1.22.** *Let $V$ to be vector space and $A : V \to V$ to be a linear operator then $\wedge^{(m-1)} A = \det(A) \cdot (A^{-1})^*$.*

*Under suitable identification, the announced formula holds for differential of a diffeomorphism of a compact finite dimensional manifold.*

*Proof.* Consider $k = m - 1$. We use the following identification between $\wedge^{(m-1)} V$ and $V$. For each $(j_1, \cdots, j_{m-1}) \in J$, we identify $e_{j_1} \wedge \cdots \wedge e_{j_{(m-1)}}$ in $\wedge^{(m-1)} V$ by $\delta_p e_p$ in $V$, where $p \notin \{j_1, \cdots, j_{m-1}\}$, $\delta_p = 1$ if $p$ is odd, and $\delta_p = -1$ if $p$ is even.

We must show that for each $(j_1, \cdots, j_{m-1}) \in J$ the exterior product $\wedge^{(m-1)} A(e_{j_1} \wedge \cdots \wedge e_{j_{(m-1)}})$ corresponds to the $\det(A) \cdot (A^{-1})^*(\delta_p e_p)$, where $\delta_p e_p$ is given as above.

Define $S := \det(A) \cdot (A^{-1})^*$, using that $A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A)$, we obtain that $S = \operatorname{cof}(A)$ where $\operatorname{cof}(A) = [(-1)^{i+j} M_{ij}]_{m\times m}$ and $M_{ij}$ is the determinant of the submatrix formed by deleting the $i$-th row and $j$-th column. We have that $M_{ij} = \det(A^{r_1\cdots r_k}_{s_1\cdots s_k})$ where $i \notin \{r_1, \cdots, r_k\}$ and $j \notin \{s_1, \cdots, s_k\}$.

Note that

$$\operatorname{cof}(A)(\delta_p e_p) = \delta_p \operatorname{cof}(A)(e_p) = \delta_p((-1)^{1+p} M_{1p}, (-1)^{2+p} M_{2p}, \cdots, (-1)^{m+p} M_{mp})_{\mathcal{B}}.$$

In case $p$ is odd, $\delta_p = 1$ and $\operatorname{cof}(A)(\delta_p e_p) = (M_{1p}, -M_{2p}, \cdots, (-1)^{m+p} M_{mp})_{\mathcal{B}}$.
We obtain that

$$\operatorname{cof}(A)(\delta_p e_p) = M_{1p} e_1 + M_{2p}(-e_2) + \cdots + M_{mp}(-1)^{m+p} e_{mp} = \\ M_{1p}(e_1 \delta_1) + M_{2p}(e_2\delta_2) + \cdots + M_{mp}(e_{mp}\delta_{mp}).$$

Using that

$$A(e_{j_1}) \wedge \cdots \wedge A(e_{j_k}) = \sum_{(i_1,\cdots,i_k)\in J} \det(A^{i_1\cdots i_k}_{j_1\cdots j_k})(e_{i_1} \wedge \cdots \wedge e_{i_k})$$

and $M_{ij} = \det(A^{r_1\cdots r_k}_{s_1\cdots s_k})$ where $i \notin \{r_1, \cdots, r_k\}$ and $j \notin \{s_1, \cdots, s_k\}$, we have that $\operatorname{cof}(A)(\delta_p e_p) \cong A(e_{j_1}) \wedge \cdots \wedge A(e_{j_k})$.

This concludes the proof. □

The result below generalizes Corollary 1.17 to arbitrary $n$ and $k$. The main difficulty here is working on the dimensions of the subbundles and its exterior powers.

**Lemma 1.23.** *The subbundle $F_\Gamma$ is volume expanding by $DX_t$ if, and only if, $\widetilde{F}$ is uniformly expanded by $\wedge^k DX_t$.*

*In particular, $E \oplus F$ is a singular hyperbolic splitting, where $F$ is volume expanding for $DX_t$ if, and only if, $\widetilde{E} \oplus \widetilde{F}$ is partially hyperbolic splitting for $\wedge^k DX_t$ such that $\widetilde{F}$ is uniformly expanded by $\wedge^k DX_t$.*

*Proof.* We consider the action of the cocycle $DX_t(x)$ on $k$-vector that is the $k$-exterior power $\wedge^k DX_t$ of the cocycle acting on $\wedge^k T_\Gamma M$.

Denote by $\|\cdot\|$ the standard norm on $k$-vectors induced by the Riemannian norm of $M$; see, e.g. [12]. We write $m = \dim M$.

Suppose that $T_\Gamma M$ admits a splitting $E_\Gamma \oplus F_\Gamma$ with $\dim E_\Gamma = m-k$ and $\dim F_\Gamma = k$.

We note that if $E \oplus F$ is a $DX_t$-invariant splitting of $T_\Gamma M$, with $\{e_1, \ldots, e_{(m-k)}\}$ a family of basis for $E$ and $\{f_1, \ldots, f_k\}$ a family of basis for $F$, then $\widetilde{F} = \wedge^k F$ generated by $\{f_{i_1} \wedge \cdots \wedge f_{i_k}\}_{1 \le i_1 < \cdots < i_k \le k}$ is naturally $\wedge^k DX_t$-invariant by construction. Then, the dimension of $\widetilde{F}$ is one with basis given by the vector $f_1 \wedge \cdots \wedge f_k$.

Assume that $F_\Gamma$ is volume expanding by $DX_t$. We must show that there exist $C$ and $\lambda > 0$ such that $|\wedge^k DX_t|_P| \geq Ce^{\lambda t}$, for all $t > 0$, where $P$ is spanned by $f_1 \wedge \cdots \wedge f_k$.

Note that

$$|| \wedge^k DX_t|_P|| = || \wedge^k DX_t(f_1 \wedge \cdots \wedge f_k)|| = ||DX_t(f_1) \wedge \cdots \wedge DX_t(f_k)||.$$

But $f_1, \cdots, f_k$ is a basis for $F$, by hypothesis there exist constants $C$ and $\lambda > 0$ such that $|\det(DX_t|_F)| \geq C.e^{\lambda t}$ for all $t > 0$. So,

$|\det(DX_t|_F)| = \text{vol}(DX_t(f_1), \cdots, DX_t(f_k)) = ||DX_t(f_1) \wedge \cdots \wedge DX_t(f_k)||.$

The reciprocal statement is straightforward.

Given a basis $\{f_1, \cdots, f_k\}$ of $F$, we have that

$$\begin{aligned} |\det(DX_t|_F)| &= \\ \text{vol}(DX_t(f_1), \cdots, DX_t(f_k)) = ||DX_t(f_1) \wedge \cdots \wedge DX_t(f_k)|| &= \\ || \wedge^k DX_t(f_1 \wedge \cdots \wedge f_k)|| = || \wedge^k DX_t|_P|| & \end{aligned}$$

where $P$ is spanned by $f_1 \wedge \cdots \wedge f_k$.

However, by hypothesis, there exist $C$ and $\lambda > 0$ such that $|| \wedge^k DX_t|_P|| \geq Ce^{\lambda t}$, for all $t > 0$. □

**Corollary 1.24.** *Assume that $E$ is uniformly contracted by $DX_t$. $E \oplus F$ is a singular-hyperbolic splitting for $DX_t$ if, and only if, $\widetilde{E} \oplus \widetilde{F}$ is partially hyperbolic splitting for $\wedge^k DX_t$ such that $\widetilde{F}$ is uniformly expanded by $\wedge^k DX_t$.*

Let $M$ be a Riemannian manifold $m$-dimensional with $\langle \cdot, \cdot \rangle$ inner product in $T_\Gamma M$, and $\langle \cdot, \cdot \rangle_*$ the inner product in $\wedge^k T_\Gamma M$ induced by $\langle \cdot, \cdot \rangle$ where $\wedge^k T_\Gamma M = \bigcup_{x \in \Gamma} \wedge^k T_x M$. So for $x \in \Gamma$, we have that $\langle \cdot, \cdot \rangle$ is defined on $T_x M$, and $\langle \cdot, \cdot \rangle_*$ is defined on $\wedge^k T_x M$.

**Lemma 1.25.** *Let $M$ be a Riemannian $m$-dimensional manifold. Then, for each inner product $[\cdot, \cdot]_*$ in $\wedge^{(m-1)} T_\Gamma M$ there exists an inner product $[\cdot, \cdot]$ on $T_\Gamma M$ such that $[\cdot, \cdot]_*$ is induced by $[\cdot, \cdot]$.*

*Proof.* Let $M$ be a Riemannian $m$-dimensional manifold with an inner product $\langle \cdot, \cdot \rangle$ in $T_\Gamma M$, and $\langle \cdot, \cdot \rangle_*$ the inner product in $\wedge^{(m-1)} T_\Gamma M$ induced by $\langle \cdot, \cdot \rangle$.

Take $[\cdot, \cdot]_{**}$ an arbitrary inner product in $\wedge^{(m-1)} T_\Gamma M$. Using that $[\cdot, \cdot]_{**}$ and $\langle \cdot, \cdot \rangle_*$ are inner products in $\wedge^{(m-1)} T_\Gamma M$ there exists a linear isomorphism $J : \wedge^{(m-1)} T_\Gamma M \to \wedge^{(m-1)} T_\Gamma M$ such that $[u, v]_{**} = \langle J(u), J(v) \rangle_*$.

Define $\varphi : GL(T_\Gamma M) \to GL(\wedge^{(m-1)} T_\Gamma M)$ given by $A \mapsto \wedge^{(m-1)} A$.

Note that $\varphi$ is an injective linear homomorphism, and due to the dimensions of the spaces, $\varphi$ is a linear isomorphism.

Hence, there exists $A \in GL(T_\Gamma M)$ such that $\wedge^{(m-1)}A = J$.

Consider $[x,y] := \langle A(x), A(y)\rangle$ for $x, y \in T_z M$ and $z \in \Gamma$. Then if $u = u_1 \wedge \cdots \wedge u_{(m-1)}$ and $v = v_1 \wedge \cdots \wedge v_{(m-1)}$ we get $[u,v]_* = \det([u_i, v_j])_{(m-1)\times(m-1)} = \det(\langle Au_i, Av_j\rangle)_{(m-1)\times(m-1)}$.

We have that

$$[u,v]_* = \det(\langle A(u_i), A(v_j)\rangle)_{(m-1)\times(m-1)} = \langle \wedge^{(m-1)}A(u), \wedge^{(m-1)}A(v)\rangle_*.$$

On the other hand,

$$[u,v]_{**} = \langle J(u), J(v)\rangle_* = \langle \wedge^{(m-1)}A(u), \wedge^{(m-1)}A(v)\rangle_*.$$

Therefore, $[\cdot,\cdot]_* = [\cdot,\cdot]_{**}$, and we are done. □

## 1.3 Proofs of main results

We are now able to prove our main results.

### 1.3.1 Proof of Theorem A

*Proof.* Consider a $m$-manifold $M$ and $\Gamma$ a compact $X_t$-invariant subset having a singular-hyperbolic splitting $T_\Gamma M = E_\Gamma \oplus F_\Gamma$ with $\dim E_\Gamma = 1$ . By Theorem 1.16 we have a $\wedge^{(m-1)}DX_t$-invariant partial hyperbolic splitting $\wedge^{(m-1)}T_\Gamma M = \widetilde{E} \oplus \widetilde{F}$ with $\dim \widetilde{F} = 1$ and $\widetilde{F}$ uniformly expanded. Following the proof of Theorem 1.16, if we write $e$ for a unit vector in $E_x$ and $\{u_1, u_2, \cdots, u_{m-1}\}$ an orthonormal base for $F_x$, $x \in \Gamma$, then $\widetilde{E}_x$ is a $(m-1)$-dimensional vector space spanned by set $\{e \wedge u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_{m-2}}$ with $i_1, \cdots, i_{m-2} \in \{1, \cdots, m-1\}\}$.

From Theorem 1.14 and the existence of adapted metrics (see e.g. [34]), there exists a field $\mathfrak{J}$ of quadratic forms so that $X$ is $\mathfrak{J}$-non-negative, $DX_t$ is strictly $\mathfrak{J}$-separated on a neighborhood $U$ of $\Gamma$, $E_\Gamma$ is a negative subbundle, $F_\Gamma$ is a positive subbundle and these subspaces are almost orthogonal. In other words, there exists a function $\delta : \Gamma \to \mathbb{R}$ such that $\tilde{\mathfrak{J}}_x - \delta(x)\mathfrak{J}_x > 0, x \in \Gamma$ and we can locally write $\mathfrak{J}(v) = \langle J(v), v\rangle$ where $J = \operatorname{diag}\{-1, 1, \cdots, 1\}$ with respect to the basis $\{e, u_1, \cdots, u_{m-1}\}$ and $\langle\cdot,\cdot\rangle$ is the adapted inner product; see [7].

By lemma 1.22, $\wedge^{(m-1)}A = \det(A) \cdot (A^{-1})^*$ with respect to the adapted inner product which trivializes $\mathfrak{J}$, for any linear transformation $A : T_x M \to T_y M$. Hence $\wedge^{m-1}DX_t(x) = \det(DX_t(x)) \cdot (DX_{-t} \circ X_t)^*$ and the infinitesimal generator $D^{(m-1)}(x)$ of $\wedge^{(m-1)}DX_t$ is the same as $\operatorname{tr}(DX(x)) \cdot Id - DX(x)^*$.

Therefore, using the identification between $\wedge^{(m-1)}T_xM$ and $T_xM$ through the adapted inner product, and Proposition 1.18

$$\begin{aligned}\hat{\mathcal{J}}_x(v) = \partial_t(-\mathcal{J})(\wedge^{(m-1)}DX_t\cdot v)\mid_{t=0} &= \langle -(J\cdot D^{(m-1)}(x) + D^{(m-1)}(x)^*\cdot J)v, v\rangle \\ &= \langle [(\mathcal{J}\cdot DX(x) + DX(x)^*\cdot\mathcal{J}) - 2\operatorname{tr}(DX(x))\mathcal{J}]v, v\rangle \\ &= (\tilde{\mathcal{J}} - 2\operatorname{tr}(DX(x))\mathcal{J})(v). \end{aligned} \tag{1.16}$$

To obtain strict $(-\mathcal{J})$-separation of $\wedge^{(m-1)}DX_t$ we search a function $\delta_{(m-1)} : \Gamma \to \mathbb{R}$ so that

$$(\tilde{\mathcal{J}} - 2\operatorname{tr}(DX)\mathcal{J}) - \delta_{(m-1)}(-\mathcal{J}) > 0 \quad \text{or} \quad \tilde{\mathcal{J}} - (2\operatorname{tr}(DX) - \delta_{(m-1)})\mathcal{J} > 0.$$

Hence it is enough to make $\delta_{(m-1)} = 2\operatorname{tr}(DX) - \delta$. This shows that in our setting $\wedge^{(m-1)}DX_t$ is always strictly $(-\mathcal{J})$-separated.

Finally, according to Proposition 1.21, to obtain the partial hyperbolic splitting of $\wedge^{(m-1)}DX_t$ which ensures singular-hyperbolicity, it is sufficient that either $\hat{\mathcal{J}}_x$ is positive definite or $\widetilde{\Delta}_a^b(x) = \int_a^b \delta_{(m-1)}(X_s(x))\,ds$ satisfies item (1) of Proposition 1.21, for all $x \in \Gamma$. This amounts precisely to the sufficient condition in the statement of Theorem A and we are done. □

Finally, we present the proof of Theorem B.

### 1.3.2 Proof of Theorem B

Let $\langle\cdot,\cdot\rangle$ to be a Riemannian metric on $TM$ and denote $\langle\cdot,\cdot\rangle_x : T_xM \times T_xM \to \mathbb{R}$ the induced inner product on $T_xM$. We denote by $\langle\cdot,\cdot\rangle_{x,*}$ the induced metric on $\wedge^k T_xM$ as in Subsection 1.1.4. In particular, $\|u\|_{x,*} := \sqrt{\langle u,u\rangle_{x,*}}$ for $u \in \wedge^k T_xM$.

Define the $k$-exterior tangent bundle $\wedge^k TM$ by $\bigcup_{x\in M}\{x\}\times\wedge^k T_xM$ and the $k$-exterior unit tangent bundle $\wedge_1^k TM$ by $\{(x,u)\in\wedge^k TM : u \in \wedge^k T_xM \text{ and } |u|_{x,*} \le 1\}$.

We are now ready to present the proof of Theorem B.

*Proof of Theorem B.* Let a singular-hyperbolic set $\Gamma$ for a $C^1$ vector field $X$ be given with a splitting $E_\Gamma \oplus F_\Gamma$ with $\dim E_\Gamma = m-k$ and $\dim F_\Gamma = k$.

Then $\widetilde{F} = \wedge^k F$ generated by $\{f_{i_1}\wedge\cdots\wedge f_{i_k}\}_{1\le i_1<\cdots<i_k\le k}$ is naturally $\wedge^k DX_t$-invariant by construction, where $\{f_1,\ldots,f_k\}$ a basis for $F$. So $\dim(\widetilde{F}) = 1$ with basis given by the vector $f_1\wedge\cdots\wedge f_k$.

By Corollary 2.2, we have a partially hyperbolic splitting $\widetilde{E}\oplus\widetilde{F}$ for $\wedge^k DX_t$ such that $\widetilde{F}$ is uniformly expanded by $\wedge^k DX_t$. Hence, from [34, Theorem 1], there exists an adapted inner product $[\cdot,\cdot]$ for $\wedge^k DX_t$ over $\Gamma$, that is, there exists $\lambda > 0$ satisfying

$[\wedge^k DX_t \mid_{\tilde{E}_x}] \cdot [\wedge^k DX_{-t} \mid_{\tilde{F}_{X_t(x)}}] \leq e^{-\lambda t}$ and $[\wedge^k DX_t \mid_{\tilde{F}_x}] \geq e^{\lambda t}, \quad \forall t > 0, x \in \Gamma$.

By Lemma 1.25, there exists an inner product $[[\cdot,\cdot]]$ on $T_\Gamma M$ such that $[\cdot,\cdot]$ is induced by $[[\cdot,\cdot]]$.

So there exists an inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $T_\Gamma M$ with induced inner product $\langle\langle\cdot,\cdot\rangle\rangle_*$ on $\wedge^{(m-1)}T_\Gamma M$ and $\lambda > 0$ such that

$\| \wedge^k DX_t \mid_{\tilde{E}_x} \|_* \cdot \| \wedge^k DX_{-t} \mid_{\tilde{F}_{X_t(x)}} \|_* \leq e^{-\lambda t}$ and $\| \wedge^k DX_t \mid_{\tilde{F}_x} \|_* \geq e^{\lambda t}, \quad \forall t > 0, x \in \Gamma$ where $\|\cdot\|$ is the norm induced by $\langle\langle\cdot,\cdot\rangle\rangle$.

Assuming the existence of this inner product defined on $T_\Gamma M$ we prove the following Lemma.

**Lemma 1.26.** *Suppose that there exists an inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $T_\Gamma M$ with induced inner product $\langle\langle\cdot,\cdot\rangle\rangle_*$ on $\wedge^{(m-1)}T_\Gamma M$ and $\lambda > 0$ such that we have the following inequalities $\| \wedge^k DX_t \mid_{\tilde{E}_x} \|_* \cdot \| \wedge^k DX_{-t} \mid_{\tilde{F}_{X_t(x)}} \|_* \leq e^{-\lambda t}$ and $\| \wedge^k DX_t \mid_{\tilde{F}_x} \|_* \geq e^{\lambda t}$ for all $t \in \mathbb{R}$ and $x \in \Gamma$. Then there exists an inner product $\langle\cdot,\cdot\rangle$ in $T_\Gamma M$ such that for all $t > 0$*

1. $|DX_t \mid_{E_x} | \cdot |DX_{-t} \mid_{F_{X_t(x)}} | \leq e^{-\lambda t}$;
2. $| \wedge^k DX_t \mid_{\tilde{E}_x} |_* \cdot | \wedge^k DX_{-t} \mid_{\tilde{F}_{X_t(x)}} |_* \leq e^{-\lambda t}$; *and*
3. $| \wedge^k DX_t \mid_{\tilde{F}_x} |_* \geq e^{\lambda t}$.

*where $|\cdot|$ is the norm induced by $\langle\cdot,\cdot\rangle$.*

*Proof.* Let $u \in E_x$ and $v \in F_{X_t(x)}$ be such that $\|u\| = 1 = \|v\|$. We observe that for a given fixed $t \in \mathbb{R}$

$$\|DX_t u\| \cdot \|DX_{-t} v\| = \| \wedge^k DX_t(u \wedge u_2 \wedge \cdots \wedge u_k)\| \cdot \| \wedge^k DX_{-t}(v \wedge v_2 \wedge \cdots \wedge v_k)\|$$

if we choose $u_2, \cdots, u_k \in T_x M$ and $v_2, \cdots, v_k \in F_{X_t(x)}$ such that:

- $\langle DX_t u, DX_t u_j\rangle = 0$ for $2 \leq j \leq k$ and $\langle DX_t u_j, DX_t u_l\rangle = \delta_{jl}$ for $2 \leq j, l \leq k$;
- $\langle DX_{-t} v, DX_{-t} v_j\rangle = 0$ for $2 \leq j \leq k$ and $\langle DX_{-t} v_j, DX_{-t} v_l\rangle = \delta_{jl}$ for $2 \leq j, l \leq k$.

Consequently we obtain

$$\begin{aligned}\|DX_t u\| \cdot \|DX_{-t} v\| &\leq \| \wedge^k DX_t\| \| \wedge^k DX_{-t}\| \|u \wedge u_2 \wedge \cdots \wedge u_k\| \cdot \|v \wedge v_2 \wedge \cdots \wedge v_k\| \\ &\leq e^{-\lambda t}\|u \wedge u_2 \wedge \cdots \wedge u_k\| \cdot \|v \wedge v_2 \wedge \cdots \wedge v_k\|.\end{aligned}$$

We note that $\|u_j\| \leq \|DX_{-t}(x)\|$ since $\|DX_t u_j\| = 1$ and analogously $\|v_j\| \leq \|DX_t(X_t x)\|$ since $\|DX_{-t} v_j\| = 1$ for $2 \leq j \leq k$ with $\|u\| = \|v\| = 1$.

We now set $R = \max\{1, \kappa_1\}$, where

$$\kappa_1 = \sup_{t\in[-1,1]} \sup_{x\in\Gamma} \|DX_t(x)\|$$

and define $B[0,R] = \{\eta \in T\Gamma : |\eta| \le R\}$ a compact subset of $T\Gamma$.

Note that if we set $t \in [-1,1]$, then we get $u, u_2, \cdots, u_k, v, v_2, \cdots, v_k \in B[0,R]$ in the argument above.

Moreover $\prod_{i=1}^k B[0,R]$ is a compact subset of $\prod_{i=1}^k T\Gamma = \sum_{p\in\Gamma} T_p\Gamma \times \overset{k}{\cdots} \times T_p\Gamma$ and let $\mathcal{I} : \prod_{i=1}^k T\Gamma \to \wedge^k T\Gamma$ be the natural injection given by

$$(w_1, \cdots, w_k) \mapsto w_1 \wedge \cdots \wedge w_k.$$

We can now define $|\cdot| = \gamma\|\cdot\|$ (or $\langle\cdot,\cdot\rangle = \gamma^2[[\cdot,\cdot]]$) where $\gamma$ is a positive number such that

$$\sup_{w\in\prod_{i=1}^k B[0,R]} \|\mathcal{I}(w)\| \le \gamma^{-1}.$$

It follows that

$$\begin{aligned}|DX_t u| \cdot |DX_{-t} v| &= \gamma\| \wedge^k DX_t(u \wedge u_2 \wedge \cdots \wedge u_k)\| \cdot \gamma\| \wedge^k DX_{-t}(v \wedge v_2 \wedge \cdots \wedge v_k)\| \\ &\le e^{-\lambda t}\gamma\|u \wedge u_2 \wedge \cdots \wedge u_k\| \cdot \gamma\|v \wedge v_2 \wedge \cdots \wedge v_k\| \le e^{-\lambda t}\end{aligned}$$

and note that the choice of $\gamma$ does not change any of the previous relations involving $\|\cdot\|$. Then for any given fixed $t \in [-1,1]$ we have obtained an adapted metric $|\cdot|$ that satisfies the statement of the lemma.

For general $t > 0$, suppose first that $t = n \in \mathbb{Z}^+$. Then by invariance of the subbundles

$$\begin{aligned}|DX_n u| \cdot |DX_{-n} v| &\le |\prod_{i=0}^{n-1}(DX_1 \circ X_i) \cdot u| \cdot |\prod_{i=0}^{n-1}(DX_{-1} \circ X_{n-i}) \cdot v| \\ &\le \prod_{i=0}^{n-1} \left(|DX_1 \mid_{E_{X_i x}}| \cdot |DX_{-1} \mid_{F_{X_{n-i} x}}|\right) \cdot |u| \cdot |v| \le |u| \cdot |v| e^{-n\lambda}.\end{aligned}$$

Now for non-integer $t > 0$ write $t = [t] + \alpha$ where $\alpha \in (0,1)$ and $[t] = \sup\{n \in \mathbb{Z}^+ : n \le t\}$ is the integer part function. Then

$$\begin{aligned}|DX_t u| \cdot |DX_{-t} v| &= |DX_{[t]} \circ DX_\alpha u| \cdot |DX_{-[t]} \circ DX_{-\alpha} v| \\ &\le |DX_\alpha u| \cdot |DX_{-\alpha} v| e^{-[t]\lambda} \le |u| \cdot |v| e^{-[t]\lambda} e^{-\alpha\lambda} = |u| \cdot |v| e^{-t\lambda}.\end{aligned}$$

We have obtained a metric $|\cdot|$ satisfying item (1) in the statement of the Lemma. Analogously, it satisfies items (2) and (3) of the statement of the Lemma, and we are done. □

From Lemma 1.26 we obtained an inner product $\langle \cdot, \cdot \rangle$ in $T_\Gamma M$ adapted to the dominated splitting $E \oplus F$ for $DX_t$, and this metric induces a metric in $\widetilde{E} \oplus \widetilde{F}$ which is an adapted metric to the partially hyperbolic splitting $\widetilde{E} \oplus \widetilde{F}$ for $\wedge^k DX_t$.

Moreover, from the definition of the inner product and exterior power, it follows that for all $t > 0$

$$|\det(DX_t \mid_{F_x})| = |(\wedge^k DX_t)(f_1 \wedge \cdot \wedge f_k)| = |(\wedge^k DX_t) \mid_{\widetilde{F}} | \geq e^{\lambda t}$$

since $F$ is spanned by $f_1, \ldots, f_k$. So $| \cdot |$ is adapted to the volume expansion along $F$.

To conclude, we are left to show that $E$ admits a constant $\omega > 0$ such that $|DX_t \mid_E \,| \leq e^{-\omega t}$ for all $t > 0$. But since $E$ is uniformly contracted, we know that $X(x) \in F_x$ for all $x \in \Gamma$.

**Lemma 1.27.** *Let $\Gamma$ be a compact invariant set for a flow $X$ of a $C^1$ vector field $X$ on $M$. Given a continuous splitting $T_\Gamma M = E \oplus F$ such that $E$ is uniformly contracted, then $X(x) \in F_x$ for all $x \in \Gamma$.*

*Proof.* See [4, Lemma 5.1] and [7, Lemma 3.3]. □

On the one hand, on each non-singular point $x$ of $\Gamma$ we obtain for $w \in E_x$

$$e^{-\lambda t} \geq \frac{|DX_t \cdot w|}{|DX_t \cdot X(x)|} = \frac{|DX_t \cdot w|}{|X(X_t(x))|} \geq \frac{|DX_t \cdot w|}{\sup\{|X(z)| : z \in \Gamma\}}.$$

Now we define $| \cdot |_* = \xi | \cdot |$, where $\xi$ is a small positive constant such that $\sup\{|X(z)|_* : z \in \Gamma\} \leq 1$. We note that the choice of the positive constant $\xi$ does not change any of the previous relations involving $| \cdot |$, except that now $|DX_t \cdot w|_* \leq e^{-\lambda t}$.

On the other hand, for $\sigma \in \Gamma$ such that $X(\sigma) = 0$, we fix $t > 0$ and, since $\Gamma$ is a non-trivial invariant set, we can find a sequence $x_n \to \sigma$ of regular points of $\Gamma$. The continuity of the derivative cocycle ensures $|DX_t \mid_{E_\sigma} |_* = \lim_{n\to\infty} |DX_t \mid_{E_{x_n}} |_* \leq e^{-\lambda t}$. Since $t > 0$ was arbitrarily chosen, we see that $| \cdot |_*$ is adapted for the contraction along $E_\sigma$. This completes the proof of Theorem B. □

# Chapter 2

# Kingman-like Theorem

As it is well-known, the Kingman Theorem is a striking tool to average the limit of a subadditivity sequence if the system is equipped with an invariant measure. The aim of this chapter is to provide a Kingman-like Theorem for an arbitrary finite measure assuming some conditions. As an application we proved a version of Birkhoff's Theorem for bounded observables.

Let us describe one interesting consequence of this Theorem. Let $X : M\times\mathbb{R} \to M$ be a continuous flow, and $M$ to be a compact metric space. Consider $X_t : M \to M$ given by $X_t(x) = X(t, x)$, and $f_t : M \to M$ defined by $f_t = X_t$. Suppose that $M$ is a compact metric space, $\varphi : M \to \mathbb{R}$ is a continuous function, and fix $x \in M$. If the following inequality holds $\limsup_n \frac{1}{n}\int_0^n \varphi \circ f_t(y_x)dt \leq \liminf_n \frac{1}{n}\int_0^n \varphi \circ f_t(x)dt$ for all $y_x \in \omega(x)$, then the limit $\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.

We emphasize that our result sheds some new light on the problem of Birkhoff average for a continuous observable in compact metric spaces.

The chapter is organized as follows: in Section 2.1 we give the statements of main results. In Section 2.2, we provide the proof of results about continuous flow on compact metric spaces. In Section 2.3 we prove the Corollary 2.6. Finally, in Section 2.4, we present the proof of Theorem C.

## 2.1 Statements of main results of Chapter

In [29], Derriennic obtained a general version of Fekete's Lemma (as we described in the introduction) and proved a generalization of Kingman's Theorem as follows.

**Theorem 2.1.** *[29, Theorem 4] Let $(M, \mathcal{A}, \mu)$ be a measure space, $f : M \to M$ be a measurable function, $\mu$ be a finite measure, $(\varphi_n)_n$ be a sequence of measurable functions where $\varphi_n : M \to \mathbb{R}$ for each $n$ in $\mathbb{N}$. If the following conditions are satisfied:*

*(i)* $\mu$ *is an invariant measure;*

*(ii)* $\varphi_n$ *is* $\mu$*-integrable for all* $n$ *in* $\mathbb{N}$*;*

*(iii) for all* $n, k \in \mathbb{N}$

$$\varphi_{n+k} - \varphi_n - \varphi_k \circ f^n \leq f^n h_k$$

*where* $(h_k)_k$ *is a sequence of positive functions such that* $\sup_k \int h_k d\mu < +\infty$*;*

*(iv)* $\inf_n \frac{1}{n} \int \varphi_n d\mu > -\infty$.

*Then the sequence* $(\frac{\varphi_n}{n})_n$ *converges* $\mu$*-almost everywhere and in* $L^1$*-norm to a* $f$*-invariant function* $\varphi$ *such that* $\int \varphi d\mu = \lim_n \frac{1}{n} \int \varphi_n d\mu$.

First of all, we introduce some definitions and notations that will be appear on text. Let $(\varphi_n)_n$ be a sequence of measurable functions where $\varphi_n : M \to \mathbb{R}$ for each $n$ in $\mathbb{N}$. We say that $(\varphi_n)_n$ is a subadditive sequence for $f$ if $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^m$ for all $m, n \geq 1$.

We consider a function $\varphi_- : M \to [-\infty, \infty]$ given by $\varphi_-(x) = \liminf_n \frac{\varphi_n(x)}{n}$. For each $\varepsilon > 0$ fixed and $k \in \mathbb{N}$ we define

$$E_k^\varepsilon = \{x \in M : \varphi_j(x) \leq j(\varphi_-(x) + \varepsilon) \text{ for some } j \in \{1, ..., k\}\}.$$

Note that $E_k^\varepsilon \subseteq E_{k+1}^\varepsilon$ and $M = \bigcup_{k=1}^{\infty} E_k^\varepsilon$.

**Theorem C.** *Let* $(M, \mathcal{A}, \mu)$ *be a measure space,* $f : M \to M$ *be a measurable function,* $\mu$ *be a finite measure. Suppose that* $(\varphi_n)_n$ *is a subadditive sequence for* $f$ *such that* $\varphi_1 \leq \beta$ *for some* $\beta \in \mathbb{R}$*. If the following conditions are satisfied:*

*(a) for all* $j \in \mathbb{N}$ *we have that* $\varphi_-(f^j(x)) = \varphi_-(x)$ $\mu-$*almost everywhere* $x$ *in* $M$*;*

*(b)* $\lim_{k\to\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ *for each* $\ell \in \mathbb{N} \setminus \{0\}$.

*Then* $\int \varphi_- d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu$*. Moreover, if there exists* $\gamma > 0$ *such that for all* $n > 0$, $\frac{\varphi_n}{n} \geq -\gamma$ *then* $\int \varphi_- d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu = \lim_n \frac{1}{n} \int \varphi_n d\mu$.

**Remark 2.2.** *Under hyphotesis of Theorem C, if* $\int \varphi_- d\mu = -\infty$ *or* $\beta \leq 0$ *then* $\int \varphi_- d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu = \lim_n \frac{1}{n} \int \varphi_n d\mu$*. (See subsection 2.4.1)*

Our goal is to provide a Kingman-like Theorem for an arbitrary measure assuming only the conditions $(a)$ and $(b)$. Moreover, we obtain the convergence of integrals even without a subadditive sequence of real numbers given by Fekete's Lemma (or same version of this result) as is usual when we work with invariant measures.

Let $(M, \mathcal{A}, \mu)$ be a measure space, $f : M \to M$ be a measurable transformation, $\mu$ be a probability measure. Let $\varphi : M \to \mathbb{R}$ be a measurable function, we consider $(\varphi_n)_n$ the additive sequence for $f$ given by $\varphi_n := \sum_{j=0}^{n-1} \varphi \circ f^j$ for each $n$ in $\mathbb{N}$, and $\varphi_-, \varphi_+$ functions defined from $M$ to $[-\infty, \infty]$ given by $\varphi_-(x) = \liminf_n \frac{\varphi_n(x)}{n}$, and $\varphi_+(x) = \limsup_n \frac{\varphi_n(x)}{n}$.

Note that for every bounded function $\varphi : M \to \mathbb{R}$ we have that $\varphi_-(f^j(x)) = \varphi_-(x)$ $\mu-$almost everywhere $x$ in $M$ for all $j \in \mathbb{N}$. In fact, by definition,

$$\varphi_-(f(x)) = \liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(f(x)) = \liminf_n \left(\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) + \frac{1}{n}[\varphi(f^n x) - \varphi(x)]\right) = \varphi_-(x).$$

**Remark 2.3.** *Under the the same hypotheses of Theorem C with condition* $(b)$ *replaced by condition* $(c)$*, that says*

$(c)$ $\mu(f^{-i}(M \setminus E_k^\varepsilon)) \leq \mu(M \setminus E_k^\varepsilon)$ *for all* $i \in \mathbb{N}$*, for any* $k \in \mathbb{N}$*, and* $\varepsilon > 0$*,*

*we obtain the conclusion of Theorem C. This follows immediately by the lemma below.*

**Lemma 2.4.** *Fixed* $\varepsilon > 0$*. If* $\mu(f^{-i}(M \setminus E_k^\varepsilon)) \leq \mu(M \setminus E_k^\varepsilon)$ *for all* $i$ *in* $\mathbb{N}$*, then* $\lim_{k \to +\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^\varepsilon)) = 0$*.*

*Proof.* Suppose that $\mu(f^{-i}(M \setminus E_k^\varepsilon)) \leq \mu(M \setminus E_k^\varepsilon)$ so

$$\frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^\varepsilon)) \leq \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(M \setminus E_k^\varepsilon) = (1 - \tfrac{k}{n})\mu(M \setminus E_k^\varepsilon),$$

and then

$$\lim_{k \to +\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(M \setminus f^{-i}(E_k^\varepsilon)) = \lim_{k \to +\infty} \mu(M \setminus E_k^\varepsilon).$$

But $\mu(E_k^\varepsilon)$ tends to 1 if $k$ tends to infinity, so $\mu(M \setminus E_k^\varepsilon)$ tends to zero as $k$ tends to infinity. □

We say that an observable $\varphi$ satisfies hypothesis $(c)$ if for all $i, k \in \mathbb{N}$ and $\varepsilon > 0$, the following inequality $\mu(f^{-i}(M \setminus E_k^\varepsilon)) \leq \mu(M \setminus E_k^\varepsilon)$ holds when we consider $(\varphi_n)_n$ an additive sequence for $f$. We observe that if the measure $\mu$ is an invariant measure, then every observable satisfies hypothesis $(c)$.

Since every bounded observable satisfies hypothesis $(a)$, we deduce Birkhoff's Theorem for finite measures and bounded observables as follows.

**Corollary D.** *Let* $(M, \mathcal{A}, \mu)$ *be a measure space,* $f : M \to M$ *be a measurable transformation,* $\mu$ *be a probability measure. If* $\varphi : M \to \mathbb{R}$ *is a bounded measurable function that satisfies the hypothesis* $(b)$ *or* $(c)$*. Then*

$$\int \varphi_- d\mu = \lim_n \frac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu = \inf_n \frac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu.$$

**Remark 2.5.** *In [22], Carvalho and Moreira introduced the notion of half-invariant measure $\mu$, that means that*

$$\mu(f^{-1}(B)) \leq \mu(B) \tag{2.1}$$

*for all measurable set $B$. Note that this implies that $(\mu(f^{-j}(B)))_{j\in\mathbb{N}}$ is a decreasing sequence. The authors showed that for any bounded observable $\varphi : M \to \mathbb{R}$ and a half-invariant measure, the limit $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$ exists for $\mu$ a.e. point $x$ in $M$. Here, Corollary D tell us that condition 2.1 can be relaxed to consider only the sets of the type $M \setminus E_k^\varepsilon$ for any $\varepsilon > 0$ and $k \in \mathbb{N}$.*

Let us mention one important consequence of Corollary D.

Let $(W, d)$ be a metric space, $g : W \to W$ be a function, and $x \in W$. The set $\mathcal{O}^+x$ is the *forward orbit of* $x$, and it is given by $\mathcal{O}^+x := \{g^n(x)\}_{n\in\mathbb{N}}$. A point $x \in W$ is a *periodic point* if there exists $m \in \mathbb{N}$ such that $g^m x = x$. More generally, we say that a point $x \in W$ is *eventually periodic* if there exists $j_0 \in \mathbb{N}$ such that $g^{j_0}x$ is a periodic point.

Let $S$ be a subset of $W$, and let $g : W \to W$ be a continuous function. The $\omega$-limit of $S$, denoted by $\omega(S, g)$, is the set of points $y \in W$ for which there are $z \in S$ and a strictly increasing sequence of natural number $\{n_k\}_{k\in\mathbb{N}}$ such that $g^{n_k}z \to y$ as $k \to \infty$. Note that $\omega(S, g) = \bigcup_{z\in S} \omega(\{z\}, g)$.

Consider $E_k^\varepsilon = \{w \in M : \frac{\varphi_j}{j}(w) \leq \varphi_-(w) + \varepsilon$ for some $j \in \{1, ..., k\}\}$, $(\varphi_n)_n$ is the additive sequence for $f$ given by $\varphi_n := \sum_{j=0}^{n-1} \varphi \circ f^j$ for each $n$ in $\mathbb{N}$, and $\varphi_-$ is the function defined from $M$ to $[-\infty, \infty)$ given by $\varphi_-(w) = \liminf_n \frac{\varphi_n(w)}{n}$.

**Corollary 2.6.** *Let $(M, \mathcal{A})$ be a measurable space for $M$ metric space, $f : M \to M$ be a measurable transformation, and $\varphi : M \to \mathbb{R}$ be a bounded measurable function. If one of the following conditions is true*

*(i)* $\lim_{k\to\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \delta_x(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ *for each $\ell \in \mathbb{N} \setminus \{0\}$ where $\delta_x$ the Dirac measure of point $x \in M$;*

*(ii) Suppose that there exists $x \in M$ such that for any $\varepsilon > 0$ there exist $j_\varepsilon, k_\varepsilon \in \mathbb{N}$ satisfying that $f^j(x) \in E_{k_\varepsilon}^\varepsilon$ for $j \geq j_\varepsilon$;*

*(iii) If $M$ is a compact metric space, and there exists $x \in M$ such that for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ satisfying that $\omega(\{x\}, f)$ is contained in the interior of $E_{k_\varepsilon}^\varepsilon$;*

*(iv) Suppose that $M$ is a compact metric space, $f, \varphi, \varphi_-$ are continuous functions, and $\omega(\{x\}, f)$ is a finite set for some $x \in M$.*

*Then the limit* $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x)$ *exists.*

**Remark 2.7.** *For each $\varepsilon > 0$ fixed and $k \in \mathbb{N}$ we define*

$$F_k^\varepsilon = \{x \in M : \varphi_j(x) \geq j(\varphi_+(x) - \varepsilon) \text{ for some } j \in \{1, ..., k\}\}.$$

*We say that an observable $\varphi$ satisfies the hypothesis $(c')$ if*

$(c')$ *for all $i, k \in \mathbb{N}$ and $\varepsilon > 0$, the following inequality $\mu(f^{-i}(M \setminus F_k^\varepsilon)) \leq \mu(M \setminus F_k^\varepsilon)$ holds when we consider $(\varphi_n)_n$ the additive sequence for $f$.*

*The next result is a direct application of Corollary D.*

**Corollary 2.8.** *Let $(M, \mathcal{A}, \mu)$ be a measure space, $f : M \to M$ be a measurable transformation, $\mu$ be a probability measure. If $\varphi : M \to \mathbb{R}$ is a measurable bounded function that satisfies the hypothesis $(c)$ and $(c')$, then there exists the limit $\widetilde{\varphi(x)} = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x)$ for $\mu$ almost every point $x$ in $M$. Moreover, the function $\widetilde{\varphi}$ defined as above is invariant under $f$, integrable and satisfies $\int \widetilde{\varphi} d\mu = \lim_n \int \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j d\mu$.*

*Proof.* Note that $\varphi_- \leq \varphi_+$, and $\gamma = -\varphi$ is also a bounded function. By Corollary D, $\lim_n \frac{1}{n}\int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu = \int \varphi_+ d\mu$. So

$$\lim_n \frac{1}{n}\int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu = \int \varphi_+ d\mu = \int \varphi_- d\mu.$$

Then $\int \varphi_- d\mu = \int \varphi_+ d\mu$, and consequently, the functions $\varphi_-$ and $\varphi_+$ coincide in $\mu$-a.e. So, $\widetilde{\varphi} = \varphi_- = \varphi_+$ for $\mu$ a.e. $x$ in $M$, i.e., $\liminf_n \frac{\varphi_n(x)}{n} = \limsup_n \frac{\varphi_n(x)}{n}$ for $\mu$ a.e. $x$ in $M$, then $\lim_n \frac{\varphi_n(x)}{n}$ exists for $\mu$ a.e. $x$ in $M$, define $\widetilde{\varphi}(x) := \lim_n \frac{\varphi_n(x)}{n}$. This completes the proof of corollary.

□

Now, we are going to introduce a version of item $(ii)$ of Corollary 2.6 for continuous flow on compact metric spaces. Let $X : M \times \mathbb{R} \to M$ be a continuous flow, and $M$ to be a compact metric space. Consider $X_t : M \to M$ given by $X_t(x) = X(t, x)$, and $f_t : M \to M$ defined by $f_t = X_t$. Let $\varphi : M \to \mathbb{R}$ be a bounded measurable function, we consider the following objects.

$\varphi_{*,-}(y) = \liminf_{n\to\infty} \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt$ for each $y \in M$;

$E_{k_\varepsilon}^{*,\varepsilon} = \{y \in M : \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt \leq \varphi_{*,-}(y) + \varepsilon \text{ for some } n \in \{1, \cdots, k\}\}$.

Given $x \in M$, we denote the Dirac measure of point $x$ by $\delta_x$.

**Corollary 2.9.** *Let $\varphi : M \to \mathbb{R}$ be a bounded measurable function, and fix $x \in M$. If for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$, then the limit $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.*

If $\varphi$ is a continuous function, we obtain an interesting criterion to provide the existence of Birkhoff's limit as follows.

**Theorem E.** *Suppose that $M$ is a compact metric space, $\varphi : M \to \mathbb{R}$ is a continuous function, and fix $x \in M$. If $\limsup\limits_n \frac{1}{n}\int_0^n \varphi \circ f_t(y_x)dt \leq \liminf\limits_n \frac{1}{n}\int_0^n \varphi \circ f_t(x)dt$ for all $y_x \in \omega(x)$, then for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$. In particular, the limit $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.*

Before we give our examples, we introduce some definition and result. We say that $x \in M$ is a $2d$-point if for any $y_x \in \omega(x)$ we have that $\omega(y_x)$ is a fixed point (i.e., there exists $q \in M$ such that $\omega(y_x) = \{q\}$ and $f_t(q) = q$ for all $t \in \mathbb{R}$). Define the fixed point under $X$ by $\operatorname{Fix} X = \{q \in M : q \text{ is a fixed point}\}$.

Let $M$ be a compact metric space $M$, and $\varphi : M \to \mathbb{R}$ be a continuous function. Recall that if $p, q \in M$ and $\omega(p) = \{q\}$ then $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(p)dt = \varphi(q)$. This allows us to consider the next result.

**Corollary 2.10.** *Suppose that $M$ is a compact metric space, $\varphi : M \to \mathbb{R}$ is a continuous function, and take a $2d$-point $x \in M$. Suppose that $\varphi$ satisfies that $\varphi|_{\omega(x)\cap \operatorname{Fix} X} \equiv \min \varphi$. Then for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$. In particular, the limit $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.*

**Example 2.11.** *In the example by Bowen (the compact subset of $\mathbb{R}^2$ denoted by $E_B$), if $(f_t(x))_{t\geq 0}$ converges to the cycle, and $\varphi$ is a continuous function on the plane, taking different values in the saddle points $A$ and $B$, the time average*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$$

*does not exist. This means that in this example there is an open set of initial states such that the corresponding orbits define non-stationary time series (whenever one uses an observable which has a different values in two saddle points).*

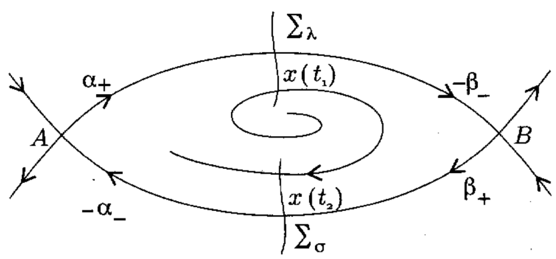

*Figure 1: Phase portrait of the example by Bowen.*

*We denote, for the example of Bowen given in figure 1, the expanding and contracting eigenvalues of the linearized vector field in* $A$ *by* $\alpha_+$ *and* $\alpha_-$, *and in* $B$ *by* $\beta_+$ *and* $\beta_-$. *We recall that the saddle points are denoted by* $A$ *and* $B$. *The condition on the eigenvalues which makes the cycle attracting is that the contracting eigenvalues dominate:* $\alpha_-\beta_- > \alpha_+\beta_+$.

*The modolus associated with the upper, respectively lower, saddle connection is denoted by* $\lambda$, *respectively* $\sigma$. *They are defined by*

$$\lambda = \alpha_-/\beta_+ \text{ and } \sigma = \beta_-/\alpha_+,$$

*their values are positive and their products is bigger than 1, assuming the cycle to be attracting. Gaunersdorfer([31], 1992) and Takens ([68],1994) proved the following.*

**Theorem 2.12.** *If* $\varphi$ *is a continuous function on* $\mathbb{R}^2$ *with* $\varphi(A) > \varphi(B)$, *and* $(f_t(x))_{t\geq 0}$ *is an orbit converging to the cycle, then we have for the partial averares of* $\varphi$:

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt = \frac{\sigma}{1+\sigma}\varphi(A) + \frac{1}{1+\sigma}\varphi(B)$$
$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt = \frac{\lambda}{1+\lambda}\varphi(B) + \frac{1}{1+\lambda}\varphi(A)$$

■

*Here, Corollary 2.10 provides some information about the existence of Birkhoff's Limit if we take a continuous function as follows.*

**Corollary 2.13.** *Suppose that* $\varphi : E_B \to \mathbb{R}$ *is a continuous function with* $\varphi(A) = \varphi(B) = \min\varphi$, *and* $(f_t(x))_{t\geq 0}$ *is an orbit converging to the cycle. Then for any* $\varepsilon > 0$ *there exist* $t_\varepsilon \in \mathbb{R}$ *and* $k_\varepsilon \in \mathbb{N}$ *satisfying that* $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ *for* $j \geq t_\varepsilon$ *and* $j \in \mathbb{N}$. *In particular, the limit* $\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ *exists.*

**Remark 2.14.** *This example also shows that for any* $x \in E_B\setminus\{A, B\}$ *such that* $(f_t(x))_{t\geq 0}$ *is an orbit converging to the cycle the Dirac measure of point* $x$, $\delta_x$, *is not an invariant measure (since* $x$ *is not a fixed point under* $X$*), but this measure satisfies that for any* $\varepsilon > 0$ *there exist* $t_\varepsilon \in \mathbb{R}$ *and* $k_\varepsilon \in \mathbb{N}$ *satisfying that* $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ *for* $j \geq t_\varepsilon$ *and* $j \in \mathbb{N}$.

## 2.2 Continuous flow on compact metric spaces

Now, we are going to provide a version of item $(ii)$ of Corollary 2.6 for continuous flow on compact metric space. Let $X : M \times \mathbb{R} \to M$ be a continuous flow, and $M$ to be a compact metric space. Consider $X_t : M \to M$ given by $X_t(x) = X(t,x)$, and $f_t : M \to M$ defined by $f_t = X_t$.

We say that a measure $\mu$ is invariant under flow $(f_t)_{t\in\mathbb{R}}$ if $\mu(E) = \mu(f_{-t}(E))$ for any mensurable set $E$ and for all $t \in \mathbb{R}$. Note that $\mu$ is an invariant measure under flow if, and only if, $\int \varphi d\mu = \int \varphi \circ f_t d\mu$ for all $\varphi : M \to \mathbb{R}$ $\mu$-integrable and for all $t \in \mathbb{R}$.

A fixed point of $(f_t)_{t\in\mathbb{R}}$ is a point $q \in M$ such that $f_t(q) = q$ for all $t \in \mathbb{R}$.

**Lemma 2.15.** *Fix $x \in M$. Then the Dirac measure of point $x$, $\delta_x$, is an invariant measure if, and only if, $x$ is a fixed point.*

*Proof.* Suppose that the Dirac measure of point $x$, $\delta_x$, is an invariant measure. Then $\delta_x(f_{-t}\{x\}) = \delta_x x = 1$ for all $t \in \mathbb{R}$, this implies that $\delta_x(f_{-t}\{x\}) = 1$, and then $x \in f_{-t}\{x\}$ for all $t \in \mathbb{R}$, so $f_t(x) = x$ for all $t \in \mathbb{R}$. $\square$

Let $\varphi : M \to \mathbb{R}$ be a bounded measurable function, we consider the following objects.

$\varphi_{*,-}(y) = \liminf\limits_{n\to\infty} \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt$ for each $y \in M$;

$E^{*,\varepsilon}_{k_\varepsilon} = \{y \in M : \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt \leq \varphi_{*,-}(y) + \varepsilon$ for some $n \in \{1, \cdots, k\}\}$.

Note that $M = \bigcup\limits_{k=1}^{\infty} E^{*,\varepsilon}_{k_\varepsilon}$ for each $\varepsilon > 0$. Given $x \in M$, we denote the Dirac measure of point $x$ by $\delta_x$. We observe the following result.

**Lemma 2.16.** *If $\delta_x$ is an invariant measure for some $x \in M$, then for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$.*

*Proof.* Fix $\varepsilon > 0$. Using that $M = \bigcup\limits_{k=1}^{\infty} E^{*,\varepsilon}_{k_\varepsilon}$, there exists $k_\varepsilon \in \mathbb{N}$ such that $x \in E^{\varepsilon,*}_{k_\varepsilon}$. But $f_t(x) = x$ for all $t \in \mathbb{R}$, and we are done. $\square$

The following is a version of item $(ii)$ of Corollary 2.6 for flows on compact metric spaces.

**Corollary 2.17.** *Let $\varphi : M \to \mathbb{R}$ be a bounded measurable function, and fix $x \in M$. If for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E^{\varepsilon,*}_{k_\varepsilon})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$, then the limit $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.*

*Proof.* Let $\varphi : M \to \mathbb{R}$ be a bounded function, define $\psi : M \to \mathbb{R}$ by $\psi(y) = \int_0^1 \varphi \circ f_t(y)dt$ for each $y \in M$. Fix $T > 0$, and note that

$$\frac{1}{T}\int_0^T \varphi \circ f_t(y)dt = \frac{1}{T}\sum_{j=0}^{[T]-1}\int_j^{j+1} \varphi \circ f_t(y)dt + \frac{1}{T}\int_{[T]}^T \varphi \circ f_t(y)dt$$

Considering $t = j + s$ for $s \in [0,1]$, we see that

$$\begin{aligned}
\int_j^{j+1} \varphi \circ f_t(y)dt &= \int_0^1 \varphi \circ f_s(f_j y)ds \\
\frac{1}{T}\int_0^T \varphi \circ f_t(y)dt &= \frac{1}{T}\sum_{j=0}^{[T]-1}\int_0^1 \varphi \circ f_t(f_j y)dt + \frac{1}{T}\int_{[T]}^T \varphi \circ f_t(y)dt \\
\frac{1}{T}\int_0^T \varphi \circ f_t(y)dt &= \frac{1}{T}\sum_{j=0}^{[T]-1} \psi \circ f_j(y) + \frac{1}{T}\int_{[T]}^T \varphi \circ f_t(y)dt \qquad (2.2)
\end{aligned}$$

Take $T = n$, by equation (2.2), $\frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f_j(y) = \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt$.

Observe that $\psi$ is a bounded function since $\varphi$ is a bounded function. Recall that $\psi_-(y) = \liminf_n \frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f_j(y)$ for $y \in M$. Now, for each $\varepsilon > 0$ and $k \in \mathbb{N}$ define

$\widetilde{E}_k^\varepsilon = \{y \in M : \frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f_j(y) \le \psi_-(y) + \varepsilon$ for some $n \in \{1, \cdots, k\}\}$.

$\widetilde{E}_k^\varepsilon = \{y \in M : \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt \le \varepsilon + \liminf_n \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt$ for some $n \in \{1, \cdots, k\}\}$, so $\widetilde{E}_k^\varepsilon = E_{k_\varepsilon}^{\varepsilon,*}$.

By hypothesis, there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $f_j(x) \in E_{k_\varepsilon}^{\varepsilon,*}$ for $j \ge t_\varepsilon$ and $j \in \mathbb{N}$. Then, by Corollary 2.6, the limit $\lim_{T\to\infty} \frac{1}{T}\sum_{j=0}^{[T]-1} \psi \circ f_j(y) = \lim_{T\to\infty} \frac{1}{[T]}\sum_{j=0}^{[T]-1} \psi \circ f_j(y)$ exists since

$$\frac{1}{T}\sum_{j=0}^{[T]-1} \psi \circ f_j(y) = \frac{1}{[T]+\beta_T}\sum_{j=0}^{[T]-1} \psi \circ f_j(y) = \frac{1}{[T](1+\frac{\beta_T}{[T]})}\sum_{j=0}^{[T]-1} \psi \circ f_j(y)$$

for some $\beta_T \in (0,1]$ such that $T = [T] + \beta_T$.

Note that

$$\Big|\frac{1}{T}\int_{[T]}^T \varphi \circ f_t(y)dt\Big| = \Big|\frac{1}{T}\int_0^{T-[T]} \varphi \circ f_t(f_{[T]}(y))dt\Big| \le \frac{1}{T}\int_0^{T-[T]} |\varphi \circ f_t(f_{[T]}(y))|dt \le \frac{1}{T}\int_0^1 |\varphi \circ f_t(f_{[T]}(y))|dt \le \frac{\|\varphi\|}{T} \to 0,$$

as $T$ tends to infinity, and we are done. □

If $\varphi$ is a continuous function, we obtain an interesting criterion to provide the existence of Birkhoff's limit as follows.

**Theorem F.** *Suppose that $M$ is a compact metric space, $\varphi : M \to \mathbb{R}$ is a continuous function, and fix $x \in M$. If $\limsup_n \frac{1}{n}\int_0^n \varphi \circ f_t(y_x)dt \le \liminf_n \frac{1}{n}\int_0^n \varphi \circ f_t(x)dt$ for all $y_x \in \omega(x)$, then for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E_{k_\varepsilon}^{\varepsilon,*})) = 1$ for $j \ge t_\varepsilon$ and $j \in \mathbb{N}$. In particular, the limit $\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi \circ f_t(x)dt$ exists.*

*Proof.* Suppose that $M$ is a compact metric space, and $\varphi : M \to \mathbb{R}$ is a continuous function (so $\varphi$ is a bounded function). Take $x \in M$, and suppose, by contradiction, that there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, and for any $t \in \mathbb{R}$ there exists $j_k \in \mathbb{N}$ with $j_k > t$ such that $f_{j_k}(x) \notin E_k^{*,\varepsilon}$.

In particular, for each $k \in \mathbb{N}$, taking $t = k$, there exists $j_k > k$ and $j_k \in \mathbb{N}$ such that $f_{j_k}(x) \notin E_k^{*,\varepsilon}$. This implies that $j_k \to +\infty$ as $k$ tends to infinity.

By compactness of $M$, there exists a subsequence of $(f_{j_k}(x))_{k\in\mathbb{N}}$ that converges to some $y_x \in \omega(x)$, suppose that

$$f_{j_{k_s}}(x) \to_s y_x \tag{2.3}$$

where $j_{k_s}$ tends to infinity as $s$ tends to infinity, $f_{j_{k_s}}(x) \notin E_{k_s}^{*,\varepsilon}$ and $j_{k_s} > k_s$. Without loss of generality, we may assume that $k_1 < k_2 < \cdots < k_s < k_{s+1} \cdots$

We recall that $E_{k_\varepsilon}^{*,\varepsilon} = \{y \in M : \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt \leq \varphi_{*,-}(y) + \varepsilon$ for some $n \in \{1, \cdots, k_\varepsilon\}\}$.

For each $s \in \mathbb{N}$, by definition of $E_{k_s}^{*,\varepsilon}$,

$$\tfrac{1}{n}\int_0^n \varphi \circ f_t(f_{j_{k_s}}(x))dt > \varphi_{*,-}(f_{t_{k_s}}(x)) + \varepsilon$$

for any $n \in \{1, \cdots, k_s\}$ since $f_{j_{k_s}}(x) \notin E_{k_s}^{*,\varepsilon}$.

Recall that $\varphi_{*,-}(z) = \liminf_{n\to\infty} \frac{1}{n}\int_0^n \varphi \circ f_t(z)dt = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f_j(z) = \psi_-(z)$, where $\psi$ is a bounded function. Then $\psi_-(f_j(z)) = \psi_-(z)$ for all $j \geq 0$ and $z \in M$, so $\varphi_{*,-}(f_j z) = \varphi_{*,-}(z)$ for all $j \geq 0$.

Using that $k_1 < k_2 < \cdots < k_s < k_{s+1} \cdots$, we see that $k_1 \in \{1, \cdots, k_s\}$ for any $s \geq 1$, and then

$$\tfrac{1}{k_1}\int_0^{k_1} \varphi \circ f_t(f^{t_{k_s}}(x))dt > \varphi_{*,-}(f^{t_{k_s}}(x)) + \varepsilon = \varphi_{*,-}(x) + \varepsilon.$$

Recall that $\frac{1}{n}\sum_{j=0}^{n-1} \psi \circ f_j(y) = \frac{1}{n}\int_0^n \varphi \circ f_t(y)dt$ where $\psi(y) = \int_0^1 \varphi \circ f_t(y)dt$ for each $y \in M$. This implies that $\frac{1}{k_1}\int_0^{k_1} \varphi \circ f_t(y)dt = \frac{1}{k_1}\sum_{j=0}^{k_1-1} \psi \circ f_j(y)$ for each $y \in M$.

**Lemma 2.18.** $\psi : M \to \mathbb{R}$ *is uniformly continuous.*

*Proof.* Take $y_0, z \in M$ and $\gamma > 0$. By compactness, $\varphi$ is uniformly continuous, then there exists $\xi > 0$ such that for every $p, q \in M$ with $d(p, q) < \xi$, we have that $|\varphi(p) - \varphi(q)| < \gamma$. Note that

$$|\psi(z) - \psi(y_0)| = |\int_0^1 \varphi \circ f_t(z)dt - \int_0^1 \varphi \circ f_t(y_0)dt| = |\int_0^1 \varphi \circ f_t(z) - \varphi \circ f_t(y_0)dt| \leq \int_0^1 |\varphi \circ f_t(z) - \varphi \circ f_t(y_0)|dt.$$

Now, $([0,1]\times M, D)$ is a compact metric space where for any $(t,z),(s,w)\in[0,1]\times M$ we define $D((t,z),(s,w)) := |t-s| + d(z,w)$, and $X_{[0,1]} = X|_{[0,1]\times M} : [0,1]\times M \to M$ is a continuous function, then $X_{[0,1]}$ is uniformly continuous. Then there exists $\delta > 0$ such that for any $(t,z),(s,w)\in[0,1]\times M$ that satisfies $D((t,z),(s,w)) < \delta$ we have that $d(X_t(z), X_s(w)) = d(f_t(z), f_s w) < \xi$.

So, for any $y_0, z \in M$ such that $d(y_0, z) < \delta$ we have that $D((t,z),(t,y_0)) = d(y_0,z) < \delta$, and then $d(X_t(z), X_t(y_0)) = d(f_t(z), f_t(y_0)) < \xi$, and we are done. □

So $\frac{1}{k_1}\int_0^{k_1} \varphi\circ f_t(\cdot)dt = \frac{1}{k_1}\sum_{j=0}^{k_1-1}\psi\circ f_j(\cdot)$ is a continuous function because it is a finite sum of continuous functions. By continuity of $\frac{1}{k_1}\int_0^{k_1}\varphi\circ f_t(\cdot)dt$, we have that

$$\frac{1}{k_1}\int_0^{k_1}\varphi\circ f_t(f^{t_{k_s}}(x))dt \to \frac{1}{k_1}\int_0^{k_1}\varphi\circ f_t(y_x)dt \geq \varphi_{*,-}(x)+\varepsilon.$$

So using that $k_1 < k_2 < \cdots < k_s < k_{s+1}\cdots$, we see that $k_\ell \in \{1,\cdots,k_s\}$ for any $s \geq \ell$ for each $\ell\in\mathbb{N}$, and then

$$\frac{1}{k_\ell}\int_0^{k_\ell}\varphi\circ f_t(f^{t_{k_s}}(x))dt \to \frac{1}{k_\ell}\int_0^{k_\ell}\varphi\circ f_t(y_x)dt \geq \varphi_{*,-}(x)+\varepsilon.$$

This implies that $\frac{1}{k_\ell}\int_0^{k_\ell}\varphi\circ f_t(y_x)dt \geq \varphi_{*,-}(x)+\varepsilon$ for any $\ell\in\mathbb{N}$, and then

$$\limsup_n \frac{1}{n}\int_0^n \varphi\circ f_t(y_x)dt \geq \varphi_{*,-}(x)+\varepsilon > \varphi_{*,-}(x) = \liminf_n \frac{1}{n}\int_0^n\varphi\circ f_t(x)dt,$$

and we are done.

□

Here, we recall the following natural lemma.

**Lemma 2.19.** *Suppose that $M$ is a compact metric space, and $\varphi : M\to\mathbb{R}$ is a continuous function. If $p,q\in M$ and $\omega(p)=\{q\}$ then $\lim_{T\to\infty}\frac{1}{T}\int_0^T\varphi\circ f_t(p)dt = \varphi(q)$.*

*Proof.* First, note that

$$\lim_{t\to+\infty} f_t(p) = q.$$

In fact, suppose that there exists $\varepsilon > 0$ and $(t_k)_k$ such that $t_k \to +\infty$ as $k$ tends to infinity and $d(f_{t_k}, q)\geq\varepsilon$. By compactness of $M$, there exists a subsequence of $(f_{t_k})_k$ that converges to some $z\in M$. Without loss of generality, $(f_{t_k})_k$ converges to $z$. Using the continuity of metric, we obtain that

$$\varepsilon \leq \lim_{k\to\infty} d(f_{t_k}, q) = d(z,q).$$

But $z \in \omega(p) = \{q\}$, and then $z = q$. This proves that $\lim_{t\to+\infty} f_t(p) = q$. This implies that $f_t(q) = q$ for any $t \in \mathbb{R}$.

Define $\psi : M \to \mathbb{R}$ by $\psi(y) = \int_0^1 \varphi \circ f_t(y)dt$ for each $y \in M$. By Lemma 2.18, $\psi$ is uniformly continuous, and then $\lim_{t\to+\infty} \psi \circ f_t(p) = \psi(q)$.

(Take $\gamma > 0$, there exists $\xi > 0$ such that for all $x, y \in M$ with $d(x, y) < \xi$ we have that $d(\psi(x), \psi(y)) < \gamma$.

Using that $\lim_{t\to+\infty} f_t(p) = q$, we have that there exists $\alpha > 0$ such that for any $t \geq \alpha$ we have that $|f_t(p) - q| < \xi$.

Then for any $t \geq \alpha$ we have that $d(\psi(f_t(p)), \psi(q)) < \gamma$.)

In particular, $\lim_{n\to+\infty} \psi \circ f_n(p) = \psi(q)$, and then $\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f_j(p) = \psi(q)$.

Fix $T > 0$, and note that

$$\frac{1}{T} \int_0^T \varphi \circ f_t(y)dt = \frac{1}{T} \sum_{j=0}^{[T]-1} \int_j^{j+1} \varphi \circ f_t(y)dt + \frac{1}{T} \int_{[T]}^T \varphi \circ f_t(y)dt$$

Considering $t = j + s$ for $s \in [0, 1]$, we see that

$$\begin{aligned}
\int_j^{j+1} \varphi \circ f_t(y)dt &= \int_0^1 \varphi \circ f_s(f_j y)ds \\
\frac{1}{T} \int_0^T \varphi \circ f_t(y)dt &= \frac{1}{T} \sum_{j=0}^{[T]-1} \int_0^1 \varphi \circ f_t(f_j y)dt + \frac{1}{T} \int_{[T]}^T \varphi \circ f_t(y)dt \\
\frac{1}{T} \int_0^T \varphi \circ f_t(y)dt &= \frac{1}{T} \sum_{j=0}^{[T]-1} \psi \circ f_j(y) + \frac{1}{T} \int_{[T]}^T \varphi \circ f_t(y)dt
\end{aligned}$$

This implies that $\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi \circ f_t(p)dt = \psi(q) = \int_0^1 \varphi \circ f_t(q)dt = \varphi(q)$.

□

Before we give our examples, we introduce some definition and result. We say that $x \in M$ is a $2d$-point if for any $y_x \in \omega(x)$ we have that $\omega(y_x)$ is a fixed point (i.e., there exists $q \in M$ such that $\omega(y_x) = \{q\}$ and $f_t(q) = q$ for all $t \in \mathbb{R}$). Define the fixed point under $X$ by $\operatorname{Fix} X = \{q \in M : q \text{ is a fixed point}\}$.

**Corollary 2.20.** *Suppose that $M$ is a compact metric space, $\varphi : M \to \mathbb{R}$ is a continuous function, and take a $2d$-point $x \in M$. Suppose that $\varphi$ satisfies that $\varphi|_{\omega(x)\cap \operatorname{Fix} X} \equiv \min \varphi$. Then for any $\varepsilon > 0$ there exist $t_\varepsilon \in \mathbb{R}$ and $k_\varepsilon \in \mathbb{N}$ satisfying that $\delta_x(f_{-j}(E_{k_\varepsilon}^{\varepsilon,*})) = 1$ for $j \geq t_\varepsilon$ and $j \in \mathbb{N}$. In particular, the limit $\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi \circ f_t(x)dt$ exists.*

*Proof.* For each $y_x \in \omega(x)$, there exists a fixed point $q_{y_x}$ such that $\omega(y_x) = \{q_{y_x}\}$, so $q_{y_x} \in \omega(x) \cap \operatorname{Fix} X$, and then $\varphi(q_{y_x}) = \min \varphi$.

Now, by Lemma 2.19, $\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi\circ f_t(y_x)dt = \varphi(q_{y_x}) = \min\varphi$.

Note that $\min\varphi \leq \liminf_n \frac{1}{n}\int_0^n \varphi\circ f_t(x)dt$, and then, by Theorem E, we are done. □

## 2.3 Proof of Corollary 2.6

Let $(M, \mathcal{A})$ be a measurable space for $M$ compact metric space, and $f : M \to M$ be a measurable transformation, $\varphi : M \to \mathbb{R}$ be a bounded measurable function.

We are going to show that if one of the following conditions is true

$(i)$ $\lim_{k\to\infty} \limsup_n \frac{1}{n}\sum_{i=0}^{n-k-1} \delta_x(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ for each $\ell \in \mathbb{N}\setminus\{0\}$ where $\delta_x$ the Dirac measure of point $x \in M$;

$(ii)$ Suppose that there exists $x \in M$ such that for any $\varepsilon > 0$ there exist $j_\varepsilon, k_\varepsilon \in \mathbb{N}$ satisfying that $f^j x \in E_{k_\varepsilon}^{\varepsilon}$ for $j \geq j_\varepsilon$;

$(iii)$ If $M$ is a compact metric space, and there exists $x \in M$ such that for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ satisfying that $(\mathcal{O}^+x)'$ is contained in the interior of $E_{k_\varepsilon}^{\varepsilon}$;

$(iv)$ Suppose that $M$ is a compact metric space, $f, \varphi, \varphi_-$ are continuous functions, and $(\mathcal{O}^+x)'$ is a finite set for some $x \in M$.

Then the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi\circ f^j(x)$ exists.

*Proof of Corollary 2.6.* Fix $\varphi : M \to \mathbb{R}$, and consider $(\varphi_n)_n$ to be the additive sequence for $f$ given by $\varphi_n := \sum_{j=0}^{n-1} \varphi\circ f^j$ for each $n$ in $\mathbb{N}$. Consider $\varphi_- : M \to \mathbb{R}$ given by $\varphi_-(w) = \liminf_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi\circ f^j(w)$ for $w \in M$.

For each $\varepsilon > 0$ fixed and $k \in \mathbb{N}$ we define

$$E_k^{\varepsilon} = \{w \in M : \varphi_j(w) \leq j(\varphi_-(w) + \varepsilon) \text{ for some } j \in \{1, ..., k\}\}.$$

Consider the measure $\mu = \delta_x$ where $\delta_x$ is the Dirac measure of point $x$.

To apply Corollary D it is sufficient to prove that $\varphi$ satisfies condition $(b)$, i.e., $\lim_{k\to\infty} \limsup_n \frac{1}{n}\sum_{i=0}^{n-k-1} \delta_x(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ for each $\ell \in \mathbb{N}\setminus\{0\}$.

$(ii)$ If $x$ is an eventually periodic point, there is nothing to show. Suppose that $x$ is not an eventually periodic point. Fixed $\varepsilon > 0$, there exist $j_\varepsilon, k_\varepsilon \in \mathbb{N}$ such that $f^j x \in E_{k_\varepsilon}^{\varepsilon}$ for $j \geq j_\varepsilon$. This implies that $(\mathcal{O}^+x) \cap M \setminus E_k^{\varepsilon}$ is a finite set for $k \geq k_\varepsilon$.

**Claim 1:** $\{j \in \mathbb{N} : x \in f^{-j}(M \setminus E_k^{\varepsilon})\}$ is a finite set.

Suppose the claim would be false. Then we could find a sequence $(j_s)_{s\in\mathbb{N}}$ such that $f^{j_s}x \in M \setminus E^{\varepsilon}_k$ for all $s \in \mathbb{N}$. So $f^{j_s}(x) \in (\mathcal{O}^+x) \cap M \setminus E^{\varepsilon}_k$ for all $s \in \mathbb{N}$. Then for $s > \#((\mathcal{O}^+x) \cap M \setminus E^{\varepsilon}_k)$ there exists $t \in \mathbb{N}$ such that $t < s$ and $f^{j_s}x = f^{j_t}x$. Using that $x$ is not an eventually periodic point, we are done.

By Claim 1, there exists $j_0 \in \mathbb{N}$ such that for $j \geq j_0$ we must have the following $x \in M \setminus f^{-j}(M \setminus E^{\varepsilon}_k)$, and then $\mu(f^{-j}(M \setminus E^{\varepsilon}_k)) = 0$ for $j \geq j_0$.

Using that $E^{\varepsilon}_k \subseteq E^{\varepsilon}_{k+1}$, we see that $\mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}})) = 0$ for all $\widetilde{k} \geq k_\varepsilon$ and $j \geq j_0$.

Now, take $\widetilde{k}$ such that $\widetilde{k}+1 > j_0$ . It easy to see that there exists $n \in \mathbb{N}$ such that $n > j_0 + \widetilde{k} + 1$, and note that

$$\tfrac{1}{n}\sum_{j=0}^{n-\widetilde{k}-1} \mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}})) = \tfrac{1}{n}\sum_{j=0}^{j_0-1} \mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}})) + \tfrac{1}{n}\sum_{j=j_0}^{n-\widetilde{k}-1} \mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}}))$$

Using that $\mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}_\varepsilon})) = 0$ for all $\widetilde{k} \geq k_\varepsilon$ and $j \geq j_0$,

$$0 \leq \tfrac{1}{n}\sum_{j=0}^{n-\widetilde{k}-1} \mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}})) = \tfrac{1}{n}\sum_{j=0}^{j_0-1} \mu(f^{-j}(M \setminus E^{\varepsilon}_{\widetilde{k}})) \leq \tfrac{j_0}{n},$$

and then

$$\limsup_{n} \tfrac{1}{n}\sum_{i=0}^{n-\widetilde{k}-1} \mu(f^{-i}(M \setminus E^{\varepsilon}_{\widetilde{k}})) = 0 \text{ for } \widetilde{k}+1 > j_0,$$

This implies that

$$\lim_{k\to\infty} \limsup_{n} \frac{1}{n}\sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E^{\varepsilon}_k)) = 0, \tag{2.4}$$

this completes the proof of item $(ii)$.

$(iii)$ Suppose that $M$ is a compact metric space. For any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\omega(\{x\}, f)$ is contained in the interior of $E^{\varepsilon}_{k_\varepsilon}$.

We are going to verify condition $(ii)$.

**Claim 2:** Consider $k \geq k_\varepsilon$. Then $(\mathcal{O}^+x) \cap M \setminus E^{\varepsilon}_k$ is a finite set.

Suppose, contrary to our claim, that there exists a sequence $\{n_s\}_{s\in\mathbb{N}}$ such that $f^{n_s}x \notin E^{\varepsilon}_k$. By compactness of $M$, there exists a subsequence of sequence $(f^{n_\ell}x)_{\ell\in\mathbb{N}}$ that converges to some $p \in M$. Without loss generality, the sequence converges to $p \in \omega(\{x\}, f)$, so $p$ is an element of interior of $E^{\varepsilon}_{k_\varepsilon}$, i.e., $p \in \operatorname{int} E^{\varepsilon}_{k_\varepsilon}$. Using that $\operatorname{int} E^{\varepsilon}_{k_\varepsilon}$ is an open set, there exists $n_p > 0$ such that for $n_s \geq n_p$ we have that $f^{n_s}x \in \operatorname{int} E^{\varepsilon}_{k_\varepsilon}$. But $\operatorname{int} E^{\varepsilon}_{k_\varepsilon} \subseteq E^{\varepsilon}_{k_\varepsilon} \subseteq E^{\varepsilon}_k$ and $f^{n_s}x \notin E^{\varepsilon}_k$ for all $s \in \mathbb{N}$, this contradiction concludes the proof of the Claim 2, and we are done.

$(iv)$ For each $\varepsilon > 0$ fixed and $k \in \mathbb{N}$ we define

$$\widehat{E^{\varepsilon}_k} = \{w \in M : \varphi_j(w) < j(\varphi_-(w) + \varepsilon) \text{ for some } j \in \{1, ..., k\}\}$$

where $\widehat{E_k^\varepsilon} \subseteq E_k^\varepsilon$.

By continuity of $f$,$\varphi$ and $\varphi_-$, we see that $\widehat{E_k^\varepsilon}$ is an open set of $M$. Using that $M = \bigcup_{k\in\mathbb{N}} \widehat{E_k^\varepsilon}$ and $\omega(\{x\}, f)$ is a finite set, there exists $k_\varepsilon$ such that $\omega(\{x\}, f) \subseteq \widehat{E_{k_\varepsilon}^\varepsilon}$. By item $(iii)$, we are done. This completes the proof of Corollary 2.6.

□

## 2.4 Proof of Theorem C

Let $(M, \mathcal{A}, \mu)$ be a measure space, $f : M \to M$ be a measurable function, $\mu$ be a finite measure. Suppose that $(\varphi_n)_n$ is a subadditive sequence for $f$ such that $\varphi_1 \leq \beta$ for some $\beta \in \mathbb{R}$. Without loss of generality, we assume that $\beta > 0$.

Under the conditions stated above, and supposing that the following conditions are satisfied:

$(a)$ for all $j \in \mathbb{N}$ we have that $\varphi_-(f^j(x)) = \varphi_-(x)$ $\mu-$almost everywhere $x$ in $M$;

$(b)$ $\lim\limits_{k\to\infty} \limsup\limits_{n} \frac{1}{n} \sum\limits_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ for each $\ell \in \mathbb{N} \setminus \{0\}$.

Then Theorem C ensures that $\inf\limits_n \frac{1}{n} \int \varphi_n d\mu = \int \varphi_- d\mu$. Moreover, if there exists $\gamma > 0$ such that for all $n > 0$, $\frac{\varphi_n}{n} \geq -\gamma$ then

$$\int \varphi_- d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu = \lim_n \frac{1}{n} \int \varphi_n d\mu.$$

The proof will be divided into two steps. In first step, we show the particular version of Theorem C when the sequence $(\frac{\varphi_n}{n})_n$ is uniformly bounded from below, i.e., there exists $\alpha > 0$ such that $\frac{\varphi_n}{n} \geq -\alpha$ for all $n \in \mathbb{N}$. In the second step, using a truncation argument we conclude from step 1 the proof of the Theorem.

We begin by proving the following theorem.

**Theorem 2.21.** *Let $(M, \mathcal{A}, \mu)$ be a measure space, $f : M \to M$ be a measurable function, $\mu$ be a finite measure. Suppose that $(\varphi_n)_n$ is a subadditive sequence for $f$ such that $\varphi_1 \leq \beta$ for some $\beta > 0$. If the following conditions are satisfied:*

*(a) for all $j \in \mathbb{N}$ we have that $\varphi_-(f^j(x)) = \varphi_-(x)$ $\mu-$almost everywhere $x$ in $M$;*

*(b) $\lim\limits_{k\to\infty} \limsup\limits_{n} \frac{1}{n} \sum\limits_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\frac{1}{\ell}})) = 0$ for each $\ell \in \mathbb{N} \setminus \{0\}$;*

*(d) there exists $\gamma > 0$ such that for all $n > 0$, $\frac{\varphi_n}{n} \geq -\gamma$.*

*Then $\lim\limits_n \frac{1}{n} \int \varphi_n d\mu = \inf\limits_n \frac{1}{n} \int \varphi_n d\mu = \int \varphi_- d\mu$.*

*Proof of Theorem 2.21.* First, without loss of generality, we consider $\beta = \gamma$. Using that $(\varphi_n)_n$ is a subadditive sequence for $f$, we obtain that $\varphi_m \leq \sum_{j=0}^{m-1} \varphi_1 \circ f^j$ for all $m \in \mathbb{N}$, but $\varphi_1 \leq \beta$, so $-\beta \leq \frac{\varphi_m}{m} \leq \beta$, and $-\beta \leq \frac{1}{m} \int \varphi_m d\mu \leq \beta$ for all $m$ in $\mathbb{N}$. In particular, $\varphi_1$ is integrable. Define $\varphi_- : M \to [-\beta, \beta]$ by $\varphi_-(x) = \liminf_n \frac{\varphi_n(x)}{n}$. So $\beta \geq \varphi_-(x) \geq -\beta$ for all $x$ in $M$, and then $\varphi_-$ is integrable.

Fixed $\varepsilon > 0$, define for each $k \in \mathbb{N}$

$$E_k^\varepsilon := \{x \in M : \varphi_j(x) \leq j(\varphi_-(x) + \varepsilon) \text{ for some } j \in \{1, ..., k\}\}$$

It is clear that $E_k^\varepsilon \subseteq E_{k+1}^\varepsilon$ for all $k$. Note that by definition of $\varphi_-$, we have that $M = \bigcup_k E_k^\varepsilon$. Define $\psi_k(x) = \varphi_-(x) + \varepsilon$ if $x \in E_k^\varepsilon$, and $\psi_k(x) = \varphi_1(x)$ if $x \notin E_k^\varepsilon$. Suppose that $x \notin E_k^\varepsilon$, then $\psi_k(x) = \varphi_1(x)$, but by $E_k^\varepsilon$'s definition we have that $\varphi_1(x) > \varphi_-(x) + \varepsilon$. It imples that $\psi_k \geqslant \varphi_- + \varepsilon$ in $M$. Now, using that $M = \bigcup_k E_k^\varepsilon$, we see that $\lim_{k\to\infty} \psi_k(x) = \varphi_-(x) + \varepsilon$ for each $x \in M$.

Now, let $L$ be a fixed and arbitrary point of accumulation of sequence $(\frac{1}{n} \int \varphi_n d\mu)_n$, so there exists $(n_t)_{t\in\mathbb{N}}$ such that $\lim_{t\to\infty} \frac{1}{n_t} \int \varphi_{n_t} d\mu = L$ and $L \in [-\beta, \beta]$. The basic idea of the proof is to verify that $\int \varphi_- d\mu \leq L \leq \lim_{k\to\infty} \int \psi_k d\mu$. Later, an easy computation will show that $\int \varphi_- d\mu = L$. Observing that $L$ is an arbitrary point of accumulation of sequence $(\frac{1}{n} \int \varphi_n d\mu)_n$, we conclude that this sequence converges to $\int \varphi_- d\mu$. This will end the proof of Theorem 2.21.

From the above we are going to show that $\int \varphi_- d\mu \leq L$ and $L \leq \lim_{k\to\infty} \int \psi_k d\mu$. First, we observe that $\int \varphi_- d\mu \leq L$. By hypothesis, there exists $\beta > 0$ such that $\frac{\varphi_n}{n} \geq -\beta$ for all $n$. We have that $\frac{\varphi_n}{n} \geq -\beta$. Define $f_n(x) := \frac{\varphi_n}{n}(x) + \beta \geq 0$ and note that

$$f(x) = \liminf_n (\frac{\varphi_n}{n}(x) + \beta) = \varphi_-(x) + \beta.$$

By Fatou's Lemma, we have that $f(x) = \varphi_-(x) + \beta$ is an integrable funcion, and

$$\int \liminf_n (f_n) d\mu \leq \liminf_n \int f_n d\mu \leq \liminf_{n_t} \int f_{n_t} d\mu$$
$$\int \varphi_-(x) + \beta d\mu \leq \liminf_{n_t} \int (\frac{\varphi_{n_t}}{n_t} + \beta) d\mu$$

Then

$$\int \varphi_-(x) d\mu \leq \liminf_{n_t} \int \frac{\varphi_{n_t}}{n_t} d\mu = \lim_{n_t} \int \frac{\varphi_{n_t}}{n_t} d\mu = L.$$

So

$$\int \varphi_-(x) d\mu \leq L. \tag{2.5}$$

Now, we show that $L \leq \lim_{k\to\infty} \int \psi_k d\mu + 2\beta h(\ell)$. We need of the following result.

**Lemma 2.22.** *For each $n > k \geq 1$ and $\mu$-a.e. $x \in M$,*

$$\varphi_n(x) \leq \sum_{i=0}^{n-k-1} \psi_k(f^i(x)) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \varphi_1\}(f^i(x))$$

*Proof.* Use the subadditivity of sequence $(\varphi_n)_n$, and the fact that $\varphi_-$ is invariant in orbit of $x$ in $\mu$-a.e., see Lemma 1 in [13]. □

Note that $\psi_k$ is integrable. We have that $-\beta \leq \frac{\varphi_n}{n}$ for all $n$, so $-\beta \leq \varphi_-$ and $-\beta \leq \varphi_1$. Now, $-\beta < -\beta + \varepsilon \leq \varphi_- + \varepsilon$, then $-\beta \leq \psi_k$.

Note that $-\beta \leq \psi_k \leq \max\{\varphi_- + \varepsilon, \varphi_1\} \leq \max\{\varphi_- + \varepsilon, \beta\}$, where $\max\{\varphi_- + \varepsilon, \beta\}$ is integrable, so $\psi_k$ is integrable. Note that

$\max\{\varphi_1, \psi_k\} \circ f^i \leq \max\{\varphi_1, \varphi_- + \varepsilon, \beta\} \circ f^i = \max\{\varphi_- + \varepsilon, \beta\} \circ f^i = \max\{\varphi_- + \varepsilon, \beta\}$ because $\varphi_-$ is invariant in orbit of $x$ in $\mu$-a.e.

But $\max\{\varphi_- + \varepsilon, \beta\}$ is integrable, then $\max\{\varphi_1, \psi_k\} \circ f^i$ is integrable too for all $i$ in $\mathbb{N}$. By Lemma 2.22,

$$\frac{1}{n} \int \varphi_n(x) d\mu \leq \frac{1}{n} \sum_{i=0}^{n-k-1} \int \psi_k(f^i(x)) d\mu + \frac{1}{n} \sum_{i=n-k}^{n-1} \int \max\{\psi_k, \varphi_1\}(f^i(x)) d\mu. \tag{2.6}$$

Define $\varphi^+ = \max\{0, \varphi\}$, and note that

$$\sum_{i=n-k}^{n-1} \int \max\{\psi_k, \varphi_1\}(f^i(x)) d\mu \leq \sum_{i=n-k}^{n-1} \int \max\{\varphi_- + \varepsilon, \varphi_1^+\}(f^i(x)) d\mu.$$

Define $S = \{x \in M : \varphi_-(x) + \varepsilon \geq \varphi_1^+(x)\}$, so

$$\sum_{i=n-k}^{n-1} \int \max\{\varphi_- + \varepsilon, \varphi_1^+\}(f^i(x)) d\mu = \sum_{i=n-k}^{n-1} [\textstyle\int_S \varphi_- + \varepsilon d\mu + \int_{M\setminus S} \varphi_1^+ \circ f^i d\mu].$$

Using that $-\beta \leq \varphi_-$ and $\int \varphi_- d\mu \leq L \in [-\beta, \infty)$, then $\int_S \varphi_- d\mu < \infty$. So,

$$\sum_{i=n-k}^{n-1} [\textstyle\int_S \varphi_- + \varepsilon d\mu + \int_{M\setminus S} \varphi_1^+ \circ f^i d\mu] \leq k[\int_S \varphi_- + \varepsilon d\mu + \beta],$$

and

$$\frac{1}{n} \sum_{i=n-k}^{n-1} \int \max\{\psi_k, \varphi_1\}(f^i(x)) d\mu \leq \frac{k}{n}(\int_S \varphi_- + \varepsilon d\mu + \beta). \tag{2.7}$$

Now, we are going to show that

$$\tfrac{1}{n} \sum_{i=0}^{n-k-1} \int \psi_k(f^i(x)) d\mu \leq (1 - \tfrac{k}{n}) \int \psi_k d\mu + 2\beta \cdot \tfrac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^\varepsilon))$$

Define $F_{i,k} := f^{-i}(E_k^{\varepsilon})$ for each $i \in \{0, ..., n-k-1\}$, so $\int \psi_k(f^i(x))d\mu = \int_{F_{i,k}} \varphi_-(f^i(x)) + \varepsilon d\mu + \int_{M\setminus F_{i,k}} \psi_k(f^i(x))d\mu$. Using that $\varphi_-$ is invariant in orbit of $x$ in $\mu$-a.e., we have that

$$\int \psi_k(f^i(x))d\mu = \int_{F_{i,k}} \varphi_-(x) + \varepsilon d\mu + \int_{M\setminus F_{i,k}} \psi_k(f^i(x))d\mu.$$

But $\varphi_-(x) + \varepsilon \leq \psi_k$ in $M$, so

$$\begin{aligned} \int \psi_k \circ f^i d\mu \leq \int_{F_{i,k}} \psi_k d\mu + \int_{M\setminus F_{i,k}} \psi_k \circ f^i d\mu = \\ \int_{F_{i,k}} \psi_k d\mu + \int_{M\setminus F_{i,k}} \psi_k d\mu - \int_{M\setminus F_{i,k}} \psi_k d\mu + \int_{M\setminus F_{i,k}} \psi_k \circ f^i d\mu = \\ \int \psi_k d\mu + \int_{M\setminus F_{i,k}} \psi_k \circ f^i d\mu + \int_{M\setminus F_{i,k}} -\psi_k d\mu = \\ \int \psi_k d\mu + \int_{M\setminus F_{i,k}} \varphi_1 \circ f^i d\mu + \int_{M\setminus F_{i,k}} -\psi_k d\mu \leq \\ \int \psi_k d\mu + \int_{M\setminus F_{i,k}} \beta d\mu + \int_{M\setminus F_{i,k}} \beta d\mu \leq \\ \int \psi_k d\mu + 2\beta\mu(M \setminus F_{i,k}). \end{aligned}$$

since $-\beta \leq \psi_k \leq \max\{\varphi_- + \varepsilon, \beta\}$. Then

$$\int \psi_k \circ f^i d\mu \leq \int \psi_k d\mu + 2\beta\mu(M \setminus F_{i,k}),$$

we obtain that

$$\frac{1}{n}\sum_{i=0}^{n-k-1} \int \psi_k \circ f^i d\mu \leq (1 - \frac{k}{n}) \int \psi_k d\mu + 2\beta \cdot \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(M \setminus F_{i,k}) \tag{2.8}$$

By (2.6), (2.7), and the inequality above we have that

$$\frac{1}{n}\int \varphi_n(x)d\mu \leq \frac{k}{n}(\int_S \varphi_- + \varepsilon d\mu + \beta) + (1 - \frac{k}{n})\int \psi_k d\mu + 2\beta \cdot \frac{1}{n}\sum_{i=0}^{n-k-1} \mu(M \setminus F_{i,k}).$$

Passing $\limsup_n$ in the previous inequality

$$\begin{aligned} L = \limsup_{n_t} \frac{1}{n_t} \int \varphi_{n_t}(x)d\mu \leq \limsup_n \frac{1}{n} \int \varphi_n(x)d\mu \leq \\ \int \psi_k d\mu + 2\beta \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\varepsilon})). \end{aligned}$$

By equation (2.5),

$$\int \varphi_- d\mu \le L \le \int \psi_k d\mu + 2\beta \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\varepsilon}))$$

Taking $\varepsilon = \frac{1}{\ell}$ for $\ell \in \mathbb{N} \setminus \{0\}$,

$$\int \varphi_- d\mu \le L \le \lim_{k\to\infty} \int \psi_k d\mu + 2\beta \lim_{k\to\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-k-1} \mu(f^{-i}(M \setminus E_k^{\frac{1}{\ell}}))$$

By hypothesis $(b)$,

$$\int \varphi_-(x) d\mu \le L \le \lim_{k\to\infty} \int \psi_k d\mu.$$

**Lemma 2.23.** $\int \varphi_- d\mu = L$

*Proof.* Using that $M = \bigcup_{k=1}^{\infty} E_k^{\frac{1}{\ell}}$, we obtain that $\psi_k \to_k \varphi_- + \frac{1}{\ell}$ in each point. But

$$-\beta \le \psi_k \le \max\{\varphi_- + \tfrac{1}{\ell}, \varphi_1^+\},$$

we define $g := \max\{\varphi_- + \frac{1}{\ell}, \beta\}$. So $g$ is integrable and $|\psi_k| \le g$. By dominated convergence theorem, we have that

$$\lim_k \int \psi_k d\mu = \int \varphi_- + \tfrac{1}{\ell} d\mu.$$

We obtain that

$$\int \varphi_- d\mu \le L \le \lim_k \int \psi_k d\mu = \int \varphi_- d\mu + \tfrac{1}{\ell}.$$

Making $\ell$ tend to infinity,

$$\int \varphi_- d\mu \le L \le \int \varphi_- d\mu$$

□

Since $\int \varphi_- d\mu = L$ for all accumulation point $L$ of the sequence $(\frac{1}{n} \int \varphi_n d\mu)_n$, we have that $\lim_n \frac{1}{n} \int \varphi_n d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu = \int \varphi_- d\mu$. This concludes the proof of Theorem 2.21.

□

Now, we are going to use a truncation argument to finish the proof of Theorem C. For each $k$ in $\mathbb{N}$ define $\varphi_n^k = \max\{\varphi_n, -kn\}$ and $\varphi_-^k = \max\{\varphi_-, -k\}$. For each $\varepsilon > 0$ fixed and $r \in \mathbb{N}$ we define $G_r^{\varepsilon} = \{x \in M : \varphi_j^k(x) \le j(\varphi_-^k(x) + \varepsilon)$ for some $j \in \{1, ..., r\}\}$.

To finish we need of the following Lemma.

**Lemma 2.24.** *The following conditions are satisfied:*

(i) $(\varphi_n^k)_n$ *is a subadditive sequence for any $k$ fixed.*

(ii) $\varphi_1^k$ *is upper bounded for any $k$ fixed.*

(iii) $(\frac{\varphi_n^k}{n})_n$ *is uniformly bounded by below for any $k$ fixed.*

(iv) $\varphi_-^k(x) = \liminf_n \frac{\varphi_n^k(x)}{n}$ *for any $k$ fixed.*

(v) *For each $j \in \mathbb{N}$ we have that $\varphi_-^k(f^j(x)) = \varphi_-^k(x)$ $\mu-a.e.$ $x$ in $M$ where $\varphi_-^k : M \to [-\infty, \infty]$ is given by $\varphi_-^k(x) = \liminf_n \frac{\varphi_n^k(x)}{n}$ for any $k$ fixed.*

(vi) $E_r^\varepsilon \subseteq G_r^\varepsilon$ *for every $\varepsilon > 0$ and $r \in \mathbb{N}$.*

(vii) *Fixed $n$, $(\varphi_n^k)_k$ is a nonincreasing monotonic sequence.*

(viii) *Fixed $n$, $\lim_k \varphi_n^k(x) = \varphi_n(x)$ for all $x$ in $M$, (then $\varphi_n^k \searrow_k \varphi_n$).*

(ix) $(\varphi_-^k)_k$ *is a nonincreasing monotonic sequence.*

(x) $\lim_k \varphi_-^k(x) = \varphi_-(x)$ *for all $x$ in $M$, (then $\varphi_-^k \searrow_k \varphi_-$).*

(xi) $(\varphi_-^k)^+(x) = (\varphi_-)^+(x)$ *for all $x$ in $M$ and for all $k$ in $\mathbb{N}$.*

■

In section 2.5, after the end of this proof, we present the demonstration of this Lemma.

By $(vi)$, $E_r^\varepsilon \subseteq G_r^\varepsilon$ for every $\varepsilon > 0$ and $r \in \mathbb{N}$. In particular, $f^{-i}(M \setminus G_r^\varepsilon) \subseteq f^{-i}(M \setminus E_r^\varepsilon)$ for all $i \geq 0$. Note that

$$\mu(f^{-i}(M \setminus G_r^\varepsilon)) \leq \mu(f^{-i}(M \setminus E_r^\varepsilon)), \text{ and then}$$

$\lim_{r\to+\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-r-1} \mu(M \setminus f^{-i}(G_r^\varepsilon)) \leq \lim_{r\to+\infty} \limsup_n \frac{1}{n} \sum_{i=0}^{n-r-1} \mu(M \setminus f^{-i}(E_r^\varepsilon))$. Therefore for each $k$ we have that the sequence $(\varphi_n^k)_n$ satisfies the hypothesis of Theorem 2.21, so

$$\int \liminf_n \frac{\varphi_n^k(x)}{n} d\mu = \int \varphi_-^k d\mu = \lim_n \int \frac{\varphi_n^k}{n} d\mu = \inf_n \int \frac{\varphi_n^k}{n} d\mu. \tag{2.9}$$

We claim that

$$\inf_k \int \varphi_n^k d\mu = \int \varphi_n d\mu. \tag{2.10}$$

To see this recall that $\varphi_n^k \searrow_k \varphi_n$ with $\varphi_n^k = \max\{\varphi_n, -kn\}$, so $\varphi_n^1 \geq \varphi_n^k$ for all $k$. Consider $\gamma_k = \varphi_n^1 - \varphi_n^k \geq 0$, and note that $\gamma_k = \varphi_n^1 - \varphi_n^k \leq \varphi_n^1 - \varphi_n^{k+1} = \gamma_{k+1}$.

Thus $(\gamma_k)_k$ is nondecreasing monotonic sequence and $\gamma_k \nearrow_k \varphi_n^1 - \varphi_n$ , and by monotone convergence theorem, $\int \varphi_n^1 - \varphi_n d\mu = \int \lim\limits_{k\to\infty} \gamma_k d\mu = \lim\limits_{k\to\infty} \int \gamma_k d\mu = \lim\limits_{k\to\infty} \int \varphi_n^1 - \varphi_n^k d\mu$, and then $\int \varphi_n d\mu = \lim\limits_{k\to\infty} \int \varphi_n^k d\mu$ and $\lim\limits_{k\to\infty} \int \varphi_n^k d\mu = \inf_k \int \varphi_n^k d\mu$

Similarly, using monotone convergence theorem,

$$\inf_k \int \varphi_-^k d\mu = \int \varphi_- d\mu \tag{2.11}$$

By (2.9), (2.10) and (2.11), we have that

$$\int \varphi_- d\mu = \inf_k (\int \varphi_-^k d\mu) = \inf_k (\inf_n \int \frac{\varphi_n^k}{n} d\mu) = \inf_n \frac{1}{n} (\inf_k \int \varphi_n^k d\mu) = \inf_n \frac{1}{n} (\int \varphi_n d\mu)$$

Then

$$\int \varphi_- d\mu = \inf_n \frac{1}{n} \int \varphi_n d\mu. \tag{2.12}$$

This concludes the proof of Theorem C.

### 2.4.1 Proof of remark 2.2

We are going to show that if $\int \varphi_- d\mu = -\infty$ or $\beta \leq 0$ then $\int \varphi_- d\mu = \inf\limits_n \frac{1}{n} \int \varphi_n d\mu = \lim\limits_n \frac{1}{n} \int \varphi_n d\mu$.

Suppose that $\int \varphi_- d\mu = \inf\limits_n \frac{1}{n} \int \varphi_n d\mu = -\infty$, then there exists a subsequence $(\frac{1}{n_k} \int \varphi_{n_k} d\mu)_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} \frac{1}{n_k} \int \varphi_{n_k} d\mu = -\infty$. Recall that $(\varphi_n)_n$ is a subadditive sequence for $f$, so $\varphi_m \leq \sum\limits_{j=0}^{m-1} \varphi_1 \circ f^j$ for all $m \in \mathbb{N}$, but $\varphi_1 \leq \beta$, and then $\frac{\varphi_m}{m} \leq \beta$ for all $m \in \mathbb{N}$.

Let $N$ be an arbitrary natural number, there exists a number $s = s(N) > 0$ such that $\beta - sN < -N$. There exists $k_0 > 0$ such that for $k > k_0$, we have that

$$\frac{1}{n_k} \int \varphi_{n_k} d\mu < -sN. \tag{2.13}$$

Let $n > n_{k_0}$, so $n = n_{k_0} + r$ with $r \geq 1$. By subadditivity of sequence $(\varphi_m)_m$,

$$\varphi_n = \varphi_{r+n_{k_0}} \leq \varphi_{n_{k_0}} + \varphi_r \circ f^{n_{k_0}}$$

$$\frac{1}{n}\varphi_n = \frac{1}{n}\varphi_{r+n_{k_0}} \leq \frac{1}{n}\varphi_{n_{k_0}} + \frac{1}{n}\varphi_r \circ f^{n_{k_0}} \leq \frac{1}{n_{k_0}}\varphi_{n_{k_0}} + \frac{1}{n} r\beta$$

Note that $\frac{1}{n} r < 1$,

$$\frac{1}{n}\int \varphi_n d\mu \leq \frac{1}{n_{k_0}}\int \varphi_{n_{k_0}} d\mu + \frac{1}{n} r\beta$$
$$\frac{1}{n}\int \varphi_n d\mu \leq -sN + \beta < -N$$

so for $n > n_{k_0}$, we obtain that $\frac{1}{n}\int \varphi_n d\mu < -N$. This shows that $\lim\limits_n \frac{1}{n}\int \varphi_n d\mu = -\infty$.

Now, suppose that $\beta \leq 0$. We are going to show that $\int \varphi_- d\mu = \inf\limits_n \frac{1}{n}\int \varphi_n d\mu = \lim\limits_n \frac{1}{n}\int \varphi_n d\mu$.

If $\inf\limits_n \frac{1}{n}\int \varphi_n d\mu = \int \varphi_- d\mu = -\infty$ there is nothing to show. Suppose that $\beta \leq 0$ and $\inf\limits_n \frac{1}{n}\int \varphi_n d\mu = \int \varphi_- d\mu \in \mathbb{R}$, then there exists a subsequence $(\frac{1}{n_k}\int \varphi_{n_k} d\mu)_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} \frac{1}{n_k}\int \varphi_{n_k} d\mu = \int \varphi_- d\mu$ and $0 \leq \frac{1}{n}\int \varphi_n d\mu - \int \varphi_- d\mu$ for all $n \in \mathbb{N}$. Recall that $(\varphi_n)_n$ is a subadditive sequence for $f$, so $\varphi_m \leq \sum\limits_{j=0}^{m-1} \varphi_1 \circ f^j$ for all $m \in \mathbb{N}$, but $\varphi_1 \leq \beta$, and then $\frac{\varphi_m}{m} \leq \beta$ for all $m \in \mathbb{N}$.

Let $\varepsilon > 0$, there exists $k_0 > 0$ such that for $k > k_0$, we have that

$$\frac{1}{n_k}\int \varphi_{n_k} d\mu - \int \varphi_- d\mu < \varepsilon. \tag{2.14}$$

Let $n > n_{k_0}$, so $n = n_{k_0} + r$ with $r \geq 1$. By subadditivity of sequence $(\varphi_m)_m$,

$$\varphi_n = \varphi_{r+n_{k_0}} \leq \varphi_{n_{k_0}} + \varphi_r \circ f^{n_{k_0}}$$
$$\frac{1}{n}\varphi_n = \frac{1}{n}\varphi_{r+n_{k_0}} \leq \frac{1}{n}\varphi_{n_{k_0}} + \frac{1}{n}\varphi_r \circ f^{n_{k_0}} \leq \frac{1}{n_{k_0}}\varphi_{n_{k_0}} + \frac{1}{n} r\beta$$

Note that $\frac{1}{n}r < 1$,

$$\frac{1}{n}\int \varphi_n d\mu - \int \varphi_- d\mu \leq \frac{1}{n_{k_0}}\int \varphi_{n_{k_0}} d\mu + \frac{1}{n} r\beta - \int \varphi_- d\mu$$
$$\frac{1}{n}\int \varphi_n d\mu - \int \varphi_- d\mu + \frac{1}{n} r\beta \leq \varepsilon + \beta \leq \varepsilon$$

so for $n > n_{k_0}$, we obtain that $\frac{1}{n}\int \varphi_n d\mu - \int \varphi_- d\mu < \varepsilon$. This shows that $\lim\limits_n \frac{1}{n}\int \varphi_n d\mu = \int \varphi_- d\mu$, and we are done.

## 2.5 Proof of Lemma 2.24

We are going to verify a sequence of technical results that were stated by Lemma 2.24. First, recall that $\varphi_n^k = \max\{\varphi_n, -kn\}$.

$(i)$ $(\varphi_n^k)_n$ is a subadditive sequence for any $k$ fixed.

We must to show that $\varphi_{n+m}^k(x) \leq \varphi_n^k(x) + \varphi_m^k \circ f^n(x)$. We have to consider the following cases.

If $\varphi_{n+m}^k(x) = -k(n+m)$ then $\varphi_{n+m}^k(x) = -k(n+m) = -kn - km \leq \varphi_n^k(x) + \varphi_m^k(f^n x)$.

If $\varphi_{n+m}^k(x) = \varphi_{n+m}(x)$ then $\varphi_{n+m}^k(x) = \varphi_{n+m}(x) \leq \varphi_n(x) + \varphi_m f^n(x) \leq \varphi_n^k(x) + \varphi_m^k \circ f^n(x)$.

This proves the item $(i)$.

$(ii)$ $\varphi_1^k$ is upper bounded for any $k$ fixed.

Just observe that $\varphi_1^k = \max\{\varphi_1, -k\} \leq \beta$.

$(iii)$ $(\frac{\varphi_n^k}{n})_n$ is uniformly bounded by below for any $k$ fixed.

Note that $\frac{\varphi_n^k}{n} = \frac{1}{n}\varphi_n^k = \max\{\frac{1}{n}\varphi_n, -k\} \geq -k$.

$(iv)$ $\varphi_-^k(x) = \liminf_n \frac{\varphi_n^k(x)}{n}$ for any $k$ fixed.

It follows from an easy computation. In fact,

$\liminf_n \frac{\varphi_n^k(x)}{n} = \liminf_n \frac{\max\{\varphi_n(x), -kn\}}{n} = \liminf_n \max\{\frac{\varphi_n(x)}{n}, -k\} =$

$= \max\{\liminf_n \frac{\varphi_n(x)}{n}, -k\} = \max\{\varphi_-(x), -k\} =: \varphi_-^k(x)$.

$(v)$ For each $j \in \mathbb{N}$ we have that $\varphi_-^k(f^j(x)) = \varphi_-^k(x)$ $\mu - a.e.$ $x$ in $M$ where $\varphi_-^k : M \to [-\infty, \infty]$ is given by $\varphi_-^k(x) = \liminf_n \frac{\varphi_n^k(x)}{n}$ for any $k$ fixed.

The proof is straightforward from condition $(a)$. So,

$\varphi_-^k(x) = \max\{\varphi_-(x), -k\} = \max\{\varphi_-(f^j x), -k\} = \varphi_-^k(f^j x)$ for all $j \in \mathbb{N}$ since $(a)$ holds.

$(vi)$ $E_r^\varepsilon \subseteq G_r^\varepsilon$ for every $\varepsilon > 0$ and $r \in \mathbb{N}$. Recall that $G_r^\varepsilon = \{x \in M : \varphi_j^k(x) \leq j(\varphi_-^k(x) + \varepsilon)$ for some $j \in \{1, ..., r\}\}$; and $E_k^\varepsilon = \{x \in M : \varphi_j(x) \leq j(\varphi_-(x) + \varepsilon)$ for some $j \in \{1, ..., k\}\}$.

To see this take $x \in E_r^\varepsilon$, so $\varphi_j(x) \leq j(\varphi_-(x) + \varepsilon)$ for some $j \in \{1, ..., k\}$. The following two cases completes the proof of item $(vi)$.

Case $\varphi_j^k(x) = \varphi_j(x)$, then $\varphi_j^k(x) = \varphi_j(x) \leq j(\varphi_-(x) + \varepsilon) \leq j(\varphi_-^k(x) + \varepsilon)$ since $\varphi_-^k(x) = \max\{\varphi_-(x), -k\}$, so $x \in G_r^\varepsilon$.

Case $\varphi_j^k(x) = -k$, we have that $\varphi_j^k(x) = -k \leq \varphi_-^k(x) \leq \varphi_-^k(x) + \varepsilon$ then $x \in G_r^\varepsilon$ with $j = 1$.

$(vii)$ Fixed $n$, $(\varphi_n^k)_k$ is a nonincreasing monotonic sequence.

We are going to show that $\varphi_n^k \geq \varphi_n^{k+1}$.

(Case I): $-kn \leq \varphi_n(x) = \varphi_n^k(x)$

Now, $-n(k+1) = -nk - n < -kn \leq \varphi_n(x) = \varphi_n^{k+1}(x)$. So $\varphi_n^{k+1}(x) = \varphi_n^k(x)$.

(Case II): $\varphi_n(x) \leq -kn = \varphi_n^k(x)$

Subcase II.1: $-(k+1)n \leq \varphi_n(x) = \varphi_n^{k+1}(x)$

Then $\varphi_n^{k+1}(x) = \varphi_n(x) \leq -kn = \varphi_n^k(x)$. So, $\varphi_n^{k+1}(x) \leq \varphi_n^k(x)$.

Subcase II.2: $\varphi_n(x) \leq -(k+1)n = \varphi_n^{k+1}(x)$

We have that $\varphi_n^{k+1}(x) = -(k+1)n < -kn = \varphi_n^k(x)$. So, $\varphi_n^{k+1}(x) < \varphi_n^k(x)$.

The item $(vii)$ is proved.

$(viii)$ Fixed $n$, we have that the $\lim_k \varphi_n^k(x) = \varphi_n(x)$ for all $x$ in $M$, (then $\varphi_n^k \searrow_k \varphi_n$).

Fix $x$ in $M$, given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have that $\varphi_n(x) > -kn$. So for $k \geq k_0$ one has $\varphi_n^k(x) = \varphi_n(x)$.

For $k \geq k_0$ we have that $|\varphi_n^k(x) - \varphi_n(x)| = 0 < \varepsilon$, this concludes the verification of $(viii)$.

$(ix)$ $(\varphi_-^k)_k$ is a nonincreasing monotonic sequence.

We have that $\varphi_-^k(x) = \max\{\varphi_-(x), -k\}$.

(Case I): $\varphi_-^k(x) = \varphi_-(x) \geq -k$

So $-(k+1) < -k \leq \varphi_-(x)$ , and then $\varphi_-^{k+1}(x) = \varphi_-(x)(= \varphi_-^k(x))$.

(Case II): $\varphi_-^k(x) = -k \geq \varphi_-(x)$

(Subcase II.1) $\varphi_-^{k+1}(x) = -(k+1) \geq \varphi_-(x)$

So $\varphi_-^{k+1}(x) = -(k+1) < -k = \varphi_-^k(x)$.

(Subcase II.2) $\varphi_-^{k+1}(x) = \varphi_-(x) \geq -(k+1)$

Then $\varphi_-^{k+1}(x) = \varphi_-(x) \leq -k = \varphi_-^k(x)$.

The item $(ix)$ is proved.

$(x)$ $\lim_k \varphi_-^k(x) = \varphi_-(x)$ for all $x$ in $M$, (and then $\varphi_-^k \searrow_k \varphi_-$).

(Case I): $\varphi_-(x) > -\infty$

Let $\varepsilon > 0$, then there exists $k_0$ in $\mathbb{N}$ such that $\varphi_-(x) > -k_0$. For $k > k_0$ we have that $\varphi_-^k(x) = \varphi_-(x) > -k_0 > -k$.

For $k > k_0$, we see that $|\varphi_-^k(x) - \varphi_-(x)| = 0 < \varepsilon$.

(Case II): $\varphi_-(x) = -\infty$

Thus $\varphi_-^k(x) = -k$ and $\lim_k \varphi_-^k(x) = \lim_k -k = -\infty = \varphi_-(x)$.

The item $(x)$ is proved.

$(xi)$ $(\varphi_-^k)^+(x) = (\varphi_-)^+(x)$ for all $x$ in $M$ and for all $k$ in $\mathbb{N}$.

We have that $\varphi_-^k = \max\{\varphi_-, -k\}$, $(\varphi_-^k)^+ = \max\{\varphi_-^k, 0\}$ and $(\varphi_-)^+ = \max\{\varphi_-, 0\}$.

(Case I): $\varphi_-^k(x) = \varphi_-(x)$

But $(\varphi_-^k)^+(x) = \max\{\varphi_-^k(x), 0\} = \max\{\varphi_-(x), 0\} = (\varphi_-)^+(x)$.

(Case II): $\varphi_-^k(x) = -k \geq \varphi_-(x)$ (so $0 > \varphi_-(x)$)

But $(\varphi_-^k)^+(x) = \max\{\varphi_-^k(x), 0\} = \max\{-k, 0\} = 0 = \max\{\varphi_-(x), 0\} = (\varphi_-)^+(x)$

This completes the proof of item $(xi)$, and we are done.

■

# Chapter 3

# Existence of invariant measures

One of the most celebrated results of invariant measure theory was proved by Krylov and Bogolyubov [17] for compact metric space. Precisely, they showed that if $f : M \to M$ is a continuous map, then $f$ admits an invariant Borel probability measure where $M$ is a compact metric space. Since there exists an invariant measure we can apply the Kingman Theorem (and so the Birkhoff Theorem) to study the statistical properties of the system. Other fundamental results that hold in the context of finite invariant measures are the Poincaré Recurrence Theorem and Kac's Theorem.

With these theorems in mind, we give necessary conditions to guarantee the existence of invariant measures in locally compact and separable metric space for continuous proper maps. Moreover, we use the Perron-Frobenius operator and the techniques developed here to obtain other criteria to guarantee the existence of invariant measures for continuous maps (not necessarily a proper map) in locally compact separable metric space.

The chapter is organized as follows: in Section 3.1 we give the statements of main results. In section 3.2, we provide some applications and examples for our results. In Section 3.3 we state some auxiliary results needed to proof of Theorem G and Theorem H. In Section 3.4 and 3.6 we present the proofs of Theorem G, and Theorem H respectively.

## 3.1 Statements of main results of Chapter

Let $M$ be a metric space, a map $f : M \to M$ is a *proper map* if the preimage of every compact set in $M$ is compact in $M$.

In what follows we consider $X$ to be a locally compact separable metric space. Let $C_0(X, \mathbb{R})$ be the set of functions that vanishes at "infinity" given by

$$C_0(X, \mathbb{R}) = C_0(X) = \{\varphi \in C(X, \mathbb{R}) : \{x \in X : |\varphi(x)| \geq \varepsilon\} \text{ is compact for all } \varepsilon > 0\}$$

where $C(X, \mathbb{R})$ is the set of all continuous function from $X$ to $\mathbb{R}$. We denote the set of all continuous functions $\varphi : X \to \mathbb{R}$ with *compact support* by $C_c(X, \mathbb{R})$, or $C_c(X)$. Our result allow us to obtain a characterization for the existence of invariant measures as follows.

**Theorem G.** *Suppose that $f : X \to X$ is a continuous proper map on locally compact separable metric space. Then the following conditions are equivalents.*

*(i) there exist $\varphi \in C_c(X; \mathbb{R})$ with $0 \leq \varphi \leq 1$ and $x_0 \in X$ such that the following number $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$;*

*(ii) there exist $\varphi \in C_c(X; \mathbb{R})$ with $0 \leq \varphi \leq 1$ and $x_0 \in X$ such that the following number $\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$;*

*(iii) there exist a probability measure $\nu$ on $X$ and an observable $\varphi \in C_0(X, \mathbb{R})$ such that $\liminf_n \frac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\nu > 0$;*

*(iv) there exists an invariant probability measure.*

Our goal is to provide a natural way to obtain invariant measures for continuous proper maps in this spaces. Moreover, our proof does not use the tightness property of Prokhorov's Theorem to obtain the convergence in the space of Borel finite measures.

**Remark 3.1.** *Actually, Theorem G holds for any measurable function $f : X \to X$ such that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$ where $X$ is a locally compact separable metric space. In Lemma 3.29, we see that continuous proper maps satisfy this condition.*

**Remark 3.2.** *Note that the following conditions are equivalents.*

*(iii) there exist a probability measure $\nu$ on $X$ and an observable $\varphi \in C_0(X, \mathbb{R})$ such that $\liminf_n \frac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\nu > 0$;*

*(iii′) there exist a probability measure $\nu$ on $X$ and an observable $\psi \in C_0(X, \mathbb{R})$ such that $\limsup_n \frac{1}{n} \int \sum_{j=0}^{n-1} \psi \circ f^j d\nu < 0$.*

**Remark 3.3.** *Using Corollary D and Theorem G, we obtain that if there exist a probability measure $\nu$ on $X$ and an observable $\varphi$ in $C_0(X, \mathbb{R})$ such that $\varphi$ satisfies hypothesis $(b)$ and $\int \varphi_- d\nu \neq 0$, then there exists an invariant probability measure, where $(X, d)$ is a locally compact separable metric space, and $f : X \to X$ is a continuous proper map.*

**Remark 3.4.** *Let $(M, \mathcal{A}, \mu)$ be a measure metric space, $f$ be a measurable transformation where $\mu$ is a finite measure (not necessarily an invariant measure under $f$). The system $(M, \mathcal{A}, \mu, f)$ is said to be* mixing *if for any bounded measurable maps $\varphi, \psi : M \to \mathbb{R}$,* *one has* $\lim_{n\to\infty}\int \varphi\circ f^n\cdot\psi d\mu=\int\varphi d\mu\int\psi d\mu$. *For a continuous proper map* $f: X\to X$ *on locally compact separable metric space, we are able to show that if there exists a probability measure* $\eta$ *(not necessarily an invariant measure under* $f$*) such that* $(X,\mathcal{A},f,\eta)$ *is a mixing system, then there exists an invariant probability measure. (See subsection 3.5.1.)*

**Remark 3.5.** *Since homeomorphisms are continuous proper maps, the conditions stated by Theorem G are also equivalents for homeomorphisms on locally compact separable metric space.*

**Remark 3.6.** *Let* $X$ *be a locally compact separable metric space and let* $\mu$ *be a Borel measure on* $X$. *We will say that* $\mu$ *is* arealike *if* $\mu(x)=0$ *for all* $x$ *in* $X$, $\mu(U)>0$ *for all nonempty open subsets* $U$ *of* $X$, *and* $\mu(K)$ *is finite for all compact subsets* $K$ *of* $X$. *In [14], Baldwin provided a topological criterion that guarantees the existence of an* arealike *invariant measure for a given fixed homeomorphism of* $X$.

*The following example admits a unique invariant measure that is not arealike measure. Consider the homeomorphism* $f:[0,+\infty)\to[0,+\infty)$ *given by* $f(x)=\frac{x}{2}$, *note that* $[0,+\infty)$ *is a locally compact separable metric space.*

We use the Perron-Frobenius operator and the techniques developed here in the proof of the Theorem G to obtain other criteria to guarantee the existence of invariant measures for locally compact separable metric space and continuous maps (not necessarily a proper map).

Let $f: X\to X$ to be a continuous function. A bounded operator $\mathcal{L}: C_0(X)\to C_0(X)$ is called *Perron-Frobenius-like operator* for $f$ if $\mathcal{L}(g)\geq 0$ whenever $g\geq 0$ for $g\in C_0(X)$, and $\mathcal{L}((g_1\circ f)g_2)=g_1\mathcal{L}(g_2)$ for all $g_1,g_2\in C_0(X)$. (In Lemma 3.36, we show that the Perron-Frobenius-like operator $\mathcal{L}$ is well defined i.e., if $f$ is a continuous function and $g_1,g_2\in C_0(X)$ then $(g_1\circ f)g_2\in C_0(X)$).

**Theorem H.** *Suppose that* $f: X\to X$ *is a continuous function, and* $\mathcal{L}: C_0(X)\to C_0(X)$ *is a Perron-Frobenius-like operator. If* $\|\mathcal{L}\|$ *is an eigenvalue of* $\mathcal{L}$, *then there exists an invariant probability measure.*

The important point to note here is the form of the boundedness of the Perron-Frobenius-like operator is used to drop the proper condition of map of Theorem G.

Our goal is to provide a relation between the existence of a Perron-Frobenius operator and the existence of invariant measures for continuous maps on locally compact separable metric spaces.

## 3.2 Applications

The characterization for the existence of invariant measures, Theorem G, allows us to obtain the following consequences.

**Example 3.7.** *Theorem G implies the Theorem of Krylov and Bogolyubov for compact metric space. In fact, suppose that $M$ is a compact metric space, and $f : M \to M$ is a continuous function. Since closed in compact is compact, we see that $f$ is a proper map, and all continuous function $\varphi : M \to \mathbb{R}$ is a continuous function with compact support. This implies that $C(X, \mathbb{R}) = C_0(X, \mathbb{R})$. Taking $\varphi = 1$, the constant function, and any non-null finite Borel measure $\mu$ of $M$, by item (i) of Theorem G, we are done.*

**Example 3.8.** *Consider the homeomorphism $f : (0, +\infty) \to (0, +\infty)$ given by $f(x) = \frac{x}{2}$, and note that $M = (0, +\infty)$ is a locally compact and separable metric space. Then does not exist an invariant measure, therefore, by Remarks 3.2 and 3.5, for any $\varphi \in C_0(M, \mathbb{R})$ and for any $x \in M$, we have that $\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) \le 0 \le \limsup_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$. Now, for any $\varphi \in C_0(M, \mathbb{R})$ with $M = (0, +\infty)$ we have that $\lim_{x\to 0} \varphi(x) = 0$. Observing that for all $x \in M$, $f^j(x)$ goes to $0$ as $j$ tends to infinity, we see that $\lim_{j\to\infty} \varphi \circ f^j(x) = 0$, and then $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = 0$.*

**Example 3.9.** *Suppose that $f : X \to X$ is a continuous proper map on locally compact separable metric space. If there exist $x \in X$, $n_0$ in $\mathbb{N}$ and a compact set $K$ contained in $X$ such that for $n > n_0$, $f^n(x) \in K$ then there exists an invariant probability measure. In fact, suppose that there exist $x_0 \in X$, $n_0$ in $\mathbb{N}$ and $K$ compact set such that for $n > n_0$, $f^n(x_0) \in K$. Without loss generality, $f^n(x_0) \in K$ for all $n \in \mathbb{N}$, since $X$ is a locally compact separable metric space. By Urysohn Lemma, Lemma 3.15, there exists a function $\varphi \in C_c(X)$ such that $0 \le \varphi(y) \le 1$ for all $y \in X$, and $\varphi(x) = 1$ for all $x \in K$, then $\varphi(f^n x_0) = 1$ for all $n \in \mathbb{N}$. We obtain that $1 = \varphi(x_0) \ge \varphi(f^n x_0) = 1$ for all $n \in \mathbb{N}$, so*

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) = 1.$$

*Using Theorem G, there exists an invariant measure, and we are done.*

In view of Example 3.9, we say that a system $(X, f)$ is *a non-trivial example* if $X$ is a locally compact separable metric space and $f : X \to X$ is a continuous proper map that admits an invariant finite measure such that for any compact set $K$ and $x \in X$ there exists $\widehat{n}(K, x) = \widehat{n} \in \mathbb{N}$ such that $f^{\widehat{n}}(x) \notin K$. The next result shows that if $X = \mathbb{R}$, it is not possible.

**Corollary 3.10.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous proper map. Then the following conditions are equivalents.*

$(i)$ *there exist $x \in \mathbb{R}$, $n_0$ in $\mathbb{N}$ and a compact set $K$ contained in $\mathbb{R}$ such that for $n > n_0$, $f^n(x) \in K$;*

$(ii)$ *there exists an invariant probability measure.*

*Proof.* By Example 3.9, we have that $(i)$ implies $(ii)$. So we are reduced to proving that $(ii)$ implies $(i)$.

Suppose that there exists an invariant probability measure. By Theorem G, there exist $\varphi \in C_c(\mathbb{R};\mathbb{R})$ with $0 \le \varphi \le 1$ and $x_0 \in X$ such that $\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$.

We have that $\operatorname{supp}\varphi$ is a compact set of $\mathbb{R}$, so there exist $c, d \in \mathbb{R}$ such that $\operatorname{supp}\varphi \subseteq [c, d]$. If $f(x) = x$ for some $x \in \mathbb{R}$, then $K = \{x\}$ is the compact set desired. Suppose that $f(x) \ne x$ for any $x \in \mathbb{R}$. By continuity, we have that $\mathbb{R}$ is either $A = \{x \in \mathbb{R} : f(x) > x\}$ or $B = \{x \in \mathbb{R} : f(x) < x\}$.

Using that $\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$, we see that $\{n : f^n(x_0) \in [c, d]\}$ is an infinite set.

(Case $I$). Suppose that $\mathbb{R} = A = \{x \in \mathbb{R} : f(x) > x\}$.

**Claim 1**: $f^i(x_0) \notin (d, +\infty)$ for all $i \in \mathbb{N}$. In fact, if $f^i(x_0) \in (d, +\infty)$ for some $i \in \mathbb{N}$, then $d < f^i(x_0) < f^{i+j}(x_0)$ for any $j \ge 0$ since $f(x) > x$ for any $x$, but $\{n : f^n(x_0) \in [c, d]\}$ is an infinite set.

**Claim 2:** For any $x \in \mathbb{R}$, if $f(x) \in [c, d]$ then $f^j(x) \notin (-\infty, c)$ for all $j \ge 1$. In fact, we have that $f^j(x) > \cdots > f^2(x) > f(x) \ge c$ for any $j \in \mathbb{N}$.

Using that $\{n : f^n(x_0) \in [c, d]\}$ is an infinite set. There exists $\widetilde{i} \in \mathbb{N}$ such that $f^{\widetilde{i}}(x_0) \in [c, d]$. By Claim 2, $f^j(f^{\widetilde{i}}(x_0)) \notin (-\infty, c)$ for all $j \ge 0$. By Claim 1, $f^j(f^{\widetilde{i}}(x_0)) \notin (d, +\infty)$ for all $j \ge 0$. We obtain that $f^{j+\widetilde{i}}(x_0) \in [c, d]$ for all $j \ge 0$.

(Case $II$). Suppose that $\mathbb{R} = B = \{x \in \mathbb{R} : f(x) < x\}$.

**Claim 3**: $f^i(x_0) \notin (-\infty, c)$ for all $i \in \mathbb{N}$. In fact, if $f^i(x_0) \in (-\infty, c)$ for some $i \in \mathbb{N}$, then $f^{i+j}(x_0) < f^i(x_0) < c$ for any $j \ge 0$ since $f(x) < x$ for any $x$, but $\{n : f^n(x_0) \in [c, d]\}$ is an infinite set.

**Claim 4:** For any $x \in \mathbb{R}$, if $f(x) \in [c, d]$ then $f^j(x) \notin (d, +\infty)$ for all $j \ge 1$. In fact, we have that $f^j(x) < \cdots < f^2(x) < f(x) \le d$ for any $j \in \mathbb{N}$.

Using that $\{n : f^n(x_0) \in [c, d]\}$ is an infinite set. There exists $\widetilde{i} \in \mathbb{N}$ such that $f^{\widetilde{i}}(x_0) \in [c, d]$. By Claim 3, $f^j(f^{\widetilde{i}}(x_0)) \notin (-\infty, c)$ for all $j \ge 0$. By Claim 4,

$f^j(f^{\tilde{i}}(x_0)) \notin (d, +\infty)$ for all $j \geq 0$. We obtain that $f^{j+\tilde{i}}(x_0) \in [c, d]$ for all $j \geq 0$, and we are done.

□

The special linear group $\mathrm{SL}(n, \mathbb{R})$ of degree $n$ over $\mathbb{R}$ is the set of $n \times n$ matrices with determinant 1, with the group operations of ordinary matrix multiplication and matrix inversion. We denote by $\mathrm{SL}(n, \mathbb{Z})$ the group of $n \times n$ matrices with integer entries and determinant equals 1. Note that $\mathrm{SL}(n, \mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}(n, \mathbb{R})$. Using Theorem G, we can prove the following result

**Corollary 3.11.** *For each $A \in \mathrm{SL}(n, \mathbb{R})$ there exist $B_A \mathrm{SL}(n, \mathbb{Z}) \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ and an observable $\varphi_A \in C_c(\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z}), \mathbb{R})$ such that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi_A(A^j B_A \mathrm{SL}(n, \mathbb{Z})) > 0$.*

In general, we can state this below result.

**Corollary 3.12.** *Suppose that $G$ is a locally compact second countable Hausdorff group, and $\Gamma$ is a lattice in $G$. For each $g \in G$ there exist $a_g\Gamma \in G/\Gamma$ and $\varphi_g \in C_c(G/\Gamma, \mathbb{R})$ such that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi_g(g^j a_g \Gamma) > 0$.*

In section 3.7, we give the proofs of Corollary 3.11 and Corollary 3.12.

## 3.3 Preliminary definitions and results

### 3.3.1 Locally compact Hausdorff spaces

Recall some definitions from Topology.

**Definition 3.13.** *Let $W$ be a topological space.*

*(i) A* neigborhood *of a point $p$ in $W$ is any open subset of $W$ which contains $p$.*

*(ii) $W$ is a* Hausdorff *space if the following condition is true: If $p \in W$, $q \in W$, and $p \neq q$ then $p$ has a neighborhood $U$ and $q$ has a neigborhood $V$ such that $U \cap V = \emptyset$.*

*(iii) $W$ is a* locally compact *if every point of $W$ has a neighborhood whose closure is compact.*

*(iv) Let $\varphi : W \to \mathbb{R}$ be a continuos function. The* support *of $\varphi$, denoted by* $\mathrm{supp}(\varphi)$*, is the closure of $\{y \in W : \varphi(y) \neq 0\}$, i.e.,* $\mathrm{supp}(\varphi) = \overline{\{y \in Y : \varphi(y) \neq 0\}}$.

*(v) Let $Z$ be a locally compact Hausdorff space. We denote the set of all continuous functions $\varphi : Z \to \mathbb{R}$ with* compact support *by $C_c(Z, \mathbb{R})$, or $C_c(Z)$.*

(vi) *A topological space* $W$ *is* $\sigma$-compact *if it is a countable union of compact sets.*

(vii) *A* second countable *space is a topological space whose topology has a countable base.*

(viii) *A topological space is called* separable *if it contains a countable, dense subset; that is, there exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.*

Let $W$ be a set equipped with a $\sigma$-algebra $\mathcal{A}$. A measure on $\mathcal{A}$ (or on $(W, \mathcal{A})$, or simply on $W$ if $\mathcal{A}$ is understood) is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) if $\{E_j\}_{j=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{A}$ then $\mu(\bigcup\limits_{j=1}^{\infty} E_j) = \sum\limits_{j=1}^{\infty} \mu(E_j)$.

Property (ii) is called *countable additivity.*

If $W$ is a set and $\mathcal{A}$ is a $\sigma$-algebra, $(W, \mathcal{A})$ is called a *measurable space* and the sets in $\mathcal{A}$ are called *measurable sets.* If $\mu$ is a measure on $(W, \mathcal{A})$, then $(W, \mathcal{A}, \mu)$ is called a *measure space.* If $W$ is a metric space, and $(W, \mathcal{A}, \mu)$ is a measure space, then $(W, \mathcal{A}, \mu)$ is called a *measure metric space.*

Let $Z$ be a locally compact Hausdorff space. We assume this terminolgies, $\mathcal{B}_Z$ will denote the Borel $\sigma$-algebra on $Z$, that is, the $\sigma$-algebra generated by open sets; measures on $\mathcal{B}_Z$ will be called Borel measures.

Let $\mu$ be a Borel measure on $Z$ and $E$ a Borel subset of $Z$. The measure $\mu$ is called *outer regular* on $E$ if

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open}\}$$

and *inner regular* on $E$ if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}.$$

If $\mu$ is outer and inner regular on all Borel sets, $\mu$ is called *regular.* A *Radon measure* on $Z$ is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Given a subset $A$ of $Z$, recall that the *indicator function* $1_A : Z \to \mathbb{R}$ is defined by setting $1_A(x) = 1$, for $x \in A$; and $1_A(x) = 0$, for $x \notin A$.

**Definition 3.14.** *[30] Let* $U \subseteq Z$ *be an open set and* $\varphi$ *be a function in* $C_c(Z)$. *If* $\varphi$ *satisfies* $0 \leq \varphi \leq 1$ *and* $\operatorname{supp}(\varphi) \subseteq U$ *we use the notation* $\varphi \prec U$.

The notation $K \prec \varphi$ will mean that $K$ is a compact subset of $X$, that $\varphi \in C_c(X)$, that $0 \leq \varphi(x) \leq 1$ for all $x \in X$, and that $\varphi(x) = 1$ for all $x \in K$.

The notation $K \prec \varphi \prec U$ will be used to indicate that $K \prec \varphi$ and $\varphi \prec U$ hold.

An useful result is the Urysohn Lemma:

**Lemma 3.15.** *[61, Lemma 2.12] Suppose $X$ is a locally compact Hausdorff space, $V$ is open in $X$, $K \subseteq V$, and $K$ is a compact. Then there exists a function $\varphi \in C_c(X)$, such that*

$$K \prec \varphi \prec V.$$

A linear functional $I$ on $C_c(Z)$ will be called *positive* if $I(\varphi) \geq 0$ whenever $\varphi \geq 0$ where $C_c(Z)$ is a normed space equipped with the uniform norm:

$$\|\varphi\| = \sup_{x \in Z} |\varphi(x)|.$$

Let $\mu$ be a Radon Measure, then $I : C_c(Z; \mathbb{R}) \to \mathbb{R}$ given by $I(\varphi) = \int \varphi d\mu$ is a positive linear functional. The Riesz Representation Theorem tell us that this reciprocal is true as follows.

**Theorem 3.16.** *[30, Theorem 7.2] If $I : C_c(Z; \mathbb{R}) \to \mathbb{R}$ be a positive linear functional. Then there is a unique Radon measure $\mu$ on $Z$ such that*

$$I(\varphi) = \int \varphi d\mu$$

*for all $\varphi \in C_c(Z)$. Moreover, $\mu$ satisfies*

$$\mu(U) = \sup\{\int \varphi d\mu : \varphi \in C_c(Z), \varphi \prec U\} \tag{3.1}$$

*for all open $U \subset Z$, and*

$$\mu(K) = \inf\{\int \varphi d\mu : \varphi \in C_c(Z), \varphi \geq 1_K\} \tag{3.2}$$

*for all compact $K \subset Z$.*

Now, recall the following results.

**Lemma 3.17.** *[30, Corollary 7.6] Let $Z$ be a locally compact Hausdorff space. If $Z$ is $\sigma$-compact, every Radon measure on $Z$ is regular.*

**Theorem 3.18.** *[30, Theorem 7.8] Let $Z$ be a second countable locally compact Hausdorff space. Then every Borel measure on $Z$ that is finite on compact sets is regular and hence Radon.*

In what follows we consider $X$ to be a locally compact separable metric space.

Let $\mu$ be a finite measure on $X$. Recall that for locally compact metric space, the condition of $\sigma$-compactness is equivalent to be second countable (that is equivalent to be a separable space). By Lemma 3.17 and Theorem 3.18, $\mu$ is a Radon measure if, only if, $\mu$ is a Borel measure. The following Lemma characterizes the bounded positive linear functionals of $C_c(X)$.

**Lemma 3.19.** *[40, p. 36] Let $I : C_c(X) \to \mathbb{R}$ be a positive linear functional, and let $\mu$ be the Radon Measure such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Then $I$ is a bounded operator if, and only if, $\mu(X) < \infty$.*

*Proof.* Suppose that $I$ is a bounded operator. By Theorem 3.16,

$$
\begin{aligned}
\mu(X) = \sup\{I(\varphi) : \varphi \in C_c(X), \varphi \prec 1_X = 1\} \leq \\
\sup\{|I(\varphi)| : \varphi \in C_c(X),\, 0 \leq \varphi \leq 1\} \leq \\
\sup_{\|\varphi\| \leq 1} |I(\varphi)| = \|I\| < \infty.
\end{aligned}
$$

Then $\mu(X) < \infty$.

Now, suppose that $\mu(X) < \infty$. Let $(\varphi_n)_n$ be a sequence in $C_c(X)$ such that uniformly converges to $\varphi$ in $C_c(X)$. By Dominated Convergence Theorem, $\lim_{n\to\infty} \int \varphi_n d\mu = \int \varphi d\mu$. So $\lim_{n\to\infty} I(\varphi_n) = \lim_{n\to\infty} \int \varphi_n d\mu = \int \varphi d\mu = I(\varphi)$, and then $I$ is a bounded operator. □

Let $C_0(X, \mathbb{R})$ be the set of functions that vanishes at infinity given by

$$C_0(X, \mathbb{R}) = \{\varphi \in C(X, \mathbb{R}) : \{x \in X : |\varphi(x)| \geq \varepsilon\} \text{ is compact for all } \varepsilon > 0\}$$

where $C(X, \mathbb{R})$ is the set of all continuous function from $X$ to $\mathbb{R}$. Note that $C_0(X)$ is the uniform closure of $C_c(X)$ (see [30, Proposition 4.35]).

Let $\mathcal{RM}$ to be the set of all finite Radon measures on $X$. By Lemma 3.19, we may consider the linear operator $\mathcal{I} : \mathcal{RM} \to C_c(X; \mathbb{R})'$ given by

$$\mathcal{I}(\mu)\varphi = \mathcal{I}_\mu \varphi = \int \varphi d\mu$$

for all $\varphi$ in $C_c(X)$ where $C_c(X; \mathbb{R})'$ is the dual of $C_c(X; \mathbb{R})$, the set of all bounded linear functional from $C_c(X; \mathbb{R})$ to $\mathbb{R}$. Then $\mathcal{I}_\mu : C_c(X; \mathbb{R}) \to \mathbb{R}$ is a bounded linear functional. But $C_0(X)$ is the uniform closure of $C_c(X)$, so there exists a continuous extension of $\mathcal{I}_\mu$ from $C_0(X)$ to $\mathbb{R}$ with norm $\|\mathcal{I}_\mu\|$. We abuse the notation and write this extension by $\mathcal{I}_\mu$ too. Then our linear operator $\mathcal{I}$ is defined from $\mathcal{RM}$ to $C_0(X; \mathbb{R})'$, i.e., $\mathcal{I} : \mathcal{RM} \to C_0(X; \mathbb{R})'$.

**Remark 3.20.** *Note that the Riesz Representation Theorem (Theorem 3.16) provides that $\mathcal{I}$ is an injective function.*

### 3.3.2 Duality

Now, we recall the standard results and definitions of functional analysis. Let $(Y, \|\cdot\|)$ be a normed space. We define the dual of $Y$ by

$$Y' = \{\varphi : Y \to \mathbb{R} \mid \varphi \text{ is a continuous and linear function}\}.$$

$$Y'' = \{\eta : Y' \to \mathbb{R} \mid \eta \text{ is a continuous and linear function}\}.$$

Define $J : Y \to Y''$ given by $J(y)(\varphi) = \varphi(y)$ for all $y$ in $Y$ and $\varphi$ in $Y'$.

**Definition 3.21.** *The weak topology in $Y$, $\sigma(Y, Y')$, is the topology spanned by continuous linear functions $\varphi$ in $Y'$.*

Recall that

$$\widetilde{V}[y, F, \varepsilon] = \bigcap_{f_i \in F} (f_i)^{-1}(B(f_i(y), \varepsilon))$$

is a basic open of $(Y, \sigma(Y, Y'))$ where $F = \{f_1, ..., f_n\}$ and $f_i \in Y'$ for $i \in \{1, ..., n\}$.

$$\widetilde{V}[f, \Gamma, \varepsilon] = \bigcap_{T_i \in \Gamma} (T_i)^{-1}(B(T_i(f), \varepsilon))$$

is a basic open of $(Y', \sigma(Y', Y''))$ where $\Gamma = \{T_1, ..., T_n\}$ and $T_i \in Y''$ for $i \in \{1, ..., n\}$.

$$V[f, \Phi, \varepsilon] = \bigcap_{i=1}^{n} (T_i)^{-1}(B(T_i(f), \varepsilon)) = \bigcap_{i=1}^{n} (J(y_i))^{-1}(B(J(y_i)(f), \varepsilon)) = \bigcap_{y_i \in \Phi} (J(y_i))^{-1}(B(f(y_i), \varepsilon))$$

is a basic open of $(Y', \sigma(Y', Y))$ where $T_i \in J(Y)$ for all $i$, $\Phi = \{y_1, ..., y_n\}$ and $y_1, ..., y_n \in Y$. We have that

$$V[f, \Phi, \varepsilon] = \bigcap_{y_i \in \Phi} (J(y_i))^{-1}(B(f(y_i), \varepsilon))$$

is a basic open of $(Y', \sigma(Y', Y))$ where $\Phi = \{y_1, ..., y_n\}$ and $y_1, ..., y_n \in Y$.

### 3.3.3 Locally compact separable metric spaces space versus Duality

Let $X$ be a locally compact separable metric space, recall that $C_0(X)$ is a normed space equipped with the uniform norm. We define the linear operator $\mathcal{I} : \mathcal{RM} \to C_0(X;\mathbb{R})'$ where $\mathcal{I}(\mu)\varphi = \mathcal{I}_\mu\varphi = \int \varphi d\mu$ for all $\varphi$ in $C_c(X;\mathbb{R})$ (note that this operator extends the previous operator $\mathcal{I}$, to see this recall the Bounded Linear Transformation Theorem, since $C_0(X;\mathbb{R})$ is the closure of $C_c(X;\mathbb{R})$), and $(C_0(X;\mathbb{R}), \|\cdot\|)$ is a Banach space. Set $(Y, \|\cdot\|) = (C_0(X;\mathbb{R}), \|\cdot\|)$ and $Y' = C_0(X;\mathbb{R})'$.

Consider $\tau_1 := \sigma(Y', Y)|_{\mathcal{I}(\mathcal{RM})}$, and take $\mathcal{I}\mu \in Y'$ for some $\mu \in \mathcal{RM}$, $\Phi = \{\varphi_1, ..., \varphi_n\} \subseteq Y$ and $\varepsilon > 0$. Then $V[\mathcal{I}\mu, \Phi, \varepsilon]$ is an open basic of $\sigma(Y', Y)$. We have that

$$V[\mathcal{I}\mu, \Phi, \varepsilon] = \bigcap_{\varphi_i \in \Phi} (J(\varphi_i))^{-1}(B(\mathcal{I}\mu(\varphi_i), \varepsilon))$$

is a basic open of $(Y', \sigma(Y', Y))$.

So,

$$V[\mathcal{I}\mu, \Phi, \varepsilon] = \{\eta \in Y' : |\eta(\varphi_i) - \textstyle\int \varphi_i d\mu| < \varepsilon \text{ for all } i \in \{1, ..., n\}\},$$

and

$$V[\mathcal{I}\mu, \Phi, \varepsilon] \cap \mathcal{I}(\mathcal{RM}) = \{I\nu \in \mathcal{I}(\mathcal{RM}) : |\textstyle\int \varphi_i d\nu - \int \varphi_i d\mu| < \varepsilon \text{ for all } i \in \{1, ..., n\}\}$$

is an open basic set of $\tau_1 = \sigma(Y', Y)|_{\mathcal{I}(\mathcal{RM})}$.

In what follows we consider $C_0(X;\mathbb{R})'$ with the $*$-weak topology. So the notation $\overline{A}$ for some $A \subseteq C_0(X;\mathbb{R})'$ will mean the closure of $A$ with respect to the $*$-weak topology of $C_0(X;\mathbb{R})' = Y'$.

**Lemma 3.22.** *$\mathcal{I}(\mathcal{RM})$ is $*$-weak closed in $C_0(X;\mathbb{R})' = Y'$.*

*Proof.* Let $T \in C_0(X;\mathbb{R})'$ be a bounded linear operator $T : C_0(X;\mathbb{R}) \to \mathbb{R}$ such that $T \in \overline{\mathcal{I}(\mathcal{RM})}$. Note that $\widehat{T} := T|_{C_c(X,\mathbb{R})} : C_c(X,\mathbb{R}) \to \mathbb{R}$ is a linear operator.

Let $\varphi$ in $C_c(X,\mathbb{R})$ such that $\varphi : X \to \mathbb{R}$ is a positive function, i.e., for all $x$ in $X$, $\varphi(x) \geq 0$. Then for all $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{RM}$ such that

$$\mathcal{I}\mu_n \in V[T, \{\varphi\}, \tfrac{1}{n}] = \{R \in C_0(X;\mathbb{R})' : |R(\varphi) - T(\varphi)| < \tfrac{1}{n}\}.$$

In other words,

$$\tfrac{1}{n} > |\mathcal{I}(\mu_n)(\varphi) - T(\varphi)| = |\textstyle\int \varphi d\mu_n - T(\varphi)|,$$

we obtain that $\lim_{n\to\infty} \int \varphi d\mu_n = T(\varphi)$. Using that $\int \varphi d\mu_n \geq 0$ for all $n$, we have that $T(\varphi) \geq 0$. But $\widehat{T} := T|_{C_c(X,\mathbb{R})}$, so $\widehat{T}(\varphi) = T(\varphi) \geq 0$, and $\widehat{T} : C_c(X,\mathbb{R}) \to \mathbb{R}$ is a positive linear functional on $C_c(X,\mathbb{R})$. By Riesz Representation Theorem (Theorem 3.16), there is a unique Radon measure $\mu$ on $X$ such that $\widehat{T}(\varphi) = \int \varphi d\mu$ for all $\varphi \in C_c(X)$. But $\widehat{T}$ is a positive bounded linear operator, by Lemma 3.19, $\mu(X) < \infty$, so $\mu \in \mathcal{RM}$. Now, $T$ is the continuous extension of $\widehat{T}$ on $C_0(X,\mathbb{R})$, and then $T = \mathcal{I}_\mu \in \mathcal{I}(\mathcal{RM})$. This completes the proof of Lemma. □

Let $\mathcal{RM}_1$ be the set of all probability Radon measure on $X$, so it is a subset of $\mathcal{RM}$.

**Remark 3.23.** *Note that* $\overline{\mathcal{I}(\mathcal{RM}_1)} \subseteq \mathcal{I}(\mathcal{RM})$.

The following classical property of locally compact separable metric spaces is required for proof of our results.

**Proposition 3.24.** *[28] Let* $(X,d)$ *be a locally compact separable metric space, then the space* $(C_c(X), \|\cdot\|)$ *is separable. Moreover,* $(C_0(X), \|\cdot\|)$ *is separable.*

Let $(X,d)$ be a locally compact separable metric space, so $C_0(X) = Y$ is a separable space, and then $(B_{Y'}, \sigma(Y',Y))$ is meatrizable where $B_{Y'} = \{\xi \in Y' : \|\xi\| \leq 1\}$. This implies that $(B_{Y'}, \sigma(Y',Y))$ is a compact metric space.

**Proposition 3.25.** $\overline{\mathcal{I}(\mathcal{RM}_1)}$ *is a compact metric space.*

*Proof.* Just note that $\mathcal{I}(\mathcal{RM}_1) \subseteq B_{Y'}$. In fact, let $\mu \in \mathcal{RM}_1$,

$$\|\mathcal{I}\mu\| = \sup_{\|\varphi\|\leq 1} |\mathcal{I}\mu(\varphi)| = \sup_{\|\varphi\|\leq 1} |\textstyle\int \varphi d\mu| \leq 1.$$

□

### 3.3.4 Dynamic of $f$

Here, we discuss the way to obtain invariant measures.

Let $g : X \to X$ be a measurable function, and let $\mu$ be a measure in $X$, we denote by $g_*\mu$ the measure defined by $g_*\mu(B) := \mu(g^{-1}B)$ for all measurable set $B$ in $X$.

Let $g : X \to X$ be a measurable function, and consider $\widehat{g_*} : \mathcal{I}(\mathcal{RM}) \to \mathcal{I}(\mathcal{RM})$ defined by $\widehat{g_*}(\mathcal{I}\eta) = \mathcal{I}(g_*\eta)$ for all $\eta$ in $\mathcal{RM}$.

Let $\nu$ be a measure in $\mathcal{RM}_1$, let $(\mu_n)_n$ be a sequence of probabilities given by

$$\mu_n = \tfrac{1}{n} \sum_{j=0}^{n-1} f_*^j \nu.$$

So $\mathcal{I}\mu_n$ in $\mathcal{I}(\mathcal{RM}_1)$ for all $n$ in $\mathbb{N}$, but $\overline{\mathcal{I}(\mathcal{RM}_1)}$ is a compact metric space, and $\mathcal{I}(\mathcal{RM}_1) \subseteq \overline{\mathcal{I}(\mathcal{RM}_1)}$. There exists a subsequence $(\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ of sequence $(\mathcal{I}\mu_n)_n$ that converges in $*$-weak topology to some $T$ in $\overline{\mathcal{I}(\mathcal{RM}_1)}$. But $\overline{\mathcal{I}(\mathcal{RM}_1)} \subseteq \mathcal{I}(\mathcal{RM})$, then there is $\mu$ in $\mathcal{RM}$ such that $\mathcal{I}\mu = T$.

Note that

$$\mathcal{I}(\mu_n)\varphi = \frac{1}{n}\sum_{j=0}^{n-1}\int \varphi \circ f^j d\nu = \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{I}f_*^j\nu(\varphi) \tag{3.3}$$

$$\widehat{f}_*(\mathcal{I}\mu_n)\varphi = \mathcal{I}(f_*\mu_n)\varphi = \mathcal{I}\mu_n(\varphi\circ f) = \tfrac{1}{n}\sum_{j=0}^{n-1}\int \varphi\circ f^{j+1}d\nu = \tfrac{1}{n}\sum_{j=1}^{n}\mathcal{I}f_*^j\nu(\varphi).$$

If $\widehat{f}_*|_{\overline{\mathcal{I}(\mathcal{RM}_1)}} : \overline{\mathcal{I}(\mathcal{RM}_1)} \to \overline{\mathcal{I}(\mathcal{RM}_1)}$ is a continuous function in $*$-weak topology, we showed that the limit of sequence of probability measures given as before provide us an invariant measure as follows.

**Lemma 3.26.** *Let $\nu$ be a probability measure in $\mathcal{RM}_1$, and $\mu$ be a finite measure such that $\mathcal{I}\mu$ is a point of accumalation of sequence $(\mathcal{I}\mu_n)_n$ where $\mu_n$ is given by $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\nu$. If $\widehat{f}_*|_{\overline{\mathcal{I}(\mathcal{RM}_1)}} : \overline{\mathcal{I}(\mathcal{RM}_1)} \to \overline{\mathcal{I}(\mathcal{RM}_1)}$ is a continuous function in $*$-weak topology, then $\mu$ is an $f$-invariant measure.*

*Proof.* We are going to show that $\mathcal{I}\mu = \widehat{f}_*(\mathcal{I}\mu)$. By hypothesis, $(\mathcal{I}\mu_{n_k})_k$ converges in $*$-weak topology to some $\mathcal{I}\mu$. Using that $\widehat{f}_*$ is a $*$-weak continuous function in a compact metric space $\overline{\mathcal{I}(\mathcal{RM}_1)}$, we obtain that $(\widehat{f}_*\mathcal{I}\mu_{n_k})_k$ converges in $*$-weak topology to $\widehat{f}_*\mathcal{I}\mu$. Then

$$\mathcal{I}(\mu_{n_k}) = \tfrac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{I}f_*^j\nu \to \mathcal{I}\mu$$

$$\widehat{f}_*\mathcal{I}\mu_{n_k} = \tfrac{1}{n_k}\sum_{j=1}^{n_k}\mathcal{I}f_*^j\nu \to \widehat{f}_*\mathcal{I}\mu$$

Let $V[\mathcal{I}\mu,\Phi,\varepsilon]$ be an arbitrary neighborhood of $\mathcal{I}\mu$ where $\Phi = \{\varphi_1,\cdots,\varphi_r\}$ such that $\varphi_i$ in $C_c(X)$ for any $i$ in $\{1,\cdots,r\}$. There exists $k_0$ in $\mathbb{N}$ such that for $k > k_0$

(a) $\frac{2}{n_k}\cdot \sup\limits_{i\in\{1,\cdots,r\}}|\varphi_i| < \frac{\varepsilon}{2}$.

(b) $|\mathcal{I}\mu_{n_k}(\varphi_i) - \mathcal{I}\mu(\varphi_i)| = |\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{I}f_*^j\nu(\varphi_i) - \mathcal{I}\mu(\varphi_i)| < \frac{\varepsilon}{2}$ for all $i$ in $\{1,\cdots,r\}$.

and note that for $k > k_0$
$|\mathcal{I}(\mu_{n_k})\varphi_i - \widehat{f}_*\mathcal{I}\mu_{n_k}\varphi_i| = |\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int\varphi_i\circ f^j d\nu - \frac{1}{n_k}\sum_{j=1}^{n_k}\int\varphi_i\circ f^j d\nu|$
$= \frac{1}{n_k}|\int\varphi_i d\nu - \int\varphi_i\circ f^{n_k}d\nu| \leq \frac{2}{n_k}\cdot\sup\limits_{i\in\{1,\cdots,r\}}|\varphi_i| < \frac{\varepsilon}{2}$,
we obtain that

$$|\mathcal{I}(\mu_{n_k})\varphi_i - \widehat{f}_*\mathcal{I}\mu_{n_k}\varphi_i| < \tfrac{\varepsilon}{2} \text{ for all } i \text{ in } \{1,\cdots,r\} \text{ if } k > k_0.$$

For $k > k_0$, and $i$ in $\{1\cdots,r\}$ fixed,

$$|\widehat{f}_*\mathcal{I}\mu_{n_k}\varphi_i - \mathcal{I}\mu(\varphi_i)| \le |\widehat{f}_*\mathcal{I}\mu_{n_k}\varphi_i - \mathcal{I}(\mu_{n_k})\varphi_i| + |\mathcal{I}(\mu_{n_k})\varphi_i - \mathcal{I}\mu(\varphi_i)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

So $\widehat{f}_*\mathcal{I}\mu_{n_k} \in V[\mathcal{I}\mu, \Phi, \varepsilon]$ for $k > k_0$, and then $(\widehat{f}_*\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ converges in $*$-weak topology to $\mathcal{I}\mu$. But $(\widehat{f}_*\mathcal{I}\mu_{n_k})_k$ converges in $*$-weak topology to $\widehat{f}_*\mathcal{I}\mu$. By unicity of limit, we obtain that $\widehat{f}_*\mathcal{I}\mu = \mathcal{I}\mu$.

Recall that $\widehat{f}_*(\mathcal{I}\mu) = \mathcal{I}f_*\mu$. We obtain that $\mathcal{I}f_*\mu = \mathcal{I}\mu$, but $\mathcal{I}$ is an injective function (see Remark 3.20), so $\mu = f_*\mu$, and we are done. □

**Remark 3.27.** *The Lemma 3.26 is an adaptation of the Lemma 2.2.4 in [73].*

## 3.4 Proof of Theorem G

Suppose that $f : X \to X$ is a continuous proper map on locally compact separable metric space. Theorem G states that the following conditions are equivalents.

- $(i)$ there exist $\varphi \in C_c(X;\mathbb{R})$ with $0 \le \varphi \le 1$ and $x_0 \in X$ such that the following number $\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$;
- $(ii)$ there exist $\varphi \in C_c(X;\mathbb{R})$ with $0 \le \varphi \le 1$ and $x_0 \in X$ such that the following number $\liminf_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x_0) > 0$;
- $(iii)$ there exist a probability measure $\nu$ on $X$ and an observable $\varphi \in C_0(X,\mathbb{R})$ such that $\liminf_n \frac{1}{n}\int\sum_{j=0}^{n-1} \varphi \circ f^j d\nu > 0$;
- $(iv)$ there exists an invariant probability measure.

To prove Theorem G, suppose that the following lemma is proved.

**Lemma I.** *Suppose that $f : X \to X$ is a continuous proper map. If there exist a probability measure $\nu$ on $X$ and an observable $\varphi$ in $C_0(X,\mathbb{R})$ such that the following number $\liminf_n \frac{1}{n}\sum_{j=0}^{n-1}\int \varphi \circ f^j d\nu > 0$, then there exists an invariant probability measure.*

Using Lemma I, we have that $(i) \to (ii) \to (iii) \to (iv)$. To deduce $(iii)$ from $(ii)$, take the Dirac measure of point $x_0$. So we are reduced to proving that $(iv)$ implies $(i)$.

Suppose that $(iv)$ holds, so there exists an invariant probability measure $\mu$. By Birkhoff's Theorem for invariant measures, for each $\varphi \in C_c(X;\mathbb{R})$ such that $0 \le \varphi \le 1$, there exists a function $\widetilde{\varphi}$ from a set of full measure $X_\varphi$ contained in $X$ to the real line $\mathbb{R}$ defined by

$$\widetilde{\varphi}(x) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$$

such that $\int \widetilde{\varphi} d\mu = \int \varphi d\mu$. We claim that for some $\psi \in C_c(X;\mathbb{R})$ such that $0 \leq \psi \leq 1$ there exists $x \in X_\psi$ such that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j(x) > 0$. Suppose the assertion of the claim is false. So for any $\varphi \in C_c(X)$ such that $0 \leq \varphi \leq 1$, we have that $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = 0$ for all $x \in X_\varphi$. Then $0 = \int 0 d\mu = \int \widetilde{\varphi} d\mu = \int \varphi d\mu$.

It implies that $\int \varphi d\mu = 0$ for all $\varphi$ in $C_c(X, \mathbb{R})$ such that $0 \leq \varphi \leq 1$. By Theorem 3.16, $\mu(X) = \sup\{\int \varphi d\mu : \varphi \in C_c(X), \varphi \prec 1_X = 1\}$, so $\mu(X) = 0$, but $\mu(X) = 1$. This completes the proof of Theorem G.

## 3.5 Proof of Lemma I

In what follows we consider $X$ to be a locally compact separable metric space.

To prove the Lemma I, we need of the following result.

**Theorem 3.28.** *Suppose that $f : X \to X$ is a measurable function such that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$. If there exist a probability measure $\nu$ on $X$ and an observable $\varphi$ in $C_0(X, \mathbb{R})$ such that $\liminf_n \frac{1}{n} \sum_{j=0}^{n-1} \int \varphi \circ f^j d\nu > 0$, then there exists an invariant probability measure.*

So, to obtain the Lemma I, we have to prove Theorem 3.28, and the following lemma about continuous proper map and continuous functions with compact support.

**Lemma 3.29.** *Suppose that $f : X \to X$ be a continuous proper map. Then $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$.*

*Proof.* Let $\psi : X \to \mathbb{R}$ be a continuous function with compact support, so $\psi \circ f$ is a continuous function. Note that

$$\{y \in X : \psi \circ f(y) \neq 0\} = f^{-1}(\{x \in X : \psi(x) \neq 0\}) \subseteq f^{-1}(\operatorname{supp} \psi)$$

So, $\operatorname{supp}(\psi \circ f) \subseteq \overline{f^{-1}(\operatorname{supp} \psi)} = f^{-1}(\operatorname{supp} \psi)$, by continuity of $f$ since $\operatorname{supp} \psi$ is closed in $X$. Now, $f$ is a proper map, and $\operatorname{supp} \psi$ is compact in $X$, so $f^{-1}(\operatorname{supp} \psi)$ is compact in $X$. But $\operatorname{supp}(\psi \circ f)$ is closed in $X$, it implies that $\operatorname{supp}(\psi \circ f)$ is compact in $X$, and we are done. □

From this moment, we are going to discuss the proof of Theorem 3.28. In the course of this, we will indicate to the reader the verification of some auxiliary results in section 3.3. Recall that $C_0(X)$ is a normed space equipped with the uniform norm, $\|\varphi\| = \sup_{x \in X} |\varphi(x)|$ for $\varphi \in C_0(X)$.

We define the linear operator $\mathcal{I} : \mathcal{RM} \to C_0(X;\mathbb{R})'$ by

$$\mathcal{I}(\mu)\varphi = \mathcal{I}_\mu \varphi = \int \varphi d\mu$$

for all $\varphi$ in $C_c(X;\mathbb{R})$ where $\mathcal{RM}$ is the set of all finite Radon measures on $X$, and $\mathcal{RM}_1$ is the set of all probability Radon measure on $X$.

Set $(Y, \|\cdot\|) = (C_0(X;\mathbb{R}), \|\cdot\|)$ and $Y' = C_0(X;\mathbb{R})'$.

Note that the Riesz Representation Theorem provides that $\mathcal{I}$ is an injective function.

In what follows we consider $C_0(X;\mathbb{R})'$ with the $*$-weak topology. So the notation $\overline{A}$ for some $A \subseteq C_0(X;\mathbb{R})'$ will mean the closure of $A$ with respect to the $*$-weak topology of $C_0(X;\mathbb{R})' = Y'$.

In our proof, an essential result is that the set $\mathcal{I}(\mathcal{RM})$ is $*$-weak closed in $C_0(X;\mathbb{R})' = Y'$ (see Lemma 3.22). This implies that $\overline{\mathcal{I}(\mathcal{RM}_1)} \subseteq \mathcal{I}(\mathcal{RM})$.

By Corollary 3.25, we conclude that $(\overline{\mathcal{I}(\mathcal{RM}_1)}), \sigma(Y', Y))$ is a compact metric space.

Let $g : X \to X$ be a measurable function, and let $\mu$ be a measure in $X$, we denote by $g_*\mu$ the measure defined by $g_*\mu(B) := \mu(g^{-1}B)$ for all measurable set $B$ in $X$.

Let $g : X \to X$ be a measurable function, and consider $\widehat{g_*} : \mathcal{I}(\mathcal{RM}) \to \mathcal{I}(\mathcal{RM})$ defined by $\widehat{g_*}(\mathcal{I}\eta) = \mathcal{I}(g_*\eta)$ for all $\eta$ in $\mathcal{RM}$.

We would like to apply the Lemma 3.26, so we have to show that $\widehat{f_*}|_{\overline{\mathcal{I}(\mathcal{RM}_1)}} : \overline{\mathcal{I}(\mathcal{RM}_1)} \to \overline{\mathcal{I}(\mathcal{RM}_1)}$ is a continuous function in $*$-weak topology. The following result give us this.

**Lemma 3.30.** *Let $X$ be a locally compact separable metric space, and $f : X \to X$ be a function such that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$. Then*

(i) *$\psi \circ f \in C_0(X, \mathbb{R})$ for all $\psi$ in $C_0(X, \mathbb{R})$*

(ii) *the map $\widehat{f_*} : \mathcal{I}(\mathcal{RM}) \to \mathcal{I}(\mathcal{RM})$ defined by $\widehat{f_*}(\mathcal{I}\eta) = \mathcal{I}(f_*\eta)$ is a continuous function in $*$-weak topology.*

(iii) *$\widehat{f_*}(\overline{\mathcal{I}(\mathcal{RM}_1)}) \subseteq \overline{\mathcal{I}(\mathcal{RM}_1)}$.*

Due to the technicality of the Lemma 3.30, we will give the proof of this at the end of this Section.

Using the Lemma 3.30, we have that $\widehat{f_*}|_{\overline{\mathcal{I}(\mathcal{RM}_1)}} : \overline{\mathcal{I}(\mathcal{RM}_1)} \to \overline{\mathcal{I}(\mathcal{RM}_1)}$ is a continuous function in $*$-weak topology, and then, by Lemma 3.26, for any $\nu$ probability measure in $\mathcal{RM}_1$, and $\mu$ be a finite measure such that $\mathcal{I}\mu$ is a point of accumalation of sequence $(\mathcal{I}\mu_n)_n$ where $\mu_n$ is given by $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\nu$, we have that $\mu$ is a $f$-invariant measure.

Now, we are going to verify certain conditions to provide that the limit of sequence $\mathcal{I}\mu_n$ given as above for some $\eta$ is not the null measure. The next result completes the proof of Lemma I.

**Lemma 3.31.** *Let $(X, d)$ be a locally compact separable metric space, and $f : X \to X$ be a function such that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$. If there exist a probability measure $\nu$ on $X$ and an observable $\varphi$ in $C_0(X, \mathbb{R})$ such that the following number $\liminf_n \frac{1}{n}\sum_{j=0}^{n-1} \int \varphi \circ f^j d\nu > 0$, then there exists an invariant probability measure.*

*Proof.* Suppose that there exist a probability measure $\nu$ on $X$ and an observable $\varphi$ in $C_0(X, \mathbb{R})$ such that $\liminf_n \frac{1}{n}\sum_{j=0}^{n-1} \int \varphi \circ f^j d\nu > 0$,

Let $\mu$ be a finite measure such that $\mathcal{I}\mu$ is a point of accumalation of sequence $(\mathcal{I}\mu_n)_n$ where $\mu_n$ is given by $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\eta$. So there exists a subsequence $(\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ of sequence $(\mathcal{I}\mu_n)_n$ that converges in $*$-weak topology to some $\mathcal{I}\mu$.

Suppose that $\mu$ is the null-measure, and consider the function $\varphi$ in $C_0(X, \mathbb{R})$. For the family of neighborhood of $\mathcal{I}(0) = 0$ given by $(V_\ell[0, \{\varphi\}, \frac{1}{\ell}])_{\ell\in\mathbb{N}}$, we have that for each $\ell$ fixed, there exists $k_\ell > 0$ such that for $k \geq k_\ell$,

$$\tfrac{1}{\ell} > |\mathcal{I}(\mu_{n_k})\varphi - \mathcal{I}(0)\varphi| = |\mathcal{I}(\mu_{n_k})\varphi| = |\tfrac{1}{n_k}\sum_{j=0}^{n_k-1} \int \varphi \circ f^j d\eta|, \text{ so}$$

$$\lim_{k\to\infty} \frac{1}{n_k}\sum_{j=0}^{n_k-1} \int \varphi \circ f^j d\eta = 0. \tag{3.4}$$

and note that

$$0 < \liminf_n \tfrac{1}{n}\sum_{j=0}^{n-1} \int \varphi \circ f^j d\nu \leq \lim_{k\to\infty} \tfrac{1}{n_k}\sum_{j=0}^{n_k-1} \int \varphi \circ f^j d\eta = 0.$$

This shows that $\mu$ is not the null-measure, and the Lemma is proved.

□

### 3.5.1 Proof of remark 3.4

Suppose that $\eta$ is a probability measure (not necessarily an invariant measure under $f$) such that $(X, \mathcal{A}, f, \eta)$ is a mixing system. We are going to show that there exists an invariant probability measure.

By definition of mixing system, for any bounded and measurable functions $\varphi, \psi : X \to \mathbb{R}$, we have that

$$\lim_{n\to\infty} \int \varphi \circ f^n \psi d\eta = \int \varphi d\eta \int \psi d\eta \tag{3.5}$$

Let $\mu$ be a finite measure such that $\mathcal{I}\mu$ is a point of accumalation of sequence $(\mathcal{I}\mu_n)_n$ where $\mu_n$ is given by $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \eta$. So there exists a subsequence $(\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ of sequence $(\mathcal{I}\mu_n)_n$ that converges in $*$-weak topology to some $\mathcal{I}\mu$.

Suppose that $\mu$ is the null-measure and fix $\varphi$ in $C_0(X, \mathbb{R})$. Consider the family of neighborhood of $\mathcal{I}(0) = 0$ given by $(V_\ell[0, \{\varphi\}, \frac{1}{\ell}])_{\ell\in\mathbb{N}}$.

Now, for each $\ell$ fixed, there exists $k_\ell > 0$ such that for $k \geq k_\ell$,

$\frac{1}{\ell} > |\mathcal{I}(\mu_{n_k})\varphi - \mathcal{I}(0)\varphi| = |\mathcal{I}(\mu_{n_k})\varphi| = |\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int \varphi \circ f^j d\eta|$, so

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \int \varphi \circ f^j d\eta = 0. \tag{3.6}$$

By equation (3.5)

$\lim_{n\to\infty} \int \varphi \circ f^n d\eta = \lim_{n\to\infty} \int \varphi \circ f^n \cdot 1_X d\eta = \eta(X) \int \varphi d\eta = \int \varphi d\eta$, then

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} \int \varphi \circ f^j d\eta = \int \varphi d\eta. \tag{3.7}$$

By equations 3.6 and 3.7, $\int \varphi d\eta = 0$ for all $\varphi$ in $C_0(X, \mathbb{R})$. By Theorem 3.16, $\eta(X) = \sup\{\int \varphi d\eta : \varphi \in C_c(X), \varphi \prec 1_X = 1\}$, so $\eta(X) = 0$, but $\eta(X) = 1$. This completes the proof of item $(i)$.

The following sequence of technical lemmas proves the Lemma 3.30.

### 3.5.2 Proof of Lemma 3.30

**Lemma 3.32.** *Suppose that $\psi \circ f \in C_c(X, \mathbb{R})$ for all $\psi \in C_c(X, \mathbb{R})$. Then $\psi \circ f \in C_0(X, \mathbb{R})$ for all $\psi$ in $C_0(X, \mathbb{R})$.*

*Proof.* Let $\psi$ in $C_0(X, \mathbb{R})$, using that $X$ is a locally compact space, we have that $C_c(X, \mathbb{R})$ is dense in $C_0(X, \mathbb{R})$, so there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of continuous functions with compact support that converges uniformly to $\psi$, and then $(\psi_n \circ f)_{n\in\mathbb{N}}$ converges uniformly to $\psi \circ f$. By hypothesis, $\psi_n \circ f \in C_c(X, \mathbb{R})$ for each $n$, so $\psi \circ f$ in $C_0(X, \mathbb{R})$, and we are done. □

**Lemma 3.33.** *[73, Lemma 2.2.1] Let $\mu$ be a finite measure in $X$, and $\phi : X \to \mathbb{R}$ be a bounded measurable function. Then*

$$\int \phi dg_*\mu = \int \phi \circ g d\mu. \tag{3.8}$$

*Proof.* Suppose that $\phi$ is a characteristic function of a mensurable set $B$ then the relation (3.8) means that $g_*\mu(B) = \mu(g^{-1}(B))$, that is true. By linearity of integral, (3.8) holds for any simple function. Finally, from the fact that every bounded measurable function can be uniformly approximated by simple fucntions, we are done. □

**Lemma 3.34.** *[73, Proposition 2.2.2] If $f : X \to X$ is a function such that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$, then the map $\widehat{f}_* : \mathcal{I}(\mathcal{RM}) \to \mathcal{I}(\mathcal{RM})$ defined by $\widehat{f}_*(\mathcal{I}\eta) = \mathcal{I}(f_*\eta)$ is a continuous function in $(\mathcal{I}(\mathcal{RM}), \tau_1)$.*

*Proof.* Let $\mathcal{I}\mu$ in $\mathcal{I}(\mathcal{RM})$ fixed, and let $V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon] \cap \mathcal{I}(\mathcal{RM})$ be an arbitrary neighborhood of $\widehat{f}_*(\mathcal{I}\mu)$ in $\mathcal{I}(\mathcal{RM})$ where $\Phi = \{\varphi_1, \cdots, \varphi_n\}$ is a finite family of $C_0(X, \mathbb{R})$.

Using that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$, by Lemma 3.32, we have that $\Psi = \{\varphi_1 \circ f, \cdots, \varphi_n \circ f\}$ is a finite family of $C_0(X, \mathbb{R})$ too. Note that

$$\widehat{f}_*(V[\mathcal{I}\mu, \Psi, \varepsilon] \cap \mathcal{I}(\mathcal{RM})) \subseteq V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon] \cap \mathcal{I}(\mathcal{RM})$$

In fact, let $\mathcal{I}\eta \in V[\mathcal{I}\mu, \Psi, \varepsilon]$, by definition

$$|\textstyle\int \varphi_i \circ f d\eta - \int \varphi_i \circ f d\mu| < \varepsilon \text{ for all } i \in \{1, ..., n\}.$$

By lemma 3.33, we obtain that $\varepsilon > |\int \varphi_i \circ f d\eta - \int \varphi_i \circ f d\mu| = |\int \varphi_i df_*\eta - \int \varphi_i df_*\mu|$ for all $i \in \{1, ..., n\}$, and then $\mathcal{I}(f_*\eta) \in V[\mathcal{I}(f_*\mu), \Phi, \varepsilon]$, but $\mathcal{I}(f_*\mu) = \widehat{f}_*(\mathcal{I}\mu)$ and $\mathcal{I}(f_*\eta) = \widehat{f}_*(\mathcal{I}\eta)$, so $\widehat{f}_*(\mathcal{I}\eta) \in V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon]$, and we are done. □

**Lemma 3.35.** $\widehat{f}_*(\overline{\mathcal{I}(\mathcal{RM}_1)}) \subseteq \overline{\mathcal{I}(\mathcal{RM}_1)}$

*Proof.* Let $T$ in $\overline{\mathcal{I}(\mathcal{RM}_1)}$, so there exists $\mu$ in $\mathcal{RM}$ such that $T = \mathcal{I}\mu$ (by remark 3.23), and let $V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon]$ be an arbitrary neighborhood of $\widehat{f}_*(\mathcal{I}\mu)$ in $C_0(X, \mathbb{R})'$ where $\Phi = \{\varphi_1, \cdots, \varphi_n\}$ is a finite family of $C_0(X, \mathbb{R})$.

Using that $\varphi \circ f \in C_c(X, \mathbb{R})$ for all $\varphi \in C_c(X, \mathbb{R})$, by Lemma 3.32, we have that $\Psi = \{\varphi_1 \circ f, \cdots, \varphi_n \circ f\}$ is a finite family of $C_0(X, \mathbb{R})$ too. Note that

$$\widehat{f}_*(V[\mathcal{I}\mu, \Psi, \varepsilon] \cap \mathcal{I}(\mathcal{RM}_1)) \subseteq V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon] \cap \mathcal{I}(\mathcal{RM}_1) \tag{3.9}$$

In fact, let $\mathcal{I}\eta \in V[\mathcal{I}\mu, \Psi, \varepsilon]$ with $\eta \in \mathcal{RM}_1$, by definition

$$|\textstyle\int \varphi_i \circ f d\eta - \int \varphi_i \circ f d\mu| < \varepsilon \text{ for all } i \in \{1, ..., n\}.$$

By lemma 3.33, we obtain that $\varepsilon > |\int \varphi_i \circ f d\eta - \int \varphi_i \circ f d\mu| = |\int \varphi_i df_*\eta - \int \varphi_i df_*\mu|$ for all $i \in \{1, ..., n\}$, and then $\mathcal{I}(f_*\eta) \in V[\mathcal{I}(f_*\mu), \Phi, \varepsilon]$, but $\mathcal{I}(f_*\mu) = \widehat{f}_*(\mathcal{I}\mu)$ and $\mathcal{I}(f_*\eta) = \widehat{f}_*(\mathcal{I}\eta)$, so $\widehat{f}_*(\mathcal{I}\eta) \in V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon]$.

Note that $\widehat{f}_*(\mathcal{I}(\mathcal{RM}_1)) \subseteq \mathcal{I}(\mathcal{RM}_1)$. First, observe that for $\nu \in \mathcal{RM}_1$, we have that $f_*\nu(X) = \nu(f^{-1}(X)) = \nu(X) = 1$, so $f_*\nu \in \mathcal{RM}_1$ if $\nu \in \mathcal{RM}_1$. Then for $\nu \in \mathcal{RM}_1$ we have that $\widehat{f}_*(\mathcal{I}\nu) = \mathcal{I}(f_*\nu)$, but $f_*\nu \in \mathcal{RM}_1$ if $\nu \in \mathcal{RM}_1$ (by Theorem 3.18), so $\widehat{f}_*(\mathcal{I}\nu) \in \mathcal{I}(\mathcal{RM}_1)$. Then $\widehat{f}_*(\mathcal{I}\eta) \in \mathcal{I}(\mathcal{RM}_1)$. This completes the proof of the inclusion (3.9).

But $T = \mathcal{I}\mu$ in $\overline{\mathcal{I}(\mathcal{RM}_1)}$, then $V[\mathcal{I}\mu, \Psi, \varepsilon] \cap \mathcal{I}(\mathcal{RM}_1) \neq \emptyset$, we obtain that

$$V[\widehat{f}_*(\mathcal{I}\mu), \Phi, \varepsilon] \cap \mathcal{I}(\mathcal{RM}_1) \neq \emptyset$$

So $\widehat{f}_*(T) = \widehat{f}_*(\mathcal{I}\mu) \in \overline{\mathcal{I}(\mathcal{RM}_1)}$, and we are done. This completes the proof of Lemma 3.30.

□

## 3.6 Proof of Theorem H

Let $X$ be a locally compact separable metric space, and $f : X \to X$ be a continuous function. A bounded operator $\mathcal{L} : C_0(X) \to C_0(X)$ is called *Perron-Frobenius-like operator* for $f$ if $\mathcal{L}(g) \geq 0$ whenever $g \geq 0$ for $g \in C_0(X)$, and $\mathcal{L}((g_1 \circ f)g_2) = g_1\mathcal{L}(g_2)$ for all $g_1, g_2 \in C_0(X)$. Now, we are going to prove the Theorem H, first we show that the Perron-Frobenius-like operator $\mathcal{L}$ is well defined as follows.

**Lemma 3.36.** *$\mathcal{L}$ is well defined, i.e., if $f$ is a continuous function and $g_1, g_2 \in C_0(X)$ then $(g_1 \circ f)g_2 \in C_0(X)$.*

*Proof.* First we prove that if $g_1 \in C_0(X)$ and $g_2 \in C_c(X)$, then $(g_1 \circ f)g_2 \in C_c(X)$. So,

$$\{x \in X : (g_1 \circ f)(x)g_2(x) \neq 0\} \subseteq \{x \in X : g_2(x) \neq 0\} \subseteq \operatorname{supp} g_2,$$

and then $\operatorname{supp}(g_1 \circ f)g_2 \subseteq \operatorname{supp} g_2$. This implies that $\operatorname{supp}(g_1 \circ f)g_2$ is a compact set, since $\operatorname{supp} g_2$ is a compact set. To finish suppose that $g_1, g_2$ in $C_0(X)$, and use that $C_c(X)$ is dense in $C_0(X)$. So, there exists a sequence $(h_n)_n$ in $C_c(X)$ that converges to $g_2$. By first part, we have that $(g_1 \circ f)h_n \in C_c(X)$ for all $n$ in $\mathbb{N}$. So $((g_1 \circ f)h_n)_n$ converges to $(g_1 \circ f)g_2$, and then $(g_1 \circ f)g_2 \in C_0(X)$, and we are done. □

Theorem H may be proved in much the same way as Theorem **??**. However, our proof makes no appeal to proper maps, and it forces us to explore the properties of Perron-Frobenius operators, topological properties of this spaces, and tools of Functional Analysis as follows.

*Proof of Theorem H.* Suppose that $X$ is a locally compact separable metric space, $f : X \to X$ is a continuous function, and $\mathcal{L} : C_0(X) \to C_0(X)$ is a Perron-Frobenius-like operator such that $\|\mathcal{L}\|$ is an eigenvalue of $\mathcal{L}$.

Recall that if $\mathcal{L} : C_0(X) \to C_0(X)$ is a Perron-Frobenius-like operator then

$$\mathcal{L}((g_1 \circ f)g_2) = g_1\mathcal{L}(g_2) \text{ for all } g_1, g_2 \in C_0(X) \text{ and}$$

$\mathcal{L}(g) \geq 0$ whenever $g \geq 0$. Note that if $\mathcal{L}$ is a Perron-Frobenius-like operator then $\alpha\mathcal{L}$ is a Perron-Frobenius-like operator for all positive real number $\alpha$.

So, without loss generality, $\mathcal{L}$ is a Perron-Frobenius-like operator such that $\|\mathcal{L}\| = 1$ is an eigenvalue of $\mathcal{L}$.

Consider the linear operator $\mathcal{I} : \mathcal{RM} \to C_0(X;\mathbb{R})'$ by

$$\mathcal{I}(\mu)\varphi = \mathcal{I}_\mu\varphi = \int \varphi d\mu$$

for all $\varphi$ in $C_c(X;\mathbb{R})$ where $\mathcal{RM}$ is the set of all finite Radon measures on $X$, and $\mathcal{RM}_1$ is the set of all probability Radon measure on $X$. Set $(Y, \|\cdot\|) = (C_0(X;\mathbb{R}), \|\cdot\|)$ and $Y' = C_0(X;\mathbb{R})'$.

In what follows we consider $C_0(X;\mathbb{R})'$ with the $*$-weak topology. So the notation $\overline{A}$ for some $A \subseteq C_0(X;\mathbb{R})'$ will mean the closure of $A$ with respect to the $*$-weak topology of $C_0(X;\mathbb{R})' = Y'$.

By boundedness of $\mathcal{L}$, the dual Banach operator $\mathcal{L}_* : C_0(X)' \to C_0(X)'$ given by $(\mathcal{L}_*T)(g) = T(\mathcal{L}g)$ for all $g \in C_0(X)$ is a bounded operator, and then $\mathcal{L}_*$ is a continuous operator in $*$-weak topology.

**Lemma 3.37.** $\mathcal{L}_*(\mathcal{I}(\mathcal{RM})) \subseteq \mathcal{I}(\mathcal{RM})$.

*Proof.* Let $\mu$ in $\mathcal{RM}$, by definition of $\mathcal{L}_*$, we note that $\mathcal{L}_*(\mathcal{I}\mu) \in C_0(X)'$, so $\mathcal{L}_*(\mathcal{I}\mu)$ is a bounded linear operator, moreover, $\mathcal{L}_*(\mathcal{I}\mu)$ is a positive operator, i.e., $\mathcal{L}_*(\mathcal{I}\mu)(\varphi) \geq 0$ whenever $\varphi \geq 0$ for $\varphi$ in $C_c(X)$. Note that $\mathcal{L}_*(\mathcal{I}\mu)(\varphi) = (\mathcal{I}\mu)(\mathcal{L}(\varphi)) = \int \mathcal{L}(\varphi)d\mu$. Now, $\mathcal{L} : C_0(X) \to C_0(X)$ is a Perron-Frobenius-like operator, so $\mathcal{L}(\varphi) \geq 0$, and then $\int \mathcal{L}(\varphi)d\mu \geq 0$. This implies that $\mathcal{L}_*(\mathcal{I}\mu)$ is a bounded positive operator, by The Riesz Representation Theorem (Theorem 3.16), there exists a finite measure $\widetilde{\mu}$ in $\mathcal{RM}$ such that

$$\int \varphi d\widetilde{\mu} = \mathcal{L}_*(\mathcal{I}\mu)(\varphi) = \int \mathcal{L}(\varphi)d\mu \text{ for all } \varphi \in C_c(X).$$

□

Recall that $\overline{\mathcal{I}(\mathcal{R}_1\mathcal{M})}$ is a compact metric space in $*$-weak topology.

Let $\nu$ be a measure in $\mathcal{R}_1\mathcal{M}$, and consider the sequence

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}_*^j(\mathcal{I}\nu).$$

By Lemma 3.37, for each $n$ in $\mathbb{N}$, there exists $\mu_n$ in $\mathcal{RM}$ such that $\mathcal{I}(\mu_n) = \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^j(\mathcal{I}\nu)$. We claim that $\mathcal{I}(\mu_n) \in \mathcal{I}(\mathcal{R}_1 M)$. In fact, by Theorem 3.16,

$$\mu_n(X) = \sup\{\int \varphi d\mu_n = \mathcal{I}\mu_n(\varphi) : \varphi \in C_c(X), \varphi \prec 1_X = 1\} \leq$$
$$\sup\{|\mathcal{I}\mu_n(\varphi)| : \varphi \in C_c(X),\, 0 \leq \varphi \leq 1\} \leq$$
$$\sup_{\|\varphi\|\leq 1} |\mathcal{I}\mu_n(\varphi)| = \|\mathcal{I}\mu_n\| = \|\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^j(\mathcal{I}\nu)\| \leq$$
$$\frac{1}{n}\sum_{j=0}^{n-1}\|\mathcal{L}_*^j\|\cdot|(\mathcal{I}\nu)| \leq \frac{1}{n}\sum_{j=0}^{n-1} 1\cdot|(\mathcal{I}\nu)| = \frac{1}{n}n|(\mathcal{I}\nu)| \leq 1$$

,

where $|(\mathcal{I}\nu)| = \sup_{\|\varphi\|=1}\int \varphi d\nu \leq \sup_{\|\varphi\|=1}\int d\nu\|\varphi\| = 1$.

Then $\mu_n(X) \leq 1$ for all $n$ in $\mathbb{N}$. So $\mu_n$ in $\mathcal{R}_1\mathcal{M}$ for all $n$ in $\mathbb{N}$, but $\overline{\mathcal{I}(\mathcal{R}_1\mathcal{M})}$ is a compact metric space. There exists a subsequence $(\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ of sequence $(\mathcal{I}\mu_n)_n$ that converges in $*$-weak topology to some $T$ in $\overline{\mathcal{I}(\mathcal{R}_1\mathcal{M})}$. But $\overline{\mathcal{I}(\mathcal{R}_1\mathcal{M})} \subseteq \overline{\mathcal{I}(\mathcal{RM})} = \mathcal{I}(\mathcal{RM})$ (see Lemma 3.22). So there exists $\mu$ in $\mathcal{RM}$ such that $T = \mathcal{I}\mu$.

Using that $1 = \|\mathcal{L}\|$, and $\mathcal{L}_*$ is a continuous function on $\mathcal{I}(\mathcal{RM})$, we are going to show that the limit of sequence of measures given as before provide us that $\mathcal{L}_*\mathcal{I}\mu = \mathcal{I}\mu$.

Note that

$$\mathcal{I}\mu_n(\varphi) = \frac{1}{n}\sum_{j=0}^{n-1}\int \mathcal{L}^j(\varphi)d\nu = \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^j\mathcal{I}\nu(\varphi), \tag{3.10}$$

and

$$\mathcal{L}_*(\mathcal{I}\mu_n)\varphi = (\mathcal{L}_*\mathcal{I}\mu_n)\varphi = \mathcal{I}\mu_n(\mathcal{L}\varphi) =$$
$$\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^{j+1}\nu(\varphi) = \frac{1}{n}\sum_{j=1}^{n}\mathcal{L}_*^j\nu(\varphi) = \frac{1}{n}\sum_{j=1}^{n}\int \mathcal{L}^j(\varphi)d\nu.$$

**Lemma 3.38.** *Let $\nu$ be a probability measure in $\mathcal{R}_1\mathcal{M}$, and $\mu$ be a finite measure such that $\mathcal{I}\mu$ is a point of accumalation of sequence $(\mathcal{I}\mu_n)_n$ where $\mathcal{I}\mu_n$ is given by $\mathcal{I}\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^j\mathcal{I}\nu$. Then $\mathcal{L}_*\mathcal{I}\mu = \mathcal{I}\mu$.*

*Proof.* We are going to show that $\mathcal{L}_*\mathcal{I}\mu = \mathcal{I}\mu$. By hypothesis, $(\mathcal{I}\mu_{n_k})_k$ converges in $*$-weak topology to some $\mathcal{I}\mu$. Using that $\mathcal{L}_*$ is a $*$-weak continuous function in $\mathcal{I}(\mathcal{RM})$, we obtain that $(\mathcal{L}_*\mathcal{I}\mu_{n_k})_k$ converges in $*$-weak topology to $\mathcal{L}_*\mathcal{I}\mu$. Then

$$\mathcal{I}\mu_{n_k} = \frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_*^j\mathcal{I}\nu \to \mathcal{I}\mu$$
$$\mathcal{L}_*\mathcal{I}\mu_{n_k} = \frac{1}{n_k}\sum_{j=1}^{n_k}\mathcal{L}_*^j\mathcal{I}\nu \to \mathcal{L}_*\mathcal{I}\mu$$

Let $V[\mathcal{I}\mu, \Phi, \varepsilon]$ be an arbitrary neighborhood of $\mathcal{I}\mu$ where $\Phi = \{\varphi_1, \cdots, \varphi_r\}$ such that $\varphi_i$ in $C_0(X)$ for any $i$ in $\{1, \cdots, r\}$. There exists $k_0$ in $\mathbb{N}$ such that for $k > k_0$

(a) $\frac{2}{n_k} \cdot \sup\limits_{i\in\{1,\cdots,r\}} |\varphi_i| < \frac{\varepsilon}{2}$.

(b) $|\mathcal{I}\mu_{n_k}(\varphi_i) - \mathcal{I}\mu(\varphi_i)| = |\frac{1}{n_k}\sum\limits_{j=0}^{n_k-1} \mathcal{L}_*^j\mathcal{I}\nu(\varphi_i) - \mathcal{I}\mu(\varphi_i)| < \frac{\varepsilon}{2}$ for all $i$ in $\{1, \cdots, r\}$.

and note that for $k > k_0$
$|(\mathcal{I}\mu_{n_k})\varphi_i - \mathcal{L}_*\mathcal{I}\mu_{n_k}\varphi_i| = |\frac{1}{n_k}\sum\limits_{j=0}^{n_k-1}\int \mathcal{L}^j(\varphi)d\nu - \frac{1}{n_k}\sum\limits_{j=1}^{n_k}\int \mathcal{L}^j(\varphi)d\nu|$
$= \frac{1}{n_k}|\int \varphi_i d\nu - \int \mathcal{L}^{n_k}(\varphi_i)d\nu| \leq \frac{1}{n_k}\cdot \sup\limits_{i\in\{1,\cdots,r\}}|\varphi_i| + \frac{1}{n_k}\cdot \sup\limits_{i\in\{1,\cdots,r\}}|\varphi_i| < \frac{\varepsilon}{2}$
we obtain that

$$|\mathcal{I}(\mu_{n_k})\varphi_i - \mathcal{L}_*\mathcal{I}\mu_{n_k}\varphi_i| < \tfrac{\varepsilon}{2} \text{ for all } i \text{ in } \{1, \cdots, r\} \text{ if } k > k_0.$$

For $k > k_0$, and $i$ in $\{1 \cdots, r\}$ fixed,

$$|\mathcal{L}_*\mathcal{I}\mu_{n_k}\varphi_i - \mathcal{I}\mu(\varphi_i)| \leq |\mathcal{L}_*\mathcal{I}\mu_{n_k}\varphi_i - \mathcal{I}(\mu_{n_k})\varphi_i| + |\mathcal{I}(\mu_{n_k})\varphi_i - \mathcal{I}\mu(\varphi_i)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

So $\mathcal{L}_*\mathcal{I}\mu_{n_k} \in V[\mathcal{I}\mu, \Phi, \varepsilon]$ for $k > k_0$, and then $(\mathcal{L}_*\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ converges in $*$-weak topology to $\mathcal{I}\mu$. But $(\mathcal{L}_*\mu_{n_k})_k$ converges in $*$-weak topology to $\mathcal{L}_*\mathcal{I}\mu$. By unicity of limit, we obtain that $\mathcal{L}_*\mathcal{I}\mu = \mathcal{I}\mu$. □

But 1 is an eigenvalue of $\mathcal{L}$, so there exists $h \neq 0$ in $C_0(X)$ such that $\mathcal{L}h = h$. Let $\widehat{y}$ in $X$ such that $h(\widehat{y}) \neq 0$. Consider $\nu = \delta_{\widehat{y}}$ the Dirac measure of point $\widehat{y}$.

Let $(\mathcal{I}\mu_n)_n$ be a sequence given by

$$\mathcal{I}\mu_n = \tfrac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_*^j\mathcal{I}\nu.$$

So $\mu_n$ in $\mathcal{R}_1\mathcal{M}$ for all $n$ in $\mathbb{N}$, but $\overline{\mathcal{I}(\mathcal{R}_1\mathcal{M})}$ is a compact metric space. There exists a subsequence $(\mathcal{I}\mu_{n_k})_{k\in\mathbb{N}}$ of sequence $(\mathcal{I}\mu_n)_n$ that converges in $*$-weak topology to some $\mathcal{I}\mu$ where $\mu$ in $\mathcal{RM}$, and by Lemma 3.38, $\mathcal{L}_*\mathcal{I}\mu = \mathcal{I}\mu$.

We claim that $\mu$ is not the null-measure. In fact, suppose that $\mu = 0$, so for every $\varphi$ in $C_0(X)$ we have that

$$\frac{1}{n_k}\sum_{j=0}^{n_k-1}\mathcal{L}_*^j\nu(\varphi)=\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int\mathcal{L}^j(\varphi)d\nu\to 0.$$

In particular, for $\varphi=h=\mathcal{L}h$,

$$\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int\mathcal{L}^jhd\nu=\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int hd\nu\to 0,$$

and then $0=\int hd\nu=\int hd\delta_{\widehat{y}}=h(\widehat{y})\neq 0$. This contradiction proves that $\mu$ is not the null measure.

We are going to verify $\eta=h\mu$ is an invariant measure where $\mathcal{L}h=h$ and $\mathcal{L}_*\mathcal{I}\mu=\mathcal{I}\mu$.

Recall that for any $g_1,g_2\in C_0(X)$, one has $\mathcal{L}((g_1\circ f)g_2)=g_1\mathcal{L}g_2$. Then for all $g:X\to\mathbb{R}$ in $C_0(X)$,

$$\mathcal{I}f_*\eta(g)=\int gdf_*\eta=\int(g\circ f)d\eta=\int(g\circ f)hd\mu=\mathcal{I}\mu((g\circ f)h)=\mathcal{L}_*\mathcal{I}\mu((g\circ f)h)=$$
$$\mathcal{I}\mu(\mathcal{L}((g\circ f)h))=\int\mathcal{L}((g\circ f)h)d\mu=\int g\mathcal{L}hd\mu=\int ghd\mu=\int gd\eta=\mathcal{I}\eta(g).$$

We obtain that $\mathcal{I}f_*\eta=\mathcal{I}\eta$, and by injectivity of $\mathcal{I}$, $f_*\eta=\eta$. This proves that $\eta$ is an invariant measure, and completes the proof of Theorem H. □

## 3.7 Topological Groups

We need of some theory to introduce the Haar measures of a locally compact Hausdorff Group. Let $G$ be a group endowed endowed with a topology $\tau$ where $e$ is the identity element of group $G$. $(G,\tau)$ is said to be a *topological group* if the multiplication map $\mathcal{P}:G\times G\to G$ defined by $(g,h)\mapsto gh$ and the inversion map $\iota:G\to G$ defined by $g\mapsto g^{-1}$ are continuous when $G\times G$ carries the product topology. The left translation defined by $g\in G$ is the map $L_g:G\to G$, $L_g(h)=gh$. Similarly, the right translation defined by $g\in G$ is the map $R_g:G\to G$, $R_g(h)=hg$. Note that the continuity of $\mathcal{P}$ ensures the continuity of $L_g$ and $R_g$ for any $g\in G$. Moreover, $L_g:G\to G$ and $R_g:G\to G$ are homeomorphism for any $g\in G$. In fact, first note that $L_g$ is one-to-one. Suppose that $L_g(h)=L_g(w)$ for some $h,w\in G$, so $gh=gw$, and then $h=w$. Now, $L_g$ is a surjective map. Take $w\in G$, but $L_g(g^{-1}w)=w$. We showed that $L_g$ is a bijective function.

Note that $T_g:G\to G$ given by $h\mapsto g^{-1}h$ is the inverse map of $L_g$. Observe that $T_gL_g(h)=T_g(gh)=g^{-1}(gh)=h$, and $L_gT_g(h)=L_g(g^{-1}h)=g(g^{-1}h)=h$. Actually, $T_g=L_{g^{-1}}$, and then $T_g$ is a continuous function. This proves that $L_g$ is a homeomorphism for any $g\in G$.

If $G$ is a topological group and $H$ is a subgroup of $G$, we equip the set of cosets $G/H$ with the *quotient topology*, i.e., $U \subseteq G/H$ is open if, and only if, $\pi^{-1}(U)$ is open in $G$ where $\pi : G \to G/H$, $g \mapsto gH$, then $\pi$ is a continuous function.

Let $W$ be a topological space.

($i$) A *neigborhood* of a point $p$ in $W$ is any open subset of $W$ which contains $p$.

($ii$) $W$ is a *Hausdorff* space if the following condition is true: If $p \in W$, $q \in W$, and $p \neq q$ then $p$ has a neighborhood $U$ and $q$ has a neigborhood $V$ such that $U \cap V = \emptyset$.

($iii$) $W$ is a *locally compact* if every point of $W$ has a neighborhood whose closure is compact.

($iv$) a *second countable* space is a topological space whose topology has a countable base.

**Lemma 3.39.** *Let $G$ to be a topological group and $H$ to be a subgroup of $G$. Then*

*($i$) $\pi$ is an open map;*

*($ii$) If $G$ is a locally compact space then $G/H$ is a locally compact space;*

*($iii$) If $G$ is a second countable space then $G/H$ is a second countable space;*

*($iv$) $\widehat{L_a} : G/H \to G/H$, $gH \mapsto agH$ is a homeomorphism for each $a \in G$.*

*Proof.* ($i$) Let $U$ to be an open set of $G$, and note that $\pi^{-1}(\pi(U)) = \bigcup_{h \in H} (R_h)^{-1}(U)$ (this implies that $\pi^{-1}(\pi(U))$ is an open set in $G$ since it is an union of open sets, and then $\pi(U)$ is an open set in $G/H$). In fact, note that if $g \in \pi^{-1}(\pi(U))$, then $\pi(g) \in \pi(U)$. There exists $x \in U$ such that $\pi(x) = \pi(g)$, that is $xH = gH$, so $xh_0 = gh_1$ for some $h_0, h_1 \in H$ and then $x = gh_2 = R_{h_2}(g)$ where $h_2 = h_1(h_0)^{-1} \in H$. Using that $x \in U$, we see that $R_{h_2}(g) \in U$, we obtain that $g \in (R_{h_2})^{-1}(U) \subseteq \bigcup_{h \in H} (R_h)^{-1}(U)$.

Now, take $y \in \bigcup_{h \in H} (R_h)^{-1}(U)$, by definition, there exists $h \in H$ such that $R_h(y) = yh \in U$, and then $yhH = \pi(yh) \in \pi(U)$. Note that $\pi(y) = yH = yhH = \pi(yh) \in \pi(U)$, so $y \in \pi^{-1}(\pi(U))$, this completes the proof of item ($i$).

($ii$) Suppose that $G$ is a locally compact space. Take $gH \in G/H$, by hyphoteses, there exist a compact set $K$ in $G$ and open set $U$ of $G$ such that $g \in U \subseteq K$, and then $\pi(g) = gH \in \pi(U) \subseteq \pi(K)$. By continuity of $\pi$, $\pi(K)$ is a compact set in $G/H$, and by openess of $\pi$, $\pi(U)$ is an open set, this completes the proof of item ($ii$).

($iii$) Suppose that $G$ is a second countable space, so there exists a countable basis $\{U_n : n \in \mathbb{N}\}$ for $G$, using that $\pi$ is an open map, we have that $\pi(U_n)$ is an open set in $G/H$. We claim that $\{\pi(U_n) : n \in \mathbb{N}\}$ is a countable basis of $G/H$. In fact, take $V$ to be an open set of $G/H$, so $(\pi)^{-1}(V)$ is an open set of $G$, so $(\pi)^{-1}(V) = \bigcup_{n\in I} U_n$ for some $I \subseteq \mathbb{N}$. Note that

$$\pi((\pi)^{-1}(V)) = \pi(\bigcup_{n\in I} U_n) \subseteq \bigcup_{n\in I} \pi(U_n),$$

using that $\pi$ is a surjective function, we see that $\pi((\pi)^{-1}(V)) = V$, and then $V \subseteq \bigcup_{n\in I} \pi(U_n)$. We claim that $V = \bigcup_{n\in I} \pi(U_n)$. Take $x \in \bigcup_{n\in I} \pi(U_n)$, there exists $y \in U_n$ such that $\pi(y) = x$ for some $n \in \mathbb{N}$. But $(\pi)^{-1}(V) = \bigcup_{n\in I} U_n$, then $x = \pi(y) \in V$. This completes the proof of item $(iii)$.

$(iv)$ Fix $a \in G$, consider $\widehat{L_a} : G/H \to G/H$, $gH \mapsto agH$. $\widehat{L_a}$ is an injective function. In fact, suppose that $\widehat{L_a}(gH) = \widehat{L_a}(bH)$, so $agH = abH$, there exist $h_0, h_1 \in H$ such that $agh_0 = abh_1$, and then $gh_0 = bh_1$. This implies that $gH = bH$. So $\widehat{L_a}$ is an injective function.

$\widehat{L_a}$ is an onto map. Just note that for any $gH \in G/H$, we have that $\widehat{L_a}(a^{-1}gH) = gH$.

Observe that $\widehat{L_{a^{-1}}} : G/H \to G/H$, $gH \mapsto a^{-1}gH$ is the inverse of $\widehat{L_a}$.

$\widehat{L_a}$ is a continuous function. Let $U$ to be an open set of $G/H$. We claim that $(\widehat{L_a})^{-1}(U)$ is an open set of $G/H$.

We have to show that $(\pi)^{-1}((\widehat{L_a})^{-1}(U))$ is an open set of $G$. Note that

$(\pi)^{-1}((\widehat{L_a})^{-1}(U)) =$
$\{x \in G : \pi(x) \in (\widehat{L_a})^{-1}(U)\} =$
$\{x \in G : \widehat{L_a}(\pi(x)) \in U\} =$
$\{x \in G : \widehat{L_a}(xH) \in U\} =$
$\{x \in G : axH \in U\} =$
$\{x \in G : \pi(ax) \in U\} =$
$\{x \in G : \pi(L_a(x)) \in U\} =$
$\{x \in G : \pi \circ L_a(x) \in U\} =$
$(\pi \circ L_a)^{-1}(U)$

Since $\pi$ and $L_a$ are continuous functions, we have that $\pi \circ L_a$ is a continuous function, and then $(\pi \circ L_a)^{-1}(U)$ is an open set in $G$. This shows that $\widehat{L_a}$ is a continuous function, so $\widehat{L_{a^{-1}}}$ is also a continuous function, and then $\widehat{L_a}$ is a homeomorphism. This completes the proof of item $(iv)$. □

**Lemma 3.40.** *Let $G$ to be a topological group and $H$ to be a subgroup of $G$. Then the following conditions are equivalents.*

*(i) if $eH \neq yH$ with $e, y \in G$, then there exist open neighborhoods $U_e, U_y$ of $e, y$ such that $U_e \cap U_yH = \emptyset$.*

*(ii)* *For all* $x, w \in G$ *such that* $xH \neq wH$ *there exist open neighborhoods* $U_x, U_w$ *of* $x, w$ *such that* $U_x \cap U_w H = \emptyset$.

*(iii)* $G/H$ *is a Hausdorff space.*

*Proof.* We are going to show that $(i)$ implies $(ii)$. Suppose that $xH \neq wH$ with $x, w \in G$, we claim that $eH \neq x^{-1}wH$. In fact, suppose that $eH = x^{-1}wH$, there exists $h \in H$ such that $e = x^{-1}wh$, and then $x = wh$. This implies that $xH = wH$, but it is a contradiction.

We have that $eH \neq x^{-1}wH$, taking $y = x^{-1}w$. By hypotheses, there exist open neighborhoods $U_e, U_y$ of $e, y$ such that $U_e \cap U_y H = \emptyset$.

Observe that $xU_e$ and $xU_y$ are open sets since $L_x : G \to G$ is a homeomorphism. Note that $x \in xU_e$ (just note that $x = xe \in xU_e$) and $w \in xU_y$ (just note that $w = x(x^{-1}w) \in xU_y$). We claim that $xU_e \cap xU_y H = \emptyset$.

In fact, suppose that $z \in xU_e \cap xU_y H$, so $z = xu = xb$ with $u \in U_e$ and $b \in U_y H$, so $x = b \in U_e \cap U_y H$, but $U_e \cap U_y H = \emptyset$. We deduced $(ii)$ from $(i)$.

Suppose that $(ii)$ holds, and take $xH \neq wH$ with $x, w \in G$. By item $(ii)$, there exist open neighborhoods $U_x, U_w$ of $x, w$ such that $U_x \cap U_w H = \emptyset$. Since $\pi$ is an open map, we have that $\pi(U_x), \pi(U_w)$ are open sets of $G/H$.

We claim that $\pi(U_x) \cap \pi(U_w) = \emptyset$. Suppose that there exists $z \in \pi(U_x) \cap \pi(U_w)$, then for some $u \in U_x$ and $v \in U_w$ we have that $z = \pi(u) = \pi(v)$. This implies that $uH = vH$, so there exists $h \in H$ such that $u = vh$. Note that $u \in U_x$ and $vh \in U_w H$, this implies that $u = vh \in U_x \cap U_w H = \emptyset$. We obtain that $\pi(U_x) \cap \pi(U_w) = \emptyset$, so $G/H$ is a Hausdorff space.

Now, suppose that $G/H$ is a Hausdorff space, and take $eH \neq yH$ with $e, y \in G$. There exist open neighborhoods $U$, $V$ of $eH, yH$ such that $U \cap V = \emptyset$. This implies that $\pi^{-1}(U) \cap \pi^{-1}V = \emptyset$ with $e \in \pi^{-1}(U)$ and $y \in \pi^{-1}(V)$, and then $\pi^{-1}(U) \cap \pi^{-1}(V)H = \emptyset$. This completes the proof of Lemma 3.40

□

We recall this technical Lemma.

**Lemma 3.41.** *Let* $G$ *to be a topological group. Then for any open neighbourhood* $U$ *of* $e$, *there exists an open neighbourhood* $V$ *of* $e$ *such that* $V = V^{-1}$ *and* $V^2 \subseteq U$.

*Proof.* Let $V = U_1 \cap U_2 \cap U_1^{-1} \cap U_2^{-1}$, where $U_1 \times U_2 \subseteq \mathcal{P}^{-1}(U)$ is a neighbourhood of $(e, e)$, $\mathcal{P} : G \times G \to G$ is the multiplication in $G$. □

**Lemma 3.42.** *Let* $G$ *to be a Hausdorff group and* $\Gamma$ *to be a closed subgroup of* $G$. *Then* $G/\Gamma$ *is a Hausdorff space.*

*Proof.* Since $L_g : G \to G$ and $R_g : G \to G$ are homeomorphisms for all $g \in G$, $g\Gamma$ and $\Gamma g$ are closed sets in $G$ for all $g \in G$. In view to apply Lemma 3.40, we have to show that for all $e, y \in G$ such that $e\Gamma \neq y\Gamma$, there exist open neighborhoods $U_e, U_y$ of $e, y$ such that $U_e \cap U_y\Gamma = \emptyset$.

Take $y \in G$ such that $\Gamma \neq y\Gamma$, so $e \in G \setminus y\Gamma$. Since $y\Gamma$ is a closed set in $G$, we have that $U = G \setminus y\Gamma$ is an open neighbourhood of $e$. By Lemma 3.41, there exists an open neighbourhood $V$ of $e$ such that $V = V^{-1}$ and $V^2 \subseteq U$. Note that $y \in Vy$ (since $y = ey \in Vy$), and $Vy$ is an open set of $G$ because $R_y : G \to G$ is a homeomorphism. We claim that $V \cap Vy\Gamma = \emptyset$.

In fact, suppose that there exists $x \in V \cap Vy\Gamma$, so $x = ayh \in V$ with $a \in V$ and $h \in \Gamma$. Then $yh = a^{-1}(ayh) \in V^{-1}V = V^2 \subseteq U$ and $U = G \setminus y\Gamma$, this contradiction completes the proof of Lemma. □

**Remark 3.43.** *If $G$ is a Hausdorff group and $\Gamma$ is a discrete subgroup of $G$, we have that $\Gamma$ is a closed set in $G$ (see e.g. [52]), and then $G/\Gamma$ is a Hausdorff space.*

Let $W$ be a topological space. If $W$ is a locally compact second countable Hausdorff space, then, by [45, Theorem 5.3, p. 33], $W$ is also a complete metric space. So, we proved the following result.

**Corollary 3.44.** *If $G$ is a locally compact second countable Hausdorff group and $\Gamma$ is a closed subgroup of $G$ then $G/\Gamma$ is a locally compact second countable metric space.*

### 3.7.1 Haar measures

A Borel measure $\mu$ on a locally compact space is called *regular* when it holds that

(*i*) every compact set is $\mu$-measurable;

(*ii*) if $A$ is measurable then $\mu(A) = \inf\{\mu(U) | A \subseteq U, U \text{ open}\}$;

(*iii*) $\mu(U) = \sup\{\mu(C) | C \subseteq U, C \text{ compact}\}$ for each open set U.

A regular Borel measure $\mu$ on a locally compact group $G$ is called a *left Haar measure* if

(*i*) $\mu$ is not the zero measure;

(*ii*) the measure of a compact set is finite;

(*iii*) for every $g \in G$ and all measurable sets $E$ the left translate $gE = L_gE = (L_{g^{-1}})^{-1}(E)$ is measurable and $\mu((L_{g^{-1}})^{-1}(E)) = \mu(E)$.

Given a left Haar measure $\mu$, by item $(iii)$ of above definition, we conclude that $\mu$ is an invariant measure under left translation $L_h : G \to G$ for any $h \in G$. We recall the Haar's Theorem as follows.

**Theorem 3.45** (Haar's Theorem)**.** *For every locally compact Hausdorff group there exists a left Haar measure.*

Let $G$ to be a topological group and $H$ to be a subgroup of $G$. We will say that a regular Borel measure $\mu$ on the quotient $G/H$ is a *left invariant Haar measure* if for all Borel sets $E \subseteq G/H$ and all $g \in G$ we have $\mu(gE) = \mu(E)$.

Given a Borel set $E \subseteq G/H$, note that $(\widehat{L_{g^{-1}}})^{-1}(E) = gE$ for any $g \in G$. If a regular Borel measure $\mu$ on the quotient is a left invariant Haar measure then $\mu(gE) = \mu(\widehat{L_{g^{-1}}})^{-1}(E)) = \mu(E) = \mu(\widehat{L_g})^{-1}(E)) = \mu(g^{-1}E)$. This means that $\mu$ is an invariant measure under $\widehat{L_g}$ for any $g \in G$, we say that $\mu$ is a *left $G$-invariant Haar measure.*

**Definition 3.46.** *Let $G$ be a locally compact Hausdorff group and $\Gamma$ be a discrete subgroup. We say that $\Gamma$ is a* ***lattice*** *in $G$ if $G/H$ carries a* ***finite*** *left $G$-invariant Harr measure.*

Let $G$ be a locally compact Hausdorff group and $\Gamma$ be a discrete subgroup. We say that $\Gamma$ is **cocompact** in $G$ if the space $G/\Gamma$ is compact.

Suppose that $G$ is a locally compact second countable Hausdorff group, and $\Gamma$ is a lattice in $G$, then $G/\Gamma$ is a locally compact second countable metric space. Then, by Corollary 3.44 and Definition 3.46, $G/\Gamma$ admits a finite invariant measure under homeomorphism $\widehat{L_b} : G/\Gamma \to G/\Gamma$ given by $g\Gamma \mapsto bg\Gamma$ for any $b \in G$. By Theorem G, we have that

**Corollary 3.47.** *Suppose that $G$ is a locally compact second countable Hausdorff group, and $\Gamma$ is a lattice in $G$. For each $g \in G$ there exist $a_g\Gamma \in G/\Gamma$ and $\varphi_g \in C_c(G/\Gamma, \mathbb{R})$ such that* $\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi_g \circ (\widehat{L_g})^j(a_g\Gamma) = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi_g(g^j a_g\Gamma) > 0.$

The special linear group $\mathrm{SL}(n,\mathbb{R})$ of degree $n$ over $\mathbb{R}$ is the set of $n \times n$ matrices with determinant 1, with the group operations of ordinary matrix multiplication and matrix inversion. We denote by $\mathrm{SL}(n,\mathbb{Z})$ the group of $n \times n$ matrices with integer entries and determinant equals 1. Note that $\mathrm{SL}(n,\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}(n,\mathbb{R})$.

Recall that $\mathrm{SL}(n,\mathbb{Z})$ is a lattice in $\mathrm{SL}(n,\mathbb{R})$. Moreover, $\mathrm{SL}(n,\mathbb{Z})$ is a noncocompact lattice in $\mathrm{SL}(n,\mathbb{R})$ (see e.g. [71, Corollary 3]). By Corollary 3.47, we have the following.

**Corollary 3.48.** *For each $A \in \mathrm{SL}(n,\mathbb{R})$ there exist $B_A\,\mathrm{SL}(n,\mathbb{Z}) \in \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ and an observable $\varphi_A \in C_c(\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}),\mathbb{R})$ such that* $\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi_A \circ (\widehat{L_A})^j(B_A\,\mathrm{SL}(n,\mathbb{Z})) =$

$\lim_n \frac{1}{n}\sum_{j=0}^{n-1} \varphi_A(A^j B_A\,\mathrm{SL}(n,\mathbb{Z})) > 0.$

# Chapter 4

# Future Perspectives

## 4.1 From Chapter 1

Motived by existence of adapted metric for a codimension one singular hyperbolic set with respect to a $C^1$ vector field on finite dimensional compact manifold, we give some conjectures.

**Conjecture 1.** *Given a singular-hyperbolic set for a $C^1$ vector field, then there exists a singular-hyperbolic adapted metric.*

In [63, Definition 3], L. Salgado has given the following notion of sectional hyperbolicity encompassing intermediate dimensions between 2 and the full dimension of the central subbundle.

**Definition 4.1.** *A compact invariant set $\Lambda$ is $p$-singular hyperbolic (or $p$-sectionally hyperbolic) for a $C^1$ flow $X$ if there exists a partially hyperbolic splitting $T_\Lambda M = E \oplus F$ such that $E$ is uniformly contracting and the central subbundle $F$ is $p$-sectionally expanding, with $2 \leq p \leq \dim(F)$.*

**Remark 4.2.** *Note that, if $L_x$ is a $p$-plane with $2 \leq p \leq \dim(F)$, we can see it as $\widetilde{v} \in \wedge^p(F_x) \setminus \{0\}$ of norm one.*

*Hence, to obtain the singular expansion we just need to show that for some $\lambda > 0$ and every $t > 0$ holds the following inequality*

$$\| \wedge^p DX_t(x).\widetilde{v}\| > Ce^{\lambda t}.$$

We do not address sectional-expanding subbundles with dimension $p$ less than the full dimension of the central subbundle here, and we conjecture that similar results should hold true.

**Conjecture 2.** *Given a p-sectional hyperbolic set* $\Gamma$ *for a* $C^1$ *vector field* $X$*, then there exists a metric such that for some constant* $\mu > 0$ *and all* $t > 0$

- $|DX_t\,|_E\,| \leq e^{\mu t}$;
- $|DX_t\,|_E\,| \leq e^{\mu t}|DX_t\,|\,F|$; *and*
- $|\wedge^p DX_t(x)|_{L_x}| > e^{\mu t}$ *for every p-dimensional linear subspace* $L_x \subset F_x, 2 \leq p < \dim F$, $x \in \Gamma$.

We stress that the contructions of adapted metrics in [8, 62], via quadratic forms, is deeply based on the dimension of the singular hyperbolic subbundles. Thus, it is not clear how to use quadratic forms to obtain adapted metrics when the codimension between the $p$-sectional hyperbolic splitting is not equal to one. This drive us to propose the next conjecture.

**Conjecture 3.** *Consider a riemannian compact manifold* $M$ *of dimension* $n \geq 4$. *If* $\Gamma \subset M$ *is a p-sectional hyperbolic set for a* $C^1$ *vector field, with* $2 < p < \dim F$, *then there exists a singular adapted metric induced by quadratic forms.*

## 4.2 From Chapter 2

In Corollary D, we showed that for any measurable bounded function $\varphi : M \to \mathbb{R}$ that satisfies the condition $(b)$ then the following limit exists

$$\int \varphi_- d\mu = \lim_n \tfrac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu = \inf_n \tfrac{1}{n} \int \sum_{j=0}^{n-1} \varphi \circ f^j d\mu.$$

Let $C^b(M;\mathbb{R})$ be the set of all bounded continuous function from $M$ to $\mathbb{R}$. In view of Corollary D, fixed a measure $\mu$, we consider the set $\mathcal{H}_\mu$ of functions of $C^b(M;\mathbb{R})$ that satisfies the condition $(b)$, namely

$$\mathcal{H}_\mu := \{\varphi \in C^b(M;\mathbb{R}) \mid \varphi \text{ satisfies the condition } (b)\}.$$

Considering $(C^b(M;\mathbb{R}), \|\cdot\|)$ the normed space where $\|\cdot\|$ is the uniform norm, i.e., $\|\varphi\| := \sup_{x\in M} |\varphi(x)|$ for any $\varphi \in C^b(M;\mathbb{R})$. We wish to investigate the properties of $\mathcal{H}_\mu$. In this sense,

**Problem 1.** *What are the topology properties of* $\mathcal{H}_\mu$ *in* $C^b(M;\mathbb{R})$*?*

Moreover, we may consider $\mathcal{H}$ the subset of $C^b(M;\mathbb{R})$ given by $\mathcal{H} = \bigcup_{\mu\in\mathcal{M}_1(M)} \mathcal{H}_\mu$ where $\mathcal{M}_1(M)$ is the set of all probabilities measures on $M$, and ask the same question for this set.

**Problem 2.** *What are the topology properties of* $\mathcal{H}$ *in* $C^b(M;\mathbb{R})$*?*

In Corollary 2.6, we gave some conditions to ensure the existence of Birkhoff's limit, so the following question is natural.

**Problem 3.** *Is it true that if* $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j(x)$ *exists then* $\lim_{k\to\infty} \limsup_n \big(\frac{1}{n}\sum_{i=0}^{n-k-1} \boldsymbol{1}_{M\setminus E_k^\varepsilon} \circ f^i(x)\big) = 0$ *for every* $\varepsilon > 0$*?*

## 4.3 From Chapter 3

Let $(M, \mathcal{A}, \mu)$ be a measure metric space, $f$ be a measurable transformation where $\mu$ is a finite measure (not necessarily an invariant measure under $f$). The system $(M, \mathcal{A}, \mu, f)$ is said to be *weakly mixing* if for any $\varphi, \psi : M \to \mathbb{R}$ bounded measurable maps, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} |\int \varphi \circ f^j \cdot \psi d\mu - \int \varphi d\mu \int \psi d\mu| = 0.$$

We recall the notions of meager set (or set of first category), and set of second category in a topological space. Given a topological space $W$, a subset $B$ of $W$ is *nowhere dense* if for each neighbourhood $U$ of $W$, the set $B \cap U$ is not dense in $U$. Equivalently, $B$ is nowhere dense if its closure contains no nontrivial open set; a subset $A$ of $W$ is *meagre* if it can be expressed as the union of countably many nowhere dense subsets of $W$. A meagre set is also called a set of *first category*; a nonmeagre set (that is, a set that is not meagre) is also called a set of *second category.*

For the unit interval in the weak topology, Halmos [35] showed that the set of all mixing measure is a set of first category in the group of measure preserve transformations, and the set of weakly mixing transformations is of the second category. Motivated by this result we consider the remark 3.4. In view of the second result of Halmos, one may conjecture the following.

**Conjecture 4.** *Let* $X$ *be a locally compact separable metric space. Suppose that* $f : X \to X$ *is a continuous proper map. If there exists a probability measure* $\eta$ *(not necessarily an invariant measure under* $f$*) such that* $(X, \mathcal{A}, f, \eta)$ *is a weakly mixing system, then there exists an invariant probability measure.*

In Theorem G, we used the properties of proper continuous functions to prove the existence of invariant measures. To drop the condition of proper maps of this theorem we consider the Perron-Frobenius operator in Theorem H. In the same spirit, one question is which hypotheses can be considered to ensure the existence of invariant measures. Precisely,

**Problem 4.** *What other conditions can be used to prove the existence of invariant measures in locally compact separable metric space using only the continuity the dynamics ?*

In Theorem H, under the assumption that the norm of Perron-Frobenius operator is an eigenvalue of Perron-Frobenius operator we prove the existence of invariant measures. One may ask whether this still true if we suppose that there exists a real eigenvalue of Perron-Frobenius operator. In other words,

**Problem 5.** *Suppose that there exists a real eingenvalue of Perron-Frobenius operator. Can we conclude that there are exist invariant measures?*

In view of Theorem G, we are going to investigate the following natural problem.

**Problem 6.** *Does there exist a similar criteria to guarantee the existence of SRB measures or Gibbs measures?*

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