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# On the Continuous Variation of Expanding Structures

Roberto Sant'Anna Sacramento

**Salvador-Bahia**
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PhD. Thesis presented to the Colegiado da Pós-Graduação em Matemática at Universidade Federal da Bahia as partial requirements to obtain the degree of Doctor of Philosophy in Mathematics.

**Advisor:** Prof. Dr. Vilton Jeovan Viana Pinheiro.

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*À minha mãe, Delma, minha irmã, Karine, à Kayla e a Augusto Bolivar (in memorian).*

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*"-Quem? O infinito? Diz-lhe que entre! Faz bem ao infinito estar entre gente."*

(Alexandre O'Neill)

# Resumo

É possível mostrar que o conjunto das medidas expansoras para transformações uniformemente expansoras é compacto e varia continuamente com a dinâmica. No presente trabalho consideramos famílias de transformações em variedades Riemannianas multidimensionais com comportamento não-uniformemente expansor. Mostramos que o conjunto de medidas expansoras para essas aplicações é $\sigma-$compacto e varia continuamente em partes compactas. Em particular concluímos que o conjunto de medidas expansoras com "parâmetros limitados" para uma dinâmica fixada é compacto. Adotamos a topologia fraca$-*$ no espaço das medidas de probabilidade.

**Palavras-chave:** Medidas expansoras; Conjunto $\sigma-$compacto; Variação contínua; Partições de Markov; Torre de Young, Expoentes de Lyapunov.

# Abstract

One can show that the set of expanding measures for uniformly expanding maps is a compact set and varies continuously with the map. In this work we consider families of transformations in multidimensional Riemannian manifolds with non-uniformly expanding behavior. We show that the set of expanding measures for these transformations is $\sigma-$compact and it varies continuously on compact pieces. In particular we conclude that the set of expanding measures with "bounded parameters" for a fixed dynamics is compact. We endow the space of probability measures with the weak$-*$ topology.

**Keywords:** Expanding measures; $\sigma-$compact set; Continuous variation; Markov partitions; Young Tower; Lyapunov exponents.

# Contents

# Introduction

One can say that the main goals of Dynamics can be narrowed down into two parts: To describe for the majority of orbits the behavior as time goes to infinity and to understand whether this limit behavior is stable under small changes in the evolution law of the system. In this work we mainly are concerned to the second part.

We study in a broader way the stability (or continuation) for certain classes of chaotic systems, namely, systems which exhibit non-uniformly expanding (NUE) behavior on the growth of the derivative for most of its orbits. We study under what conditions one can observe the continuity of the set of expanding measures with respect to the dynamics. The context of expanding measures presents a more general setting than the context involving only Lebesgue measure. Roughly speaking, a probability measure is called expanding if it gives full weight to the set of points displaying non-uniform expanding behavior (see Definition 1.2.2 for more details). Examples 9.6 and 9.7 of [24] shows us that even in lower dimensions the context of expanding measures is much richer than the context involving Lebesgue measures: There are systems which do not admit invariant measures that are absolutely continuous with respect to the Lebesgue measure and, even more, these systems present zero Lyapunov exponent for Lebesgue almost every point, but they possesses an uncountable number of ergodic invariant probabilities whose supports are the whole manifold and whose Lyapunov exponents are positive (see Remark 2.4.1 and Theorem 5.2.1). Roughly speaking, our results ensure that considering a NUE system then NUE behavior holds for every dynamic close enough (this is one direction) and we also consider the opposite direction: if a dynamic is accumulated by NUE dynamics, this limit dynamic also presents NUE behavior.

We remark that, although some similarities in a few statements, our approach can not be regarded as statistical stability, where one attempts to express stability in terms of persistence of statistical properties of the system. Roughly speaking, in this case we can compare the average along the orbit with the average of the system in the ambient space. But one distinctive feature in the statistical stability case is that one can follows the continuation of a measure with an specific property (for instance, the absolutely continuous invariant measure with respect to Lebesgue, see [13, 31] for the uniformly hyperbolic case, [9, 2, 4] for the non-uniformly expanding case and [15, 33] for the partially hyperbolic case). On the other hand, in our approach we do not follow the continuation of measures only with some reference property. One can expect that when we perturb the dynamics we obtain for each dynamic close infinitely many expanding measures close to the original expanding measure.

We need some preliminary definitions.

**Definition 0.0.1.** *Let $Y$ be a metric space and let $K(Y)$ be the collection of compact subsets of $Y$. We define the* Hausdorff distance *on $K(Y)$ by:*

$$d_H(A,B) := \inf\{\varepsilon \geq 0;\ A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}, \tag{1}$$

*where $A_\varepsilon = \bigcup_{a\in A}\{y \in Y;\ d(a,y) \leq \varepsilon\}$ is the $\varepsilon-$*neighborhood of the compact subset *$A \subset Y$.*

**Definition 0.0.2.** *Let $X$ and $Y$ be metric spaces. We say that a map $\Gamma : X \longrightarrow K(Y)$ is a* family of compact sets parameterized by $X$*. If $\Gamma$ is continuous at some $x \in X$, we say that it is a* family of compact sets parameterized by $X$ *continuous at $x$. If $\Gamma \in C^0(X, K(Y))$ it is said to be a* continuous family of compact sets parameterized by $X$.

In this case we endow $K(Y)$ with the topology given by the Hausdorff distance.

If $\Gamma : X \longrightarrow K(Y)$ is a continuous family of compact sets parameterized by $X$ then, for $x_1, x_2 \in X$ close enough, the compact sets $\Gamma(x_1)$ and $\Gamma(x_2)$ will be close with respect to the Hausdorff distance. It means that for each point of $\Gamma(x_1)$ there is a point of $\Gamma(x_2)$ close enough and vice versa.

**Definition 0.0.3.** *We say that a metric space $Y$ is $\sigma-$*compact *if $Y = \bigcup_{j\in\mathbb{N}} Y_j$ can be written as the countable union of infinitely many compact sets $Y_j \subset Y$.*

We denote by $d_*$ the distance in the space $\mathcal{M}^1(K)$ of probability measures on a compact metric space $K$, which is defined in the following way:

Since $K$ is a compact metric space, there is a countable subset $(\varphi_i)_{i\in\mathbb{N}}$ of $C^0(K)$ which is dense in the unit ball $B^1 := \{\varphi \in C^0(K); \|\varphi\|_0 \leq 1\}$. Given $\mu, \nu \in \mathcal{M}^1(K)$, we define:

$$d_*(\mu,\nu) = \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_K \varphi_i d\mu - \int_K \varphi_i d\nu \right|.$$

# The uniformly expanding case

Lets analyze first what happens in the uniformly expanding/hyperbolic case and then we will try to understand what happens beyond the uniformly hyperbolic scenario at least for some class of transformations. Consider a uniformly hyperbolic diffeomorphism $f : M \longrightarrow M$ defined on a compact Riemannian $d$–dimensional manifold. Denote the set of all invariant hyperbolic probability measures for $f$ by $\mathcal{M}^1_{f,hyp}$, that is, the set of probabilities which gives full weight to the hyperbolic set of $f$ (which will be assumed to be all the manifold $M$). Since every point belongs to the hyperbolic set, it is clear that $\mathcal{M}^1_{f,hyp}$ coincides with $\mathcal{M}^1_f$ (the set of every invariant probability measures for $f$). We conclude that $\mathcal{M}^1_{f,hyp}$ is a compact set of $\mathcal{M}^1(M)$, the set of every borelian probability measures on $M$ endowed with the weak$-*$ topology. In an analogous way, if we admit that $f$ is an uniformly expanding endomorphism, we can define the set of expanding invariant probabilities for $f$ as the set of probabilities that gives full weight to the expanding set of $f$ (which is, again, all the manifold $M$).

**Theorem 0.0.4.** *Denote by $X$ the set of $C^2$ uniformly hyperbolic diffeomorphisms $g : M \longrightarrow M$, $Y_g := \mathcal{M}^1_{g,hyp}$ and $Y := \mathcal{M}^1(M)$. Then the map*

$$\Gamma : X \longrightarrow K(Y), \Gamma(g) := Y_g$$

*is a continuous family of compact sets parameterized by $X$.*

Before we prove Theorem 0.0.4, it is worth to say some words about structural stability. The notion of structural stability, was proposed by Andronov and Pontryagin, [11], back in the thirties, and since then much effort has been made to characterize systems with such property.

**Definition 0.0.5.** *We say that a diffeomorphism $f : M \longrightarrow M$ is* structurally stable *if there is a neighborhood $V$ of $f$ such that for each $g \in V$ there is a homeomorphism $h : M \longrightarrow M$ such that*

$$h \circ f = g \circ h.$$

In the $60'$s Palis and Smale conjectured in [23] that: a diffeomorphism (or flow) is structurally stable if, and only if, it is Axiom A and satisfies the strong transversality condition. The conjecture proved to be true by the work of authors such as J. Robbin, J. C. Robinson and R. Mañe (see [28, 29, 18]). This result can be improved: one can show that the conjugation can be required to depend nicely on the perturbation (see [16]): The neighborhood $V \ni f$ can be taken in such a way that for each $g \in V$, the conjugation $h = h_g$ and there is $K > 0$ (uniform on $V$) such that

$$\sup_{x \in M} d(h_g(x), x) \leq K \sup_{x \in M} d(g(x), f(x)). \tag{2}$$

Also, M. Shub studied structural stability for uniformly expanding endomorphisms (see [32]): He extended the results on structural stability above for uniformly expanding endomorphism of a compact manifold. In this way, we may enounce the following result, witch is a version of Theorem 0.0.4 for uniformly expanding endomorphisms.

**Theorem 0.0.6.** *Denote by $X$ the set of $C^2$ uniformly expanding endomorphisms $g : M \longrightarrow M$, $Y_g := \mathcal{M}^1_{g,exp}$ and $Y := \mathcal{M}^1(M)$. Then the map*

$$\Gamma : X \longrightarrow K(Y), \Gamma(g) := Y_g$$

*is a continuous family of compact sets parameterized by $X$.*

**Proof of Theorem 0.0.6:** Since $f$ is uniformly expanding, we conclude by the Corollary of Theorem $\alpha$ of [32] that there is a neighborhood $V \ni f$ such that for each $g \in V$ there is a homeomorphism $h_g$ satisfying $h_g \circ f = g \circ h_g$ and 2. Consider $\mu \in Y_f = \mathcal{M}^1_{f,hyp}$. It is a straightforward fact that the probability measure $\nu$ defined as $\nu := \mu \circ h^{-1}$ belongs to $Y_g = \mathcal{M}^1_{g,hyp}$. Consider $\varepsilon > 0$. We will show that there is $\delta > 0$ such that $d_0(g, f) < \delta \Rightarrow d_*(\mu, \nu) < \varepsilon$.

Consider $i_0 > 0$ such that $\sum_{i=i_0}^{+\infty} \frac{1}{2^i} < \varepsilon/4$ (because $\sum_{i=1}^{+\infty} \frac{1}{2^i}$ is a convergent series) and $\delta > 0$ suitable to the uniform continuity of each $\varphi_i$, $i \in \{1, \cdots, i_0\}$. Precisely, $\delta > 0$ is such that

$$\forall x_1, x_2 \in M,\ d(x_1, x_2) < \delta \Rightarrow |\varphi_i(x_1) - \varphi_i(x_2)| < \varepsilon / \left(2 \cdot \sum_{k=1}^{i_0-1} \frac{1}{2^k}\right), \forall i \in \{1, \cdots, i_0 - 1\}, \quad (3)$$

which is possible since we have a finite number of $\varphi_i' s$ and each $\varphi_i$ is uniformly continuous on the compact $M$).

Thus, if $x \in M$ is such that $d(h_g(x), x) < \delta$ and $\nu = \mu \circ h_g^{-1}$, we have that

$$\begin{aligned} d_*(\mu, \nu) &= \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i d\mu - \int_M \varphi_i d\nu \right| \\ &= \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i d\mu - \int_M \varphi_i d\mu \circ h_g^{-1} \right| \\ &= \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i d\mu - \int_M \varphi_i \circ h_g d\mu \right| \\ &= \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i - \varphi_i \circ h_g d\mu \right| \\ &= \sum_{i=1}^{i_0-1} \frac{1}{2^i} \left| \int_M \varphi_i - \varphi_i \circ h_g d\mu \right| + \sum_{i=i_0}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i - \varphi_i \circ h_g d\mu \right| \\ &< \varepsilon/2 + 2 \cdot \sum_{i=i_0}^{+\infty} \frac{1}{2^i} < \varepsilon/2 + 2 \cdot \varepsilon/4 = \varepsilon. \end{aligned}$$

In the last inequality we used the fact that $|\varphi_i(x) - \varphi_i \circ h_g(x)| < 2,\ \forall x \in M,\ \forall i \in \mathbb{N}$, because the functions $\varphi_i$ are taken in the unit ball $B^1$.

So, in view of 2 it is enough to take $g \in V$ satisfying $\sup_{x \in M} d(g(x), f(x)) < \delta/K$ and we are done.

In an analogous way we prove that given a measure $\nu \in Y_g$ the measure $\mu := \nu \circ h_g$ belongs to $Y_f$ and given $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{x \in M} d(g(x), f(x)) < \delta \Rightarrow d_*(\mu, \nu) < \varepsilon$.

With this we conclude that, denoting by $d$ also the distance in the space $X$, $d(g, f) < \delta \Rightarrow d_H(Y_g, Y_f) < \varepsilon$ (where, in this case, $d$ denotes the distance in $X$ associated to the $C^2$ topology) and the proof is complete.

□

**Remark 0.0.7.** *The proof of Theorem 0.0.4 is analogous to the proof of Theorem 0.0.6, since it is based on the notion of structural stability.*

Now that we understand that the set of expanding measures is a compact set that varies continuously with the dynamics, when we restrict ourselves to uniformly expanding diffeomorphisms, we will study this phenomenon in another context: Non-uniformly expanding maps (NUE maps). We will conclude in Main Theorem (see Chapter 1) that in fact the set of expanding measures for a NUE map may not be compact, in general, but it is $\sigma-$compact. Furthermore, this set varies continuously in compact sets, in the sense of Definition 0.0.2. We will identify some parameters that allows us to control the continuous variation of the set of expanding measures and obtain classes of compact sets inside it. By fixing the parameters, we will conclude that although the whole set of expanding measures may not vary continuously, the set of measures with "bounded parameters" vary continuously (see Theorems A and B and Definition 1.4.1). Theorems A and B both work in complementary directions. This is connected to the nature of Hausdorff distance: in order to check if two compact sets are close, we need to compare distance between points in these sets in both directions (see Equation 1). Theorems A and B are the core results in this work and they are used to prove our Main Theorem, where we state that one can observe the continuation of some subsets of the set of expanding measures (or, equivalently, of the set of measures with positive Lyapunov exponents). These subsets are described in Definition 1.4 and in a more detailed way in the proof of Main Theorem.

The text is organized as follows. In Chapter 1 we present preliminary definitions and results and explain formally the statement of the main results in this work. In Chapter 2 we point out some of the main tools utilized in the proofs of the main theorems. We analyze specially the relation of Markov maps with NUE maps and expanding measures. In Chapter 3 we prove some results involving the stability for return maps. This is the core of the technical results present in this work, where we understand how the perturbation of a dynamics affects the associated induced Markov maps and their properties. We dedicate Chapters 4 and 5 to the proof of the main theorems. Finally, in Chapter 6 we point out future perspectives for this work.

# Chapter 1

# Preliminaries and statement of main results

## 1.1 Preliminaries

Let $M$ be a compact Riemannian manifold of dimension $d \geq 1$ and $f : M \longrightarrow M$ a map defined on $M$.

The map $f$ is called *non-flat* if it is a local $C^{1+}$ diffeomorphism (i.e., $C^{1+\alpha}$ with $\alpha > 0$) in the whole manifold except in a *non-degenerated critical/singular set* $\mathcal{C} \subset M$, that is, a subset for which there is $\beta > 0$ such that $f$ behaves like a polynomial of degree $\beta$ close to it.

**Definition 1.1.1.** *We say that* $\mathcal{C} \subset M$ *is a* non-degenerated critical/singular set *if* $\exists \beta, B > 0$ *such that the following conditions hold:*

*1*

$$\frac{1}{B}\mathrm{dist}(x,\mathcal{C})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq B\mathrm{dist}(x,\mathcal{C})^{-\beta}$$

*for all* $v \in T_xM$.

*For every* $x, y \in M\backslash\mathcal{C}$ *with* $\mathrm{dist}(x,y) < \mathrm{dist}(x,\mathcal{C})/2$ *we have:*

*2*

$$|\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\|| \leq \frac{B}{\mathrm{dist}(x,\mathcal{C})^{\beta}}\mathrm{dist}(x,y).$$

**Definition 1.1.2.** *A map* $f : M \longrightarrow M$ *is called* non-flat *if it is a* $C^{1+\alpha}$ *local diffeomorphism except in a non-degenerate critical/singular set* $\mathcal{C} \subset M$.

A measure $\mu$ is called $f$*−non-singular* if $f_*\mu \ll \mu$, where $f_*\mu = \mu \circ f^{-1}$ is the push-forward of $\mu$ by $f$. Consider a non-flat map $f$ with critical/singular set $\mathcal{C} \subset M$. A finite measure $\mu$ is called $f$*−non-flat* if it is $f$−non-singular, $\mu(\mathcal{C}) = 0$, the Jacobian $J_\mu f(x)$

is well defined and positive for $\mu$–almost every $x \in M$, and for $\mu$-almost every $x, y \in M\backslash$ with $\text{dist}(x,y) < \text{dist}(x,\mathcal{C})/2$ we have

$$\left\| \log \frac{J_\mu f(x)}{J_\mu f(y)} \right\| \leq \frac{B}{\text{dist}(x,\mathcal{C})^\beta}.$$

Unless otherwise stated, we deal in this work with non-flat dynamics and non-singular measures.

**Definition 1.1.3.** *We say that a point $x \in M$* has all Lyapunov exponents positive *if*

$$\limsup_{n \in \mathbb{N}} \frac{1}{n} \log \|(Df^n(x))^{-1}\|^{-1} > 0. \tag{1.1}$$

*Additionally, we say that $\mu$* has all of its Lyapunov exponents positive *if 1.1 holds for $\mu$–almost every point $x \in M$.*

## 1.2 Non-uniformly expanding (NUE) maps

**Definition 1.2.1.** *A positively invariant set $\mathcal{H} \subset M$ (i.e., $f(\mathcal{H}) \subset \mathcal{H}$) is called* $(\lambda, \ell)$–expanding, $\lambda \geq 0$, *if there exists $\ell \in \mathbb{N}$ with*

$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D\widetilde{f}(\widetilde{f}^i(x))^{-1}\|^{-1} > \lambda, \tag{1.2}$$

*for every $x \in \mathcal{H}$ (where $\widetilde{f} = f^\ell$), and $\mathcal{H}$ satisfies the slow approximation condition, i.e., for each $\epsilon > 0$ there is $\delta > 0$ such that*

$$\limsup_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(f^j(x), \mathcal{C}) \leq \varepsilon \tag{1.3}$$

*for every $x \in \mathcal{H}$, where $dist_\delta(x,y)$ denotes the $\delta$–truncated distance from $x$ to $\mathcal{C}$, defined as*

$$\begin{cases} \text{dist}_\delta(x,\mathcal{C}) = \text{dist}(x,\mathcal{C}), \text{ if } \text{dist}(x,\mathcal{C}) \leq \delta \\ \text{dist}_\delta(x,\mathcal{C}) = 1, \text{ if } \text{dist}(x,\mathcal{C}) > \delta \end{cases}$$

When $\mathcal{C} = \varnothing$, $\mathcal{H}$ is $(\lambda, \ell)$–expanding if 1.2 holds for every $x \in \mathcal{H}$.

**Definition 1.2.2.** *A probability measure $\mu$ is called* $(\lambda, \ell)$–expanding measure *(with respect to $f$) if $\mu$ is $f$-non singular ($f_*\mu \ll \mu$) and there exists a $(\lambda, \ell)$–expanding set $\mathcal{H}$ such that $\mu(M\backslash\mathcal{H}) = 0$. In this case we also say that $f$ is (non-uniformly) expanding.*

We may drop the indexes writing that $\mu$ is a $\lambda$–expanding measure (or even an expanding measure) and $\mathcal{H}$ is a $\lambda$–expanding set (or even an expanding set), when there is no chance of misunderstanding.

We denote the set of all ergodic invariant expanding probability measures for $f$ by $\mathcal{M}^1_{\mathfrak{exp}}(f)$.

If $\mu \in \mathcal{M}_{\mathfrak{exp}}(f)$, then by 1.2 the *expansion time function*

$$\mathcal{E}_\lambda(x) = \min\left\{N \geq 1;\ \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\|^{-1} \geq \lambda,\ \forall n \geq N\right\} \tag{1.4}$$

is defined and finite for $\mu$–almost every point $x \in M$. Also, the *recurrence time function*

$$\mathcal{R}_{\varepsilon,\delta}(x) = \min\left\{N \geq 1;\ \frac{1}{n}\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^i(x), \mathcal{C}) \leq \varepsilon,\ \forall n \geq N\right\} \tag{1.5}$$

is defined and finite for $\mu$–almost every point $x \in M$. We define the *tail set*

$$\Gamma_n(\lambda, \varepsilon, \delta) = \{x;\ \mathcal{E}(x) > n \text{ or } \mathcal{R}_{\varepsilon,\delta}(x) > n\}\,. \tag{1.6}$$

This is the set of points which at time $n$ have not yet achieved either the uniform exponential growth of derivative or the uniform slow recurrence. If $\mathcal{C} = \varnothing$, we ignore the recurrence time function in the definition of the tail set. We may drop the indexes and write $\Gamma_n$ instead of $\Gamma_n(\lambda, \varepsilon, \delta)$ when there is no chance of misunderstanding.

## 1.3 Hyperbolic times

**Definition 1.3.1.** *Let us fix $0 < b = \frac{1}{3}\min\{1, 1/\beta\}$. Given $0 < \sigma < 1$ and $\varepsilon > 0$, we say that $n$ is a $(\sigma, \varepsilon)$-*hyperbolic time *for a point $x \in M$ (with respect to the non-flat map $f$ with a $\beta$-non-degenerated critical/singular set $\mathcal{C}$) if for all $1 \leq k \leq n$ we have*

$$\prod_{j=n-k}^{n-1} \|(Df \circ f^j(x))^{-1}\| \leq \sigma^k \text{ and } \operatorname{dist}_\varepsilon(f^{n-k}(x), \mathcal{C}) \geq \sigma^{bk}. \tag{1.7}$$

*We denote the set of points of $M$ such that $n \in \mathbb{N}$ is a $(\sigma, \varepsilon)$-hyperbolic time by $H_n(\sigma, \varepsilon, f)$ or, shortly, by $H_n(f)$.*

We point out that in the case $\mathcal{C} = \varnothing$ the definition of $(\sigma, \delta)$–hyperbolic time reduces to the first condition in 1.7 and we simply call it a *$\sigma$–hyperbolic time.*

The following results (Propositions 1.3.2 and 1.3.5), whose proofs can be found in [3] and [6], give the main properties of hyperbolic times that we shall utilize.

**Proposition 1.3.2** (Geometric properties of hyperbolic times)**.** *Given $\sigma \in (0, 1)$ and $\varepsilon > 0$, there exists $\delta > 0$, which depends only on $\sigma, \varepsilon$ and on the map $f$, such that if $x \in H_n(\sigma, \varepsilon, f)$ then there is a neighborhood $V_n(x)$ of $x$ satisfying:*

1. *$f^n$ maps $\overline{V_n(x)}$ diffeomorphically onto the ball $\overline{B_\delta(f^n(x))}$;*

2. *dist$(f^{n-j}(y), f^{n-j}(z)) \leq \sigma^{j/2}$ dist$(f^n(y), f^n(z))$; $\forall y, z \in V_n(x)$ and $1 \leq j \leq n$,*

The sets $V_n(x)$ are called *hyperbolic pre-balls* and their images, $f^n(V_n(x)) = B_\delta(f^n(x))$, *hyperbolic balls*. See Figure 1.1. We may refer to items 1) and 2) of Proposition 1.3.2 as *geometric version of hyperbolic times*. Since we are dealing with $(\lambda,\ell)$–expanding measures, sometimes one can consider a geometric version of hyperbolic times not just for $f$ but also for $\widetilde{f} := f^\ell$, i.e., for every $y, z$ in the hyperbolic pre-ball $V(x)$ of $x$ one has:

$$\text{dist}(\widetilde{f}^{n-j}(y), \widetilde{f}^{n-j}(z)) \leq \sigma^{j/2}\text{dist}(\widetilde{f}^{n}(y), \widetilde{f}^{n}(z));\ \forall y, z \in V_n(x) \text{ and } 1 \leq j \leq n,$$

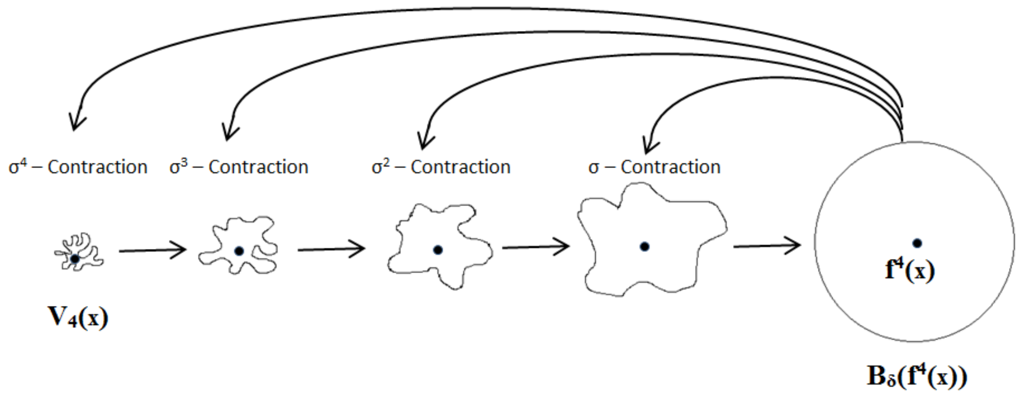



Figure 1.1: Example where $n = 4$ is a hyperbolic time for $x$.

The following Lemma is a straightforward consequence of the definition of hyperbolic times.

**Lemma 1.3.3.** *Hyperbolic times satisfy the following property:*

*If* $p \in H_j(\sigma,\delta,f)$ *and* $f^j(p) \in H_l(\sigma,\delta,f)$ *then* $p \in H_{j+l}(\sigma,\delta,f)$.

**Remark 1.3.4.** *a) If* $n$ *is a hyperbolic time for* $x \in M$*, chain rule and Proposition 1.3.2 immediately give us that* $\|(Df^n(x))^{-1}\| < \sigma^n$.

*b) If* $g : M \longrightarrow M$ *is differentiable at a point* $p \in M$ *then*

$$\|(Dg(p))^{-1}\|^{-1} = \liminf_{x \longrightarrow p} \frac{d(g(x), g(p))}{d(x,p)}.$$

*In fact, if we see in local coordinates then* $g$ *is differentiable at* $p$ *if, and only if,* $\dfrac{1}{\|x-p\|}(g(x) - g(p) - Dg(p)\cdot(x-p)) \longrightarrow 0$ *when* $x \longrightarrow p$*. But since*

$$\begin{aligned} \frac{1}{\|x-p\|}\|g(x)-g(p)-Dg(p)\cdot(x-p)\| &\geq \tfrac{\|Dg(p)\cdot(x-p)\|}{\|x-p\|} - \tfrac{\|g(x)-g(p)\|}{\|x-p\|} \geq \\ \frac{\|Dg(p)^{-1}\|^{-1}\cdot\|x-p\|}{\|x-p\|} - \frac{\|g(x)-g(p)\|}{\|x-p\|} &= \|Dg(p)^{-1}\|^{-1} - \tfrac{d(g(x),g(p))}{d(x,p)}, \end{aligned}$$

*we get that* $\|Dg(p)^{-1}\|^{-1} - \dfrac{d(g(x),g(p))}{d(x,p)} \longrightarrow 0$ *when* $x \longrightarrow p$, *and we are done.*

*In particular, we obtain that if there are* $f : M \longrightarrow M$, $x \in M$, $0 < \sigma < 1$ *and* $n > 0$ *is such that items* (1) *and* (2) *of Proposition 1.3.2 holds, then*

$$\|(Df^n(x))^{-1}\|^{-1} > \sigma^{-n} \geq \sigma^{-1}.$$

*As we will see in the proof of Claim 4.1.3, this gives us some sort of an equivalence between the analytical definition of hyperbolic times in 1.7 and the geometric property they have as described in Proposition 1.3.2. We use the term "some sort of equivalence" in last sentence because we may recover the analytical feature of hyperbolic times from the geometrical not for* $f$ *but for some iterate* $f^\ell$ *of it.*

c) *By using Birkhoff's Theorem, we get that if* $\mu$ *is an ergodic* $f-$*invariant probability and there is* $\lambda > 0$ *with* $\int \log(\|(Df)^{-1}\|^{-1})d\mu > \lambda$ *then there exists* $H \subset M$ *with* $\mu(H) > 0$ *such that 1.2 holds for* $\mu-$*almost every point* $x \in M$.

d) *As a consequence of the definition of non-degenerated critical set, we obtain that if* $\log\,\mathrm{dist}(x,\mathcal{C})$ *is* $\mu-$*integrable then* $\log(\|(Df)^{-1}\|^{-1})$ *is also* $\mu-$*integrable. In fact, by condition* (1) *of Definition 1.1.1 we can obtain that there exists* $\rho > \beta$ *such that*

$$|\log\|(Df)^{-1}\|^{-1}| \leq \rho|\log\,\mathrm{dist}(x,\mathcal{C})|$$

*for all* $x$ *in a small open neighborhood* $V$ *of* $\mathcal{C}$. *To obtain this it is enough to take some* $\rho$ *greater than* $\beta$ *(remember that condition* (1) *of Definition 1.1.1 yields* $\frac{1}{B}\mathrm{dist}(x,\mathcal{C})^\beta \leq \|Df(x)^{-1}\| \leq B\mathrm{dist}(x,\mathcal{C})^{-\beta}$*). Since* $\log\|(Df)^{-1}\|^{-1}$ *is bounded on the compact set* $M\backslash V$, *this function must be integrable with respect to* $\mu$ *on* $M$ *as long as* $\log\,\mathrm{dist}(x,\mathcal{C})$ *is* $\mu-$*integrable.*

Let

$$\varphi_{\mathfrak{exp},+}(x) := \limsup_{n\in\mathbb{N}} \frac{1}{n}\sharp\{1 \leq j \leq n;\ x \in H_j(f)\} \tag{1.8}$$

denote the frequency of hyperbolic times for $x \in M$.

### 1.3.1 Frequency of hyperbolic returns

It is well known that if $\mathcal{H}$ is an expanding set for a map $f : M \longrightarrow M$ then every point $x$ of $\mathcal{H}$ has infinitely many hyperbolic times. Indeed, they have uniformly bounded positive frequency of hyperbolic times, as we state in the following result.

**Proposition 1.3.5.** *Given* $\lambda > 0, \ell \in \mathbb{N}$ *and a* $(\lambda,\ell)-$*expanding measure for* $f$ *there exists* $\theta' > 0$, $\sigma > 0$ *and* $\varepsilon_0 > 0$ *such that for* $\mu-$*a.e.p.* $x \in M$ *and* $\varepsilon \in (0,\varepsilon_0]$

$$\limsup_{n\in\mathbb{N}} \frac{1}{n}\sharp\{1 \leq j \leq n;\ x \in H_j(\sigma,\varepsilon,f^\ell)\} > \theta'.$$

We remark that in this case we can take $\sigma = e^{-\lambda/4}$. The proof of this Proposition can be found in Alves [3] and Alves, Bonatti, Viana [6]. However we will include this proof here to emphasize the dependence of the frequency $\theta'$ on $f$ and $\lambda$. This result is an application of a Lemma due to Pliss (see A.0.1).

**Proof:** Lets suppose that 1.2 holds for some $x \in M$. Then, for $N \in \mathbb{N}$ large enough we have that

$$\sum_{j=0}^{N-1} \log \|Df(f^j(x))^{-1}\|^{-1} \geq \lambda N.$$

If we take $\beta > 0$ given by Definition 1.1.1 and fix any $\rho > \beta$, we get by condition 1 of the same Definition that

$$\left|\log \|Df(x)^{-1}\|\right| \leq \rho \left|\log \operatorname{dist}(x, \mathcal{C})\right| \tag{1.9}$$

for every $x$ in a neighborhood $V$ of $\mathcal{C}$.

Fix $\varepsilon_1 > 0$ so that $\rho\varepsilon_1 \leq \lambda/2$. By 1.3 (hypothesis of slow recurrence to the critical set) we can take $r_1 > 0$ such that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_1}(f^j(x), \mathcal{C}) \geq -\varepsilon_1 N. \tag{1.10}$$

Fix any $K_1 \geq \rho|\log r_1|$ large enough so that $\log \|Df(y)^{-1}\|^{-1} \leq K_1, \ \forall y \in M \backslash V$. Then let $J$ be the subset of times $1 \leq j \leq N$ such that $\log \|Df(f^{j-1}(x))^{-1}\|^{-1} > K_1$ and define:

$$a_j = \begin{cases} \log \|Df(f^{j-1}(x))^{-1}\|^{-1} & \text{if} \quad j \notin J \\ 0 & \text{if} \quad j \in J \end{cases}$$

By construction, $a_j \leq K_1$ for $1 \leq j \leq N$. Note that if $j \in J$ then $f^{j-1}(x) \in V$. Moreover, for each $j \in J$ we have:

$$\rho|\log r_1| \leq K_1 < \log \|Df(f^{j-1}(x))^{-1}\| < \rho|\log \operatorname{dist}(f^{j-1}(x), \mathcal{C})|,$$

which shows that $\operatorname{dist}(f^{j-1}(x), \mathcal{C}) < r_1$ for every $j \in J$. In particular we have:

$$\operatorname{dist}_{r_1}(f^{j-1}(x), \mathcal{C}) = \operatorname{dist}(f^{j-1}(x), \mathcal{C}) < r_1, \forall j \in J.$$

Therefore, by 1.9 and 1.10,

$$\sum_{j \in J} \log \|Df(f^{j-1}(x))^{-1}\|^{-1} \leq \rho \sum_{j \in J} |\log \operatorname{dist}(f^{j-1}(x), \mathcal{C})| \leq \rho\varepsilon_1 N.$$

See that $\varepsilon_1$ was chosen in such a way that the last term is smaller than $\lambda N/2$. As a consequence we have that

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} \log \|Df(f^{j-1}(x))^{-1}\|^{-1} - \sum_{j \in J} \log \|Df(f^{j-1}(x))^{-1}\|^{-1} \geq \frac{\lambda}{2} N.$$

Now, set $c_1 = \lambda/4$, $c_2 = \lambda/2$ and $A = K_1$. Applying Lemma A.0.1 to the numbers $a_1, \cdots, a_N$, we obtain $\theta_1 > 0$ and $l_1 \geq \theta_1 N$ times $1 \leq p_1 < p_2 \cdots < p_{l_1} \leq N$ such that

$$\sum_{j=n+1}^{p_i} \log \|Df(f^{j-1}(x))^{-1}\|^{-1} \geq \sum_{j=n+1}^{p_i} a_j \geq \frac{\lambda}{4}(p_i - n) \tag{1.11}$$

for every $0 \leq n < p_i$ and $1 \leq i \leq l_1$.

Now, fix $\varepsilon_2 > 0$ small enough so that $\varepsilon_2 < \theta_1 b\lambda/4$, where $b$ is as in the definition of hyperbolic times, and let $r_2 > 0$ be such that

$$\sum_{j=0}^{N-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \geq -\varepsilon_2 N. \tag{1.12}$$

Let $c_1 = -b\lambda/4$, $c_2 = -\varepsilon_2$, $A = 0$ and $\theta_2 = \dfrac{c_2 - c_1}{A - c_1} = 1 - \dfrac{4\varepsilon_2}{\lambda}$.

Consider the numbers $a_j = \log \operatorname{dist}_{r_2}(f^{j-1}(x), \mathcal{C})$, with $1 \leq j \leq N$. Applying again Lemma A.0.1 we conclude that there are $l_2 \geq \theta_2 N$ times $1 \leq q_1 < \cdots < q_{l_2} \leq N$ such that

$$\sum_{j=n}^{q_i-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \geq -\frac{b\lambda}{4}(q_i - n) \tag{1.13}$$

for every $0 \leq n < q_i$ and $1 \leq i \leq l_2$.

We can easily see that our condition on $\varepsilon_2$ means that $\theta_1 + \theta_2 > 1$. Let $\theta' = \theta_1 + \theta_2 - 1$. Then there exists $l = (l_1 + l_2 - N) \geq \theta' N$ times $1 \leq n_1 < \cdots < n_l \leq N$ at which 1.11 and 1.13 occur simultaneously:

$$\sum_{j=n}^{n_i-1} \log \|Df(f^j(x))^{-1}\|^{-1} \geq \frac{\lambda}{4}(n_i - n)$$

and

$$\sum_{j=n}^{n_i-1} \log \operatorname{dist}_{r_2}(f^j(x), \mathcal{C}) \geq -\frac{b\lambda}{4}(n_i - n),$$

for every $0 \leq n < n_i$ and $1 \leq i \leq l$. Letting $\sigma = e^{-\lambda/4}$ we easily obtain from the inequalities above that

$$\prod_{j=n_i-k}^{n_i-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \text{ and } \operatorname{dist}_{r_2}(f^{n_i-k}(x), \mathcal{C}) \geq \sigma^{bk}$$

for every $1 \leq i \leq l$ and $1 \leq k \leq n_i$. In other words, all those $n_i$ are $(\sigma, \delta)$–hyperbolic times for $x$, with $\delta = r_2$.

$\square$

**Remark 1.3.6.** *From the proof of Proposition 1.3.5 on easily sees that condition 1.3 in the definition of non-uniformly expanding map is not needed in all its strength for the proof work. Actually, the only places where we have used 1.3 are 1.12 and 1.10. Hence, it is enough that 1.3 holds for* $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ *and* $\delta = \max\{r_1, r_2\}$.

**Remark 1.3.7.** *Note that the proof of Proposition 1.3.5 gives more precisely that if for some $x \in M$ and $N \in \mathbb{N}$ one has*

$$\sum_{j=0}^{N-1} \log \|Df(f^j(x))^{-1}\|^{-1} \geq \lambda N \text{ and } \sum_{j=0}^{N-1} \log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \geq -\varepsilon N$$

*(where $\varepsilon$ and $\delta$ are chosen according to Remark 1.3.6), then there exist integers $0 < n_1 < \cdots < n_l < N$, $\theta' > 0$ and $\delta > 0$ such that $l > \theta' N$ and $x \in H_{n_i}(\sigma, \delta, f)$ for each $1 \leq i \leq l$.*

Now, following [24], consider $x \in M$ and a subset $\mathcal{U}(x) \subset \mathcal{O}^+(x)$ of the positive orbit of $x$.

**Definition 1.3.8.** *The collection $\mathcal{U} = (\mathcal{U}(x))_{x \in M}$ is called* asymptotically invariant *if for every $x \in M$,*

*1* $\sharp\{j \in \mathbb{N};\ f^j(x) \in \mathcal{U}(x)\} = \infty$, *and*

*2* $\mathcal{U}(x) \cap \mathcal{O}^+_f(f^n(x)) = \mathcal{U}(f(x)) \cap \mathcal{O}^+_f(f^n(x))$ *for every big $n \in \mathbb{N}$.*

*Define $\omega_f(x)$ as usual (the set of accumulation points of $\mathcal{O}^+_f(x)$) and $\omega_{f,\mathcal{U}}(x)$ as the set of accumulation points of $\mathcal{U}(x)$*

**Remark 1.3.9.** *We can conclude by using Lemma 1.3.3 that the collection of hyperbolic iterates in the orbit of each $x \in \mathcal{H}$, i.e., the collection $h = (h(x))_{x \in \mathcal{H}}$ of sets $h(x) := \{f^n(x);\ n \geq 1,\ x \in H_n(\sigma, \delta, f)\}$, is an example of asymptotically invariant collection.*

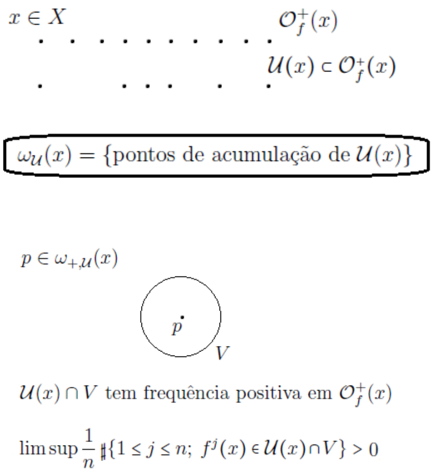
**Definition 1.3.10.** *The collection $\mathcal{U}$ has* positive frequency *if* $\limsup \frac{1}{n} \sharp\{1 \leq j \leq n; f^j(x) \in \mathcal{U}(x)\} > 0$ *for every $x \in \mathcal{H}$.*

In this case we define the set of $\mathcal{U}$-frequently visited points of the orbit $\mathcal{O}^+(x)$ as the set of points $p \in M$ such that $\limsup\limits_{n \longrightarrow +\infty} \dfrac{1}{n} \sharp\{1 \leq j \leq n; f^j(x) \in \mathcal{U}(x) \cap V\} > 0$ for every neighborhood $V$ of $p$. This set is denoted by $\omega_{+,f,\mathcal{U}}$.

**Notation 1.3.11.** *In the sequel we denote*

$$\varphi^V_{\mathcal{U}}(x) := \limsup_{n \longrightarrow +\infty} \frac{1}{n} \sharp\{1 \leq j \leq n; f^j(x) \in \mathcal{U}(x) \cap V\} > 0 \tag{1.14}$$

*The function $\varphi^V_{\mathcal{U}}(x)$ denotes the frequency of visits of the orbit of $x$ to the set $V$, but not considering every iterates, only those who belong to the collection $\mathcal{U}$ (we refer to this by using the term $\mathcal{U}-$visits of $x$ to the set $V$). In an analogous way, if $x \in V$, $\varphi^V_{\mathcal{U}}(x)$ denotes the frequency of $\mathcal{U}-$returns of the orbit of $x$ to the set $V$. If necessary, we use the notation $\varphi^V_{\mathcal{U},f}$ to emphasize the dynamics used in the required context.*




**Proposition 1.3.12.** *Assume that $(f, \mu)$ is ergodic. There is a compact set $\mathcal{A} \subset M$ such that $\omega_f(x) = \mathcal{A}$ for $\mu-$a.e.p. $x \in M$.*

**Proposition 1.3.13.** *Let $\mathcal{U} = (\mathcal{U}(x))_{x \in M}$ be an asymptotically invariant collection and let $\mathcal{A}$ be the attractor associated to $M$ (as in Proposition 1.3.12). There exists a compact $\mathcal{A}_{\mathcal{U}} \subset M$ such that $\omega_{f,\mathcal{U}}(x) = \mathcal{A}_{\mathcal{U}}$ for $\mu-$a.e.p. of $M$. Furthermore, if $\mathcal{U}$ has positive frequency then there is also a compact set $\mathcal{A}_{\mathcal{U},+} \subset M$ such that $\omega_{f,\mathcal{U},+}(x) = \mathcal{A}_{\mathcal{U},+}$ for $\mu-$a.e.p. of $M$.*

A proof for the last two results can be found in Section 3 of [24]. The set $\mathcal{A}$ is called an ergodic attractor whereas the sets $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\mathcal{U},+}$ are called $\mathcal{U}-$ergodic attractor and statistical $\mathcal{U}-$ergodic attractor, respectively. In our context one cannot expect that both $\mathcal{A}_{\mathcal{U}}$ and $\mathcal{A}_{\mathcal{U},+}$ have positive measure, but one always have $\mu(\mathcal{A}) > 0$ (see Proposition 3.12 of [24]).

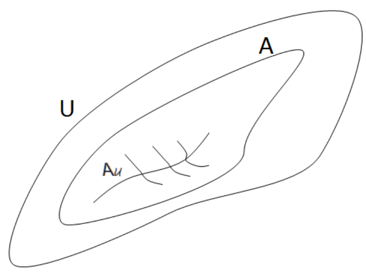



Figure 1.2: $U$ is an ergodic component with attractor $A$ and $\omega_{\mathcal{U}}$-limit set $A_{\mathcal{U}}$.

**Remark 1.3.14.** *It is worth to note that Propositions 1.3.12 and 1.3.13 are valid even if $\mu$ is an ergodic measure not necessarily invariant. When we consider the case where $\mu$ is an ergodic invariant measure, one can show that the ergodic attractor coincides with the support* $\text{supp}\mu$ *of $\mu$.*

**Remark 1.3.15.** *Consider a non-flat expanding map $f : M \longrightarrow M$ and $\mu \in \mathcal{M}^1_{\mathfrak{erp}}(f)$. For each $x \in M$ consider the collection of hyperbolic images of $x$, $h = (h(x))_{x \in M}$, defined as $h(x) = \{f^n(x), x \in H_n(\sigma, \varepsilon, f)\}$ (see Definition 1.3.1). It is easy to see (by using Lemma 1.3.3) that in fact the collection $h$ is an asymptotically invariant collection. Thus by Proposition 1.3.13 we obtain an $h-$ergodic attractor, that we call* a hyperbolic ergodic attractor *and denote by $\mathcal{A}^f_{hyp}$, such that $\omega_{f,h}(x) = \mathcal{A}^f_{hyp}$ for $\mu-$a.e.p. $x \in M$. Furthermore there is an $h-$statistical ergodic attractor, that we call* a statistical hyperbolic ergodic attractor *and denote by $\mathcal{A}^f_{hyp,+}$, such that $\omega_{f,h,+}(x) = \mathcal{A}^f_{hyp,+}$ for $\mu-$a.e.p. $x \in M$.*

**Lemma 1.3.16.** *Consider $\mu \in \mathcal{M}_{\mathfrak{erp}}(f)$ and suppose that the frequency of hyperbolic times of $x$ is bounded from below by $\theta' > 0$ for $\mu-$a.e. point $x \in M$. Then, there exists $N > 1$ and a set $B \subset M$ with $\varphi^B_{h,f}(x) \geq \theta' \cdot \frac{1}{N}$ for $\mu-$a.e.p. $x \in M$.*

**Proof:** Consider a finite cover of $M$ by open balls $B_1, B_2, \cdots B_N$. Let $\mathcal{A}_{f,+,hyp}$ be the statistical attractor for $f$ on $M$ given by Proposition 1.3.13. We can see by definition of $\mathcal{A}_{f,+,hyp}$ that if $B_i \cap \mathcal{A}_{f,+,hyp} = \varnothing$ then $\varphi^{B_i}_{f,hyp}(x) = 0$ for $\mu-$ a.e.p. $x \in M$. In this way, we can assume with no loss of generality that $B_i \cap \mathcal{A}_{f,+,hyp} \neq \varnothing$ $\forall 1 \leq i \leq N$ (we are not counting the sets $B_i$ such that $B_i \cap \mathcal{A}_{f,+,hyp} = \varnothing$). Since the collection $h = (h(x))_{x \in M}$ of the hyperbolic iterates in the orbit of each $x \in M$ is asymptotically invariant (see Definition 1.3.8 and Remark 1.3.9), we obtain that $\varphi^{B_i}_{f,hyp}$ is $f-$invariant $\forall 1 \leq i \leq N$. Then, from the ergodicity of $\mu$ we know that for each $j$ there is $k_j > 0$ such that $\varphi^{B_j}_{f,hyp} \equiv k_j(\mod \mu)$.

Now, consider a point $x \in M$ typical for $\mu$. Since the proportion of hyperbolic iterates in $\mathcal{O}^+_f(x)$ is higher than $\theta'$ we have

$$\varphi^{B_1 \cup \cdots \cup B_N}_{f,hyp}(x) \geq \theta'.$$

Take $B \in \{B_1, \cdots, B_N\}$ with

$$\varphi^B_{f,hyp}(x) = \max_{1 \leq j \leq k}\{\varphi^{B_j}_{f,hyp}(x)\}.$$

Then:

$$\theta' \leq \varphi^{B_1 \cup \cdots \cup B_N}_{f,hyp}(x) \leq \sum_{j=1}^{N} \varphi^{B_j}_{f,hyp}(x) \leq N \cdot \varphi^B_{f,hyp}(x),$$

that is, $\varphi^B_{f,hyp}(x) \geq \theta' \cdot \dfrac{1}{N}$.

□

We stress that by Lemma 1.3.16 we conclude that for the set $B$ obtained, not only the frequency of hyperbolic visits but the frequency of hyperbolic returns is bounded by $\dfrac{\theta'}{N}$. Now fix $0 < r < \delta$, where $\delta$ is the radius of hyperbolic balls given by Proposition 1.3.2. If we take a finite cover of $M$ with balls of radius $r$ one can see, arguing by induction on the dimension of the manifold, that the cardinality of this cover can be taken as

$\left(\dfrac{\mathrm{diam}M+1}{2\sqrt{d}r}\right)^d$, where $d$ is the dimension of the compact Riemannian manifold $M$. Then we may assume that $N$ in Lemma 1.3.16 satisfies

$$N = \left(\frac{\mathrm{diam}M+1}{2\sqrt{d}r}\right)^d \tag{1.15}$$

Consider an ergodic invariant expanding probability $\mu \in \mathcal{M}^1_{\mathfrak{exp}}(f)$.

**Definition 1.3.17.** *Denote* $\mathcal{B} := \{B_r(p), p \in M,\ B_r(p) \cap \mathcal{A}^f_{hyp,+} \neq \varnothing\}$ *as the collection of every balls with radius $r$ that intersects the statistical attractor* $\mathcal{A}^f_{hyp,+}$ *given by Remark 1.3.15. We define the* $\mu-$frequency of hyperbolic returns *as*

$$\theta_f := \sup_{B_r(p)\in\mathcal{B}} \left\{\varphi^{B_r(p)}_{f,hyp}\right\}.$$

See that we only need to consider balls that intersect the statistical attractor because the frequency of hyperbolic visits to open sets that do not intersect this attractor is always zero. Also, since the set $B$ obtained by Lemma 1.3.16 satisfies $\varphi^B_{f,hyp}(x) \geq \left(\dfrac{\mathrm{diam}M}{2\sqrt{d}r}\right)^{-d} \cdot \theta'$ for $\mu-$almost every $x \in B$, we conclude that

$$\theta_f \geq \left(\frac{\mathrm{diam}M+1}{2\sqrt{d}r}\right)^{-d} \cdot \theta' \tag{1.16}$$

As we will see later, the frequency of hyperbolic returns $\theta_f$ plays a key role in our constructions, since it is related to the integrability of return time functions with respect to suitable measures. For further details, see Chapters 3 and 4.

### 1.3.2 First hyperbolic time

Consider a non-flat map $f$ and suppose that there exists $(\sigma,\delta)-$hyperbolic times for almost every point with respect to a given $f-$invariant ergodic reference measure $\mu$. This allows us to introduce a map $h : M \longrightarrow \mathbb{Z}^+$ defined $\mu-$almost everywhere which assigns to $x \in M$ its first $(\sigma,\delta)-$hyperbolic time (in other words, $h(x) := \min\{n \in \mathbb{N};\ x \in H_n(\sigma,\delta,f)\}$).

## 1.4 Main Results

Denote by $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2},\ell,\delta,\theta,f)$ the set of all ergodic invariant expanding measures associated to a map $f$ for which the following requests are satisfied:

1 The frequency of hyperbolic returns $\theta_f$ is bounded from below by $\theta > 0$,

2 Almost every point $x \in M$ belongs to some pre-ball $V_n(x')$ that expands with respect to $\widetilde{f} := f^\ell$ in a rate controlled by $\sigma^{1/2}$, i.e.:

$$\mathrm{dist}(\widetilde{f}^{n-j}(y), \widetilde{f}^{n-j}(z)) \leq \sigma^{j/2}\mathrm{dist}(\widetilde{f}^{n}(y), \widetilde{f}^{n}(z));\ \forall y, z \in V_n(x') \text{ and } 1 \leq j \leq n,$$

where $0 < \sigma < 1$, and

3 The size of hyperbolic balls for the points in last assertion is at least $\delta$.

**Definition 1.4.1.** *If $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$ we say that $\mu$ is* an expanding measure for $f$ with bounded parameters.

**Remark 1.4.2.** *One can see that, by controlling the parameters of an expanding measure as in Definition 1.4.1, we can prevent that perturbations of this measure lack expanding behavior, that is, we have compactness on the set of measures with bounded parameters (see Theorem C).*

**Remark 1.4.3.** *See that the definition of expanding measure with bounded parameters is based on the geometric properties of hyperbolic times, and not in the analytical definition of them.*

**Theorem A.** *Consider $0 < \sigma_n < 1$ with $\sigma_n \longrightarrow \sigma_0$, $\delta, \theta > 0$, $\ell \in \mathbb{N}$ and a convergent sequence $f_n \longrightarrow f_0$ in the $C^1$ topology. For each $n \geq 1$ consider $\mu_n \in \mathcal{M}_{\mathfrak{exp}}(\sigma_n^{1/2}, \ell, \delta, \theta, f_n)$. Then there are a subsequence $\mu_{n_j}$ and an expanding measure $\mu_0 \in \mathcal{M}_{\mathfrak{exp}}(\sigma_0^{1/2}, \ell, \delta, \theta, f_0)$ such that $\mu_{n_j} \longrightarrow \mu_0$ in the weak$-*$ topology.*

**Theorem B.** *Consider a convergent sequence $f_n \longrightarrow f_0$ in the $C^1$ topology and a measure $\mu_0 \in \mathcal{M}_{\mathfrak{exp}}(\sigma_0^{1/2}, \ell, \delta, \theta, f_0)$. Then for $n \in \mathbb{N}$ big enough there exists $0 < \sigma_n < 1$ and $\mu_n \in \mathcal{M}_{\mathfrak{exp}}(\sigma_n^{1/2}, \ell, \delta, \theta, f_n)$ with $\sigma_n \longrightarrow \sigma_0$ such that $\mu_n \longrightarrow \mu_0$ in the weak$-*$ topology.*

The following result is a consequence of Theorem A.

**Theorem C.** *Let $g : M \longrightarrow M$ be a non-flat map. Then the set of expanding measures with bounded parameters $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, g)$ is compact in the weak$-*$ topology.*

**Proof:** It is enough to take the constant sequence of dynamics $g_n = g$ and apply Theorem A.

□

Denote $\mathcal{M}_+^1(f)$ as the set of all ergodic $f-$invariant probabilities whose all Lyapunov exponents are positive and $X$ as the set of non-flat maps endowed with the $C^1$ topology.

**Definition 1.4.4.** *We say that $\mathcal{M}_+^1(f)$* varies continuously in compact sets at *$f$ if it can be written as a nested union of compact sets $\mathcal{M}_{f,\imath}$ such that:*

*1* $\mathcal{M}_{f,1} \subset \mathcal{M}_{f,2} \subset \cdots \subset \mathcal{M}_{f,\imath} \subset \cdots$ *and* $\mathcal{M}^1_+(f) = \bigcup_{\imath \geq 1} \mathcal{M}_{f,\imath}$,

*2 For each* $\imath \in \mathbb{N}$ *there is a family of compact sets parameterized by some neighborhood of* $f$ $V \subset X$ $\Gamma_\imath : V \longrightarrow K(\mathcal{M}^1(M))$ *which is continuous at* $f$*. For* $g \in V$ *we denote* $\Gamma_\imath(g) = \mathcal{M}_{g,\imath}$.

**Main Theorem.** *Consider a non-flat map* $f : M \longrightarrow M$ *with exceptional set* $\mathcal{C} \subset M$ *which is constituted only by critical points or only by singular points. If* $\mathcal{M}^1_+(f) \neq \emptyset$ *then it varies continuously on compact sets at* $f$*.*

See that the continuity in item 2 of Definition is not necessarily uniform on $\imath \in \mathbb{N}$. Furthermore, its clear in the definition of continuous variation above that the set $\mathcal{M}^1_+(f)$ is $\sigma-$compact. It will be clear in the proof of Main Theorem how the sets $\mathcal{M}_{f,\imath}$ are chosen (in fact, the choice has to do with a proper adjustment of the parameters for a measure with bounded parameters and requirement that they form a nested collection of sets).

Theorems A and B both work in complementary directions. In Theorem A we construct an expanding measure for $f_0$ assuming that dynamics close to $f_0$ have some expanding measure. In Theorem B we construct an expanding measure for dynamics close enough to $f_0$ assuming that $f_0$ already has an expanding measure. In addition, in both Theorems the expanding measures for $f_n$ and $f_0$ are close, as long as $f_n$ is close enough to $f_0$. These Theorems are useful to obtain the continuous variation of expanding measures on compact sets, as we can see in Main Theorem. It is useful to see that the sets of measures with bounded parameters in Theorems A and B are in fact sets of expanding measures. Furthermore, by Theorem C we see that these sets are also compact sets.

## 1.5 Examples and applications

Before we proceed to the proof of the mains results, lets understand what happens with the set of expanding measures in some simple examples.

For start, one can see that even for small perturbations the set of invariant measures for a dynamics can change drastically. Denote $I = [0, 1]$.

**Example 1.5.1.** *Consider* $f$ *as the identity map on the interval* $I$ *(Figure 1.3) and* $g$ *as some close perturbation of* $f$ *as in Figure 1.4. While* $f$ *has an uncountable number of invariant measures,* $g$ *only has two invariant measures.*

Theorem 0.0.4 allows us to conclude that for every uniformly expanding map the set of expanding measures is a compact set and varies continuously with the map. This result is valid even for uniformly hyperbolic maps. However, outside the uniformly hyperbolic scenario this statement does not hold.

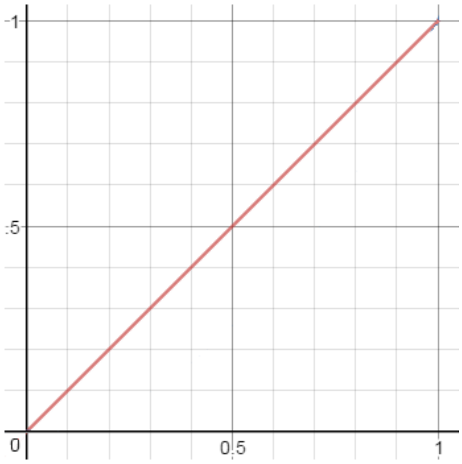



Figure 1.3:

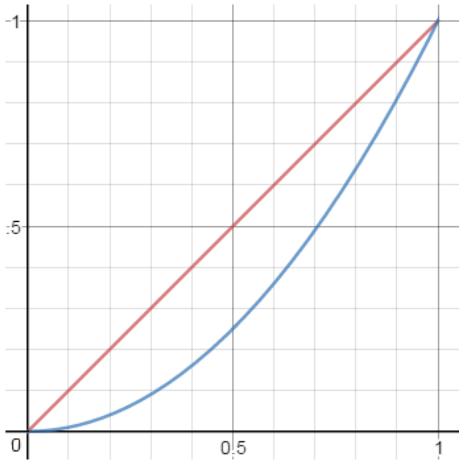



Figure 1.4:

**Example 1.5.2.** *Consider the complete tent map $f : I \longrightarrow I$ given by $f(t) = 1 - 2|t - 1/2|$ (see Figure 1.5). Then the set of invariant measures coincides with the set of invariant expanding measures. In fact, the obstruction to expanding behavior in this map is associated to the pre-orbit of $t_0 = 1/2$, denoted by $\mathcal{O}^-_f(t_0)$. But since $t_0$ is not a fixed point for $f$, any measure supported in a subset $U \subset \mathcal{O}^-_f(t_0)$ is not an invariant measure.*

*Then we conclude that the set of invariant expanding measures for $f$ is a compact set.*

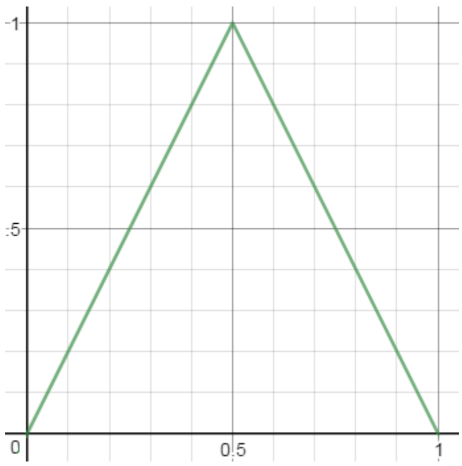



Figure 1.5:

**Example 1.5.3.** *Let $a > 0$ be such that $\dfrac{a}{4} < 1$ and consider the map $f : I \longrightarrow I$ given by:*

$$f(t) = \begin{cases} at^3 + (2-a)t^2 + \frac{a}{4}t & \text{if} \quad 0 \le t < 1/2 \\ 1 - 2|t - 3/4| & \text{if} \quad 1/2 \le t \le 1 \end{cases}$$

*(See Figure 1.6). We can see that the set of invariant expanding measures is compact (it is the set of invariant measures for the portion associated to the tent map) but $f$ is not an uniformly expanding map, since we have a negative Lyapunov exponent at $t = 0$ (see Section 2.4). So, the compactness of the set of expanding maps is not a sufficient condition for a map to be uniformly expanding.*

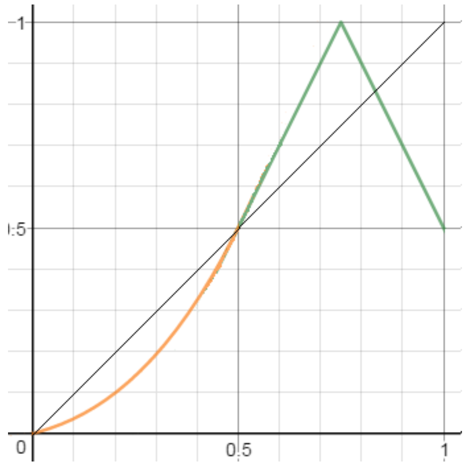



Figure 1.6:

One can see that in the non-uniformly expanding context, it may happens that the set of expanding measures is not compact. As we can identify by Theorem C, when we control some parameters associated to the measures, namely, the rate of expansion, the frequency of hyperbolic returns and the size of hyperbolic balls, then we obtain compactness. We refer to this kind of object as *an expanding measure with bounded parameters* (see formal definitions about this concept in Section 1.4). In the next example we explore the case when we do not control the rate of expansion.

**Example 1.5.4.** *Consider the map $f : I \longrightarrow I$ whose graph is given by Figure 1.7. For instance, we can say that this graph is obtained from a rotation of the graph of $g(t) = t^4 \cdot \sin \dfrac{\pi}{t}$. See that, for each $n \geq 1$, $\mu_n := \delta_{\frac{1}{2n}}$ is an expanding measure. However, $\mu_n \longrightarrow \mu_0 := \delta_0$, which is not an expanding measure for $f$. Then, even in lower dimension, the set of expanding measures is not compact.*

Although outside the uniformly hyperbolic context we cannot expect that the set of expanding measures is compact (or even that it varies continuously), Main Theorem tells us that for NUE maps this set is $\sigma-$compact and varies continuously on compact sets. By Theorem C we see that, even though the set of all expanding measures is not necessarily compact, when we control the parameters associated to the expanding measure, namely, the rate of expansion, the first iterate when we observe expansion, the frequency of hyperbolic returns and the size of hyperbolic balls, the set of measures with

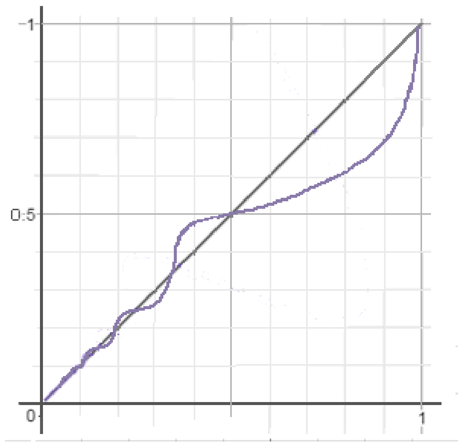



Figure 1.7:

bounded parameters will be compact. This is an important step in the construction of the compact pieces in the decomposition required in Definition 1.4.

In [24], Pinheiro gives us examples of some classes of maps which exhibits plenty of expanding measures. Between them we can cite:

**Example 1.5.5.** *Let $f : I \longrightarrow I$ be given by*

$$f(t) = \begin{cases} g(t), & t < 1/2 \\ 1 - g(1-t), & t \geq 1/2, \end{cases}$$

*where $g(t) = t + 2t^2$. By using Theorem 5 of [24] we can conclude that $f$ has an uncountable number of ergodic invariant expanding measures.*

**Example 1.5.6.** *Let $F : I^2 \longrightarrow I^2$ be given by*

$$F(x,y) = (f(x), (1+x)\phi(y)),$$

*where $f$ is as in Example 1.5.5 and $\phi(y) = 1/2 - |y - 1/2|$ is the "tent" map of slope one. Again, applying Theorem 5 of [24] we conclude that $F$ has an uncountable number of ergodic invariant expanding measures.*

**Example 1.5.7.** *An important class of non-uniformly expanding dynamical systems (with critical sets) in dimension greater than one was introduced by Viana in [34]. This class of maps can be described as follows: Consider $a_0 \in (1,2)$ such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $b : S^1 \longrightarrow \mathbb{R}$ be a Morse function, for instance, $b(s) = 2\pi s$. For fixed small $\alpha > 0$, consider the map $F : S^1 \times \mathbb{R} \longrightarrow S^1 \times \mathbb{R}$ defined as*

$$F(s,x) = (g(s), q(s,x)),$$

*where $g$ is the uniformly expanding map of the circle defined by $g(s) = d \cdot s (\operatorname{mod} \mathbb{Z})$ for some $d \geq 16$ and $q(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$.*

*Its possible to perform an analogous construction in higher dimensions.*

*By applying Theorem 7 of [24] we can conclude that $F$ admits an uncountable collection of ergodic invariant measures with all Lyapunov exponents positive (see Section 1.1 for precise definitions).*

In conclusion, we can apply our main results to these maps and obtain that every small perturbation of them in the $C^1$ topology gives rise to maps exhibiting expanding measures and the set of expanding measures varies continuously in compact sets as in Definition 1.4.

# Chapter 2

# Tools and strategies

In this Chapter we present some of the main tools and strategies used in the proof of the main theorems. Some of the results in this section are largely known in the literature but we include some proofs in here for completeness and to see how constants depend on each other.

## 2.1 Markov maps and liftable measures

Consider a mensurable map $F : U \longrightarrow U$ defined in a Borel set $U \subset M$ and a countable collection $\mathcal{P} = \{P_1, P_2, P_3, \cdots\}$ of Borel subsets of $U$ satisfying:

**Definition 2.1.1.** *We say that $\mathcal{P}$ is a* Markov partition *and $(F, \mathcal{P})$ is a* full Markov map *if the following assumptions are satisfied:*

- *$int(P_i) \cap int(P_j) = \emptyset$, if $i \neq j$*
- *$F|_P$ is a homeomorphism and can be extended to a homeomorphism sending $\overline{P}$ onto $\overline{U}, \forall P \in \mathcal{P}$;*
- *$\lim_j diam(\mathcal{C}_j(x)) = 0$, $\forall x \in \cap_{j \geq 0} F^{-j}(\cup_i P_i)$, where $\mathcal{C}_j(x) = \{y;\ \mathcal{P}(F^s(y)) = \mathcal{P}(F^s(x))\ \forall 0 \leq s \leq j\}$ and $\mathcal{P}(x)$ denotes the element of $\mathcal{P}$ which contains $x$. We call $\mathcal{C}_j(x)$ as the $j-$cylinder containing $x$ in the Markov partition $\mathcal{P}$.*

**Definition 2.1.2.** *Consider a function $F : U \longrightarrow U$ and a full Markov partition $\mathcal{P}$ with respect to $F$. The pair $(F, \mathcal{P})$ is called a* full Markov map induced by *$f$ defined on $U$ if there exists a function $R : U \longrightarrow \mathbb{N} = \{0, 1, 2, 3, \cdots\}$ (the inducing time) such that*

- *$\{R \geq 1\} = \{x \in U;\ R(x) \geq 1\} = \bigcup_{P \in \mathcal{P}} P$,*
- *$R|_P$ is constant $\forall P \in \mathcal{P}$,*

- $F(x) = f^{R(x)}(x)\ \forall x \in U$.

*We often say* induced Markov map *instead of* full Markov map induced by $f$ *when there is no chance of misunderstanding.*

**Definition 2.1.3.** *Given a full induced Markov map $(F, \mathcal{P})$, an ergodic $f$-invariant probability $\mu$ is said to be* liftable *to $F$ if there exists a finite $F$-invariant measure $\nu \ll \mu$ such that*

$$\mu = \sum_{P \in \mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P),$$

*where $R$ is the inducing time of $F$, $\nu|_P$ denotes the measure given by $\nu|_P(A) = \nu(A \cap P)$ and $f_*^j$ is the* push-forward *by $f^j$ $(f_*^j \nu = \nu \circ f^{-j})$.*

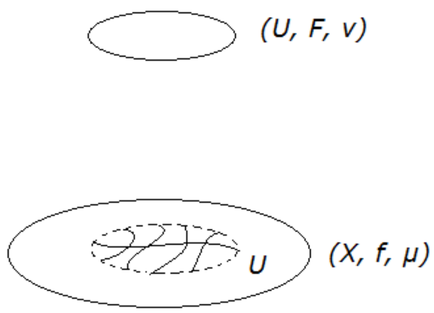



Figure 2.1:

**Definition 2.1.4.** *We say that an induced Markov map $(F, \mathcal{P})$ defined on an open set $Y \subset X$ is* compatible with a measure $\mu$ *if*

1. $\mu(Y) > 0$;
2. *$\mu$ is $F$-non-singular;*
3. $\mu(\bigcup_{P \in \mathcal{P}} P) = \mu(Y)$ *(in particular, $\mu(\partial P) = 0\ \forall P \in \mathcal{P}$).*

**Definition 2.1.5.** *We say that an induced Markov map $(F, \mathcal{P})$ defined on a set $Y \subset X$ has* bounded distortion with respect to a measure $\mu$ *(or, in a simplest way, has $\mu$-bounded distortion) if*

- *$(F, \mathcal{P})$ is compatible with $\mu$;*
- *$\mu$ has Jacobian with respect to $F$ and*

- $\exists K > 0$ *such that*

$$\left|\log \frac{J_\mu F(x)}{J_\mu F(y)}\right| \leq K \operatorname{dist}(F(x), F(y)),$$

*for* $\mu$ *almost every point* $x, y \in P$ *and for all* $P \in \mathcal{P}$

The problem of lift a measure was studied by several authors in the last years. Among them, Aaronson gives in [1] a condition to lift a measure $f$-invariant to the level of the induced Markov map (namely, bounded distortion for the measure), as we can see in Theorem 2.1.7. In [24] Pinheiro removes the bounded distortion condition, replacing it by a statistical condition (see Theorem 2.1.8). Also, we can project a given measure that is invariant for the induced Markov map, as long as its return time is integrable, as we can see in next Theorem.

**Theorem 2.1.6** (Folklore 1). *Let* $(F, \mathcal{P})$ *be a full induced Markov map for* $f$ *defined on some* $Y \subset M$ *and let* $R$ *be its inducing time. If* $\nu$ *is a finite* $F$*-invariant measure such that* $\int R d\nu < \infty$ *then*

$$\eta = \sum_{P \in \mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P) \left( = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}}) \right)$$

*is a finite* $f$*-invariant measure.*

**Proof:** We give here a sketch of the proof of this Theorem, which is based on Young towers (see [36]).

Consider $f : M \longrightarrow M$ and a full Markov map $(F, \mathcal{P})$ on $\Delta$. Define the set

$$\widehat{\Delta} := \{(z, n) \in \Delta \times \{0, 1, 2, \ldots\}; n < R(z)\},$$

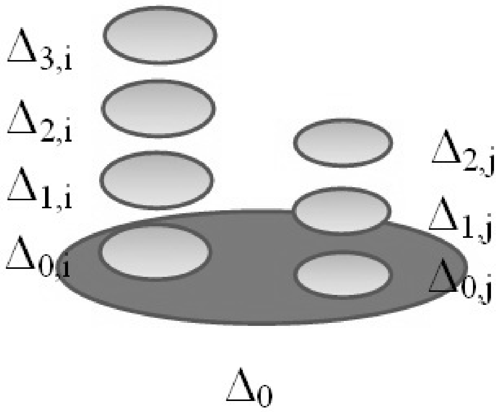
and consider the following dynamics on $\widehat{\Delta}$:

$$\mathbb{F}(x, l) = \begin{cases} (x, l+1) & \text{se } l+1 < R(x) \\ (f^{R(x)}(x), 0) & \text{se } l+1 = R(x) \end{cases} \tag{2.1}$$

See that $\widetilde{\nu}$ is $\mathbb{F}$–invariant. Indeed, suppose that $\nu$ is a $F$-invariant measure and consider the measure $\widetilde{\nu}$ defined on $\widehat{\Delta}$ given by $\widetilde{\nu}(A \times \{n\}) := \nu(A)$, for all $\mu$-mensurable subset $A \subset \Delta_{0,i}$ and $0 \leq n < R_i$. We may have two cases to consider:

- $n > 0$

  In this case, we have $\mathbb{F}^{-1}(A \times \{n\}) = (A \times \{n-1\})$ and hence $\widetilde{\nu}(\mathbb{F}^{-1}(A \times \{n\})) = \widetilde{\nu}((A \times \{n-1\})) = \mu(A) = \widetilde{\mu}(A \times \{n\})$.




- $n = 0$

  In this case:

  $\widetilde{\nu}(\mathbb{F}^{-1}(A \times \{n\})) = \widetilde{\nu}(\bigcup_{i\in\mathbb{N}}(F^{-1}(A) \cap \Delta_{0,i} \times \{R_i\})) = \sum_{i\in\mathbb{N}} \widetilde{\nu}(F^{-1}(A) \cap \Delta_{0,i} \times \{R_i\}) = \sum_{i\in\mathbb{N}} \nu(F^{-1}(A) \cap \Delta_{0,i}) = \nu(F^{-1}(A)) = \nu(A) = \widetilde{\nu}(A \times \{n\})$

We define the *projection* of the tower $\widehat{\Delta}$ on $M$ as the map $\pi : \widehat{\Delta} \longrightarrow M$ given by

$$\pi(x, n) = f^n(x).$$

See that, defined that way, the projection $\pi$ is continuous and satisfies:

$$f \circ \pi = \pi \circ \mathbb{F}, \tag{2.2}$$

To conclude the demonstration, it is enough to ensure the next steps:

- Clearly we have that $\pi_* \widetilde{\nu}$ is a $f$-invariant measure, because $\widetilde{\nu}$ is a $\mathbb{F}$−invariant measure.

- $\widetilde{\nu}(\widehat{\Delta}) = \int R d\nu$

  Indeed, $\widetilde{\nu}(\widehat{\Delta}) = \widetilde{\nu}(\bigcup_{i=1}^{\infty} \bigcup_{k=0}^{R_i-1} \Delta_{0,i} \times \{k\}) = \sum_{i=1}^{\infty} \widetilde{\nu}(\bigcup_{k=0}^{R_i-1} (\Delta_{0,i} \times \{k\})) = \sum_{i=1}^{\infty} R_i \nu(\Delta_{0,i}) = \sum_{i=1}^{\infty} \int_{\Delta_{0,i}} R d\nu = \int R d\nu$

- It is easy to check that:

$$\pi_* \widetilde{\nu} = \sum_{i\in\mathbb{N}} \sum_{k=0}^{R_i-1} f_*^k(\nu|_{\Delta_{0,i}}) \left( = \sum_{i\in\mathbb{N}} f_*^k(\nu|_{\{R>k\}}) \right). \tag{2.3}$$

Defining $\eta := \pi_* \widetilde{\nu}$ we conclude the proof.

□

**Theorem 2.1.7** (Folklore 2). *Let $\mu$ be a $f$-non-singular measure. If $(F,\mathcal{P})$ is a full induced Markov map for $f$ with $\mu$-bounded distortion then there exists an ergodic $F$-invariant probability $\nu \ll \mu$ which density belongs to $L^\infty(\mu)$. Indeed,* $\log \frac{d\nu}{d\mu} \in L^\infty(\mu|_{\{\frac{d\nu}{d\mu}>0\}})$.

*Furthermore, if the induction time $R$ of $F$ is integrable with respect to $\nu$, then* $\eta = \sum_{P\in\mathcal{P}} \sum_{j=0}^{R(P)-1} f_*^j(\nu|_P)$ *is a finite $f$-invariant ergodic measure absolutely continuous w.r.t. $\mu$.*

**Proof:** The second part of this Theorem is obtained directly from Theorem 2.1.6. See Lemma 4.4.1 of [1] for a proof of the first part.

□

Next Theorem, whose proof can be found in [24], ensures that we can lift a measure as long as some statistical condition is satisfied (replacing the hypothesis of bounded distortion as in Theorem 2.1.7).

**Theorem 2.1.8.** *Let $(F,\mathcal{P})$ be a full induced Markov map for $f$ defined on an open set $B \subset X$. Let $R$ be the inducing time of $F$ and $\mu$ an ergodic $f$-invariant probability measure such that $\mu(\{R=0\}) = 0$ and $\mathcal{O}_f^+(x) \cap \mathcal{O}_f^+(y) \neq \emptyset \Rightarrow \mathcal{O}_F^+(x) \cap \mathcal{O}_F^+(y) \neq \emptyset$ for $\mu$-a.e.p. $x, y \in B$. If there exists $\Theta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sharp\{0 \le j < n;\ f^j(x) \in \mathcal{O}_F^+(x)\} \ge \Theta \tag{2.4}$$

*for $\mu$ almost every $x \in B$, then there is a non trivial ($\not\equiv 0$), finite and $F$-invariant measure $\nu$ such that $\nu(Y) \le \mu(Y)$ for all Borel set $Y \subset B$ and such that $\int R d\nu \le \Theta^{-1}$.*

The following definition is useful to obtain a set of points where in each neighborhood the frequency of hyperbolic visits is positive for all $f_n$.

**Definition 2.1.9.** *Consider for each $j \in \mathbb{N}$ a set $\emptyset \neq A_j \subset M$. We define $A := \lim_{j\in\mathbb{N}} A_j$ as the* limit set of the sequence $A_j$ *given by* $A := \{x \in M,\ \exists x_{j_k} \in A_{j_k}\ \forall k \in \mathbb{N}$ such that $x_{j_k} \longrightarrow x$ when $k \longrightarrow +\infty\}$.

See that, since $M$ is compact, $A \neq \emptyset$. We can say that $A$ is the set of all accumulation points for sequences where each term belongs to one set $A_j$.

## 2.2 Nested sets

In this section we will see the notion of nested sets, as well as their main properties. Most of the material in this section is adapted from Sections 2 and 5 of [24]. We put it here as a matter of completeness for this text and to understand the dependence of some structures in the proof of the main theorems.

Nested sets are a generalization to the multidimensional case of nice intervals, introduced by Martens in [19]. Consider, for instance, a map $f : [0,1] \circlearrowleft$. A nice interval is an open interval $I$ such that the future orbit $\mathcal{O}^+(\partial I)$ of the boundary of $I$ doesn't return to $I$, that is, $\mathcal{O}^+(\partial I) \cap I = \varnothing$. These intervals are easy to construct when we consider interval maps. For example, two consecutive points of a periodic orbit define a nice interval. The main property of a nice interval we are interested in is that there are no linked pre-images of a nice interval, that is, if $I_1$ and $I_2$ are sent homeomorphically onte an open nice interval by $f^{n_1}$ and $f^{n_2}$ respectively then $I_1 \cap I_2 = \varnothing$, $I_1 \subset I_2$ or $I_2 \subset I_1$). As we can see, nice intervals become particularly useful when dealing with partitions. The same happens with nested sets, as we can see below.

Let $f : M \longrightarrow M$ be a map defined in a manifold $M$. A set $P \subset M$ is said to be a *regular pre-image of order* $n \in \mathbb{N}$ of a set $K \subset M$ if $f^n$ sends $P$ homeomorphically onto $K$. Lets denote by $\mathrm{ord}(P)$ the order of $P$ (with respect to $K$).

In this section we fix a collection $\mathcal{E}_0$ of open connected subsets of $M$. For each $n \in \mathbb{N}$ and $V \in \mathcal{E}_0$, consider a collection $\mathcal{E}_n(V)$ of regular pre-images of $K$ with order $n$. Note that we are not considering that the collection $\mathcal{E}_n(V)$ contains necessarily all regular pre-images of $V$ with order $n$. Define $\mathcal{E}_n = (\mathcal{E}_n(V))_{V \in \mathcal{E}_0}$.

**Definition 2.2.1.** *We say that the sequence* $\mathcal{E} = \{\mathcal{E}_n\}_n$ *is a* dynamically closed family of regular pre-images *if* $f^\ell(E) \in \mathcal{E}_{n-\ell}$ $\forall E \in \mathcal{E}_n$ *e* $\forall 0 \le \ell \le n$.

This condition ensures that if a regular pre-image of order $n$ of $V$ doesn't belong to $\mathcal{E}_n(V)$ then the regular pre-images of $V$ with order higher than $n$ don't belong to $\mathcal{E}(V)$ too. See Figure 2.2.

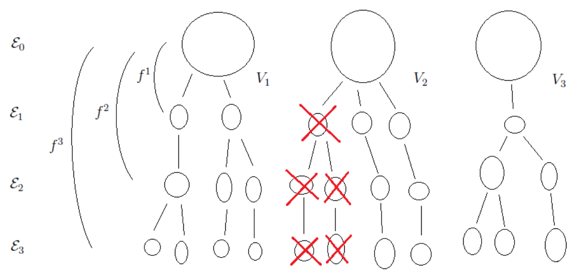



Figure 2.2: Scheme showing the construction of a dinamically closed family of regular pre-images. If we eliminate a set $\mathcal{E}_1$, we must eliminate all of its corresponding pre-images in $\mathcal{E}_2$ and $\mathcal{E}_3$ as well.

Given $Q \in \mathcal{E}_n$, we denote $f^n|_Q$ by $f^Q$ and denote by $f^{-Q}$ the $\mathcal{E}$-inverse branch, $(f^n|_Q)^{-1}$. Let $\mathcal{E} = (\mathcal{E}_n)_n$ be a dynamically closed family of regular pre-images. A set $P$ is called $\mathcal{E}$*−pre-image* of a set $W \subset X$ if there exists $n \in \mathbb{N}$ and $Q \in \mathcal{E}_n$ such that $\overline{W} \subset f^n(Q)$ and $P = f^{-Q}(W)$, where $\overline{W}$ is the closure of $W$.

**Remark 2.2.2.** *See that if two distinct $\mathcal{E}-$pre-images, $\mathcal{X}_1$ and $\mathcal{X}_2$, of a set $\mathcal{X} \subset X$ have the same order then they cannot intersect.*

**Definition 2.2.3.** *We say that two open sets $U_1$ and $U_2$ are linked if $U_1 \backslash U_2$, $U_2 \backslash U_1$ and $U_1 \cap U_2$ are not empty sets.*

Note that two open connected sets, $U_1$ and $U_2$, are linked if, and only if, $\partial U_1 \cap U_2$ and $U_1 \cap \partial U_2$ both are not empty sets.

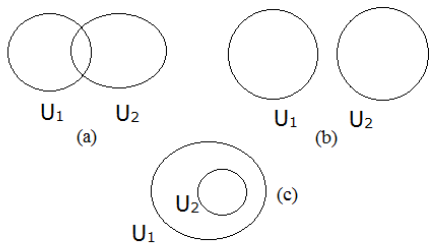



Figure 2.3: Some situations where two open connected sets may or may not intersect. Just in case (a) they are linked.

**Definition 2.2.4.** *A set $V$ is called $\mathcal{E}-$nested if it is open and it is not linked with any $\mathcal{E}-$pre-image of itself.*

The main property of a nested set is that any $\mathcal{E}-$pre-images $P_1$, $P_2$ of this set are not linked, as we can see in the next result.

**Proposition 2.2.5.** *If $V$ is a $\mathcal{E}-$nested set and $P_1$, $P_2$ are $\mathcal{E}-$pre-images of $V$, then the following assertions hold:*

1. *$P_1$ and $P_2$ are not linked*
2. *If $P_1 \cap P_2 \neq \emptyset$ and $P_1 \neq P_2$ then $ord(P_1) \neq ord(P_2)$;*
3. *If $P_1 \subsetneq P_2$ with $ord(P_1) < ord(P_2)$ then $V$ is contained in a $\mathcal{E}-$pre-image of itself with order bigger than zero (that is, $f^{ord(P_2)-ord(P_1)}(V) \subset V$).*

**Proof:** We show first that $P_1$ and $P_2$ are not linked sets. Let $k_j = \text{ord}(P_j)$, with $j \in \{1, 2\}$. We have two cases to consider. If $k_1 = k_2$, we have by Remark 2.2.2 that $P_1$ and $P_2$ cannot intersect, so they are not linked. If $k_1 \neq k_2$ (we may assume, for example, that $k_1 < k_2$, the other case is analogous), suppose by contradiction that $P_1$ and $P_2$ are linked. Consider also $p_j \in P_j \cap \partial P_{3-j}$ and $Q_j \in \mathcal{E}_{k_j}$ such that $P_j = f^{-Q_j}(V)$.

Since $\mathcal{E}$ is a dynamically closed family of pre-images of elements of $\mathcal{E}_0$, we have that $Q := f^{k_1}(Q_2) \in \mathcal{E}_{k_2-k_1}$ and that $P := f^{k_1}(P_2) = f^{-Q}(V)$ is an $\mathcal{E}$−pre-image of $V$. Besides, by construction we have that $f^{k_1}(P_1) = V$. Since $f^{k_1}(p_1) \in f^{k_1}(P_1) \cap \partial(f^{k_1}(P_2)) = V \cap \partial P$ and $f^{k_1}(p_2) \in f^{k_1}(P_2) \cap \partial(f^{k_1}(P_1)) = P \cap \partial V$, it follows that $P$ and $V$ are linked sets, what is impossible since $V$ is a $\mathcal{E}$−nested set. This proves *(1)*.

Suppose that $P_1 \cap P_2 \neq \varnothing$ and $P_1 \neq P_2$. By Remark 2.2.2 we have that $P_1$ and $P_2$ must have different orders, because they are $\mathcal{E}$−pre-images of the same set. This concludes *(2)*.

To conclude *(3)*, lets suppose that $P_1 \subsetneqq P_2$ and $k_1 < k_2$. Then $V = f^{k_1}(P_1) \subset f^{k_1}(P_2)$, that is, $V$ is contained in an $\mathcal{E}$−pre-image of itself with order $k_2 - k_1 > 0$. Since $f^{k_1}(P_2)$ is an $\mathcal{E}$−pre-image of $V$, we conclude that $f^{k_2-k_1}(V) \subset f^{k_2}(P_2) = V$.

□

### 2.2.1 Construction of $\mathcal{E}$−nested sets

We consider in this section an open connected subset $A$ of $M$ which is not contained in any $\mathcal{E}$−pre-image of itself with order bigger than zero. One can notice that this assumption is suitable in the non-uniformly expanding context, since in this case we expect to see pre-images with smaller size, in the sense of diameter.

A finite sequence $\mathcal{K} = (P_0, P_1, \cdots, P_n)$ of $\mathcal{E}$−pre-images of $A$ is called chain of pre-images of $A$ beginning in $A$ if

1. $0 < \operatorname{ord}(P_0) < \operatorname{ord}(P_1) < \cdots < \operatorname{ord}(P_{n-1}) < \operatorname{ord}(P_n)$;
2. $A$ and $P_0$ are linked sets;
3. $P_{j-1}$ and $P_j$ are linked sets, $\forall 1 \leq j \leq n$;

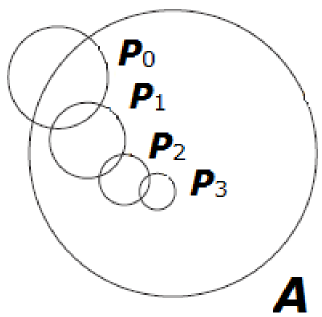



Figure 2.4: A chain $(P_0, P_1, P_2, P_3)$ of $\mathcal{E}$−pre-images beginning in $A$.

**Definition 2.2.6.** *Denote $ch_{\mathcal{E}}(A)$ as the collection of all chains of $\mathcal{E}$−pre-images of $A$ beginning in $A$. We denote by $A^*$ to the subset of $A$ defined as*

$$A^* = A \setminus \overline{\bigcup_{(P_j)_j \in ch_{\mathcal{E}}(A)} \bigcup_j P_j}. \tag{2.5}$$

Since $A$ is open and connected, we easily obtain that if $(P_0, P_1, \ldots, P_n) \in ch_{\mathcal{E}}(A)$, then $\bigcup_{j=n_0}^{n_1} P_j$ is an open connected set $\forall 0 \le n_0 \le n_1 \le n$. Also, we can see that $A^*$ is open, but not necessarily connected.

**Remark 2.2.7.** *If we use two distinct dynamics for this construction, lets say $f$ and $g$, we will use the notation $A^{*,f}$ and $A^{*,g}$ to distinguish whether we are talking about chains of pre-images by $f$ or by $g$.*

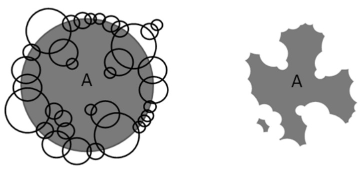



Figure 2.5: On the left we see a ball $A$ (in gray) and the boundaries of the $\mathcal{E}$−pre-images of $A$ in $ch_{\mathcal{E}}(A)$. On the right we depict $A^*$.

We proceed now with an abstract construction of a nested set. We can see in the next result that $A^*$ (or at least one of its connected components) is in fact a nested set. The proofs for next Proposition and its Corollary can be found in [24], Section 2.

**Proposition 2.2.8.** *Consider $A \subset M$ and suppose that $A^* \neq \emptyset$. If $A'$ is a connected component of $A^*$ then $A'$ is an $\mathcal{E}$−nested set.*

The previous gives a way to construct nested sets, but this depends on $A^*$ be non-empty. A way to ensure this condition is to show that all chains have small diameter (we consider the diameter of a chain $(P_j)_j$ as the diameter of the set $\bigcup_j P_j$). This is shown in next Corollary. Another way to ensure that $A^* \neq \emptyset$ is by assuming that the pre-images of a set are separated enough, as we can see below in Lemma 2.2.13.

**Corollary 2.2.9.** *Let $0 < \varepsilon < 1/2$ and let $A = B_r(p)$ be a connected open ball with radius $r$ centered in $p \in M$ such that $f^n(A) \not\subset A$, $\forall n > 0$. Suppose that every chain of $\mathcal{E}$−pre-images of $A$ has diameter smaller than $2\varepsilon r$. Then $A^*$ contains the ball $B_{r(1-2\varepsilon)}(p)$. Moreover, the connected component $A'$ of $A^*$ that contains $p$ is an $\mathcal{E}$−nested set containing $B_{r(1-2\varepsilon)}(p)$.*

### 2.2.2 Nested sets adapted to expanding structures

Here we present some concepts and results that allow us to bring together the notion of dynamically closed family of pre-images, nested sets and hyperbolic times. This should be useful to obtain Markov partitions in the non-uniformly expanding context and other important results that will follow.

Let $0 < \sigma < 1$ and $\delta > 0$ and for $f : M \longrightarrow M$ let $\mathcal{H}$ be the set of all points in $M$ with positive frequency of $(\sigma, \delta)$−hyperbolic times, that is, the set of points for which (1.8) holds. For example, if there exists $\mu \in \mathcal{M}_{\mathfrak{erp}}(f)$ then by Proposition 1.3.5 there exists $\mathcal{H} \subset M$ with $\mu(\mathcal{H}) = 1$ satisfying such a property. Denote by $\mathcal{E}_{\mathcal{H}} = (\mathcal{E}_{\mathcal{H},n})_n$ the collection of all $(\sigma, \delta)$−hyperbolic pre-balls, where $\mathcal{E}_{\mathcal{H},n} = \{V_n(x);\ x \in H_n(\sigma, \delta, f)\}$ is the collection of all $(\sigma, \delta)$−hyperbolic pre-balls of order $n$. By using Lemma 1.3.3, its easy to verify that the collection of all $(\sigma, \delta)$−hyperbolic pre-balls is a dynamically closed family of pre-images as in Definition 2.2.1.

Given $x \in M$ and $0 < r < \delta$, let $(B_r(x))^*$ be the set defined by (2.5) associated to $\mathcal{E}_{\mathcal{H}}$. If $x \in (B_r(x))^*$, it follows from Proposition 2.2.8 (by taking $A = \{B_r(x)\}$) that the connected component of $(B_r(x))^*$ that contains $x$ is an $\mathcal{E}_{\mathcal{H}}$−nested set.

**Definition 2.2.10.** *If $x \in (B_r(x))^*$, we define the $(\sigma, \delta)$−*hyperbolic nested ball *with respect to $f$ with radius $r$ and center at $x$ as the connected component of $(B_r(x))^*$ which contains $x$. We denote such a set as $B_r^*(x)$. We may use the notation $B_r^{*,f}(x)$, when it is necessary to emphasize the dynamics used in the construction of the $(\sigma, \delta)$−hyperbolic nested ball.*

Note that, since we have contraction in any hyperbolic time, $B_r(x)$ cannot be contained in any hyperbolic pre-image of itself (with order bigger than zero), that is, one cannot find a $\mathcal{E}_{\mathcal{H}}$−pre-image $P \in M$ of $B_r(x)$ such that $B_r(x) \subset P$ (and hence $\operatorname{diam}(B_r(x)) < \operatorname{diam}(P)$), otherwise we would obtain some $E \in \mathcal{E}_{\mathcal{H},n}$ and a contractive behavior (for the past) between $E \supset P$ and $f^n(E) \supset B_r(x)$. This would give us that $\operatorname{diam}(B_r(x)) \geq \operatorname{diam}(P)$, a contradiction. Hence the set $A = \{B_r(x)\}$ in Definition 2.2.10 above is indeed an open set according the hypothesis of Section 2.2.1.

**Remark 2.2.11.** *Note that, since two distinct $\mathcal{E}_{\mathcal{H}}$−pre-images of a set with the same order cannot intersect (see Remark 2.2.2), we obtain that the order of the elements of a chain of pre-images beginning in $B_r(x)$ is strictly decreasing, that is, if $(P_0, P_1, \cdots, P_n)$ is a chain of $\mathcal{E}_{\mathcal{H}}$−pre-images of $B_r(x)$ then $0 < \operatorname{ord}(P_0) < \cdots < \operatorname{ord}(P_n)$.*

**Definition 2.2.12.** *We say that $f$ is* backward separated *if for all $x \in M$ we have:*

$$\operatorname{dist}\left(x, \bigcup_{j=1}^{n} f^{-j}(x) \setminus \{x\}\right) > 0;\ \forall n \geq 1. \tag{2.6}$$

As an example of the previous definition, we can give a map $f$ with bounded number of pre-images: $(\sup \sharp\{f^{-1}(x);\ x \in M\} < +\infty)$. In fact, if there exists $k \in \mathbb{N}$ such that $(\sup \sharp\{f^{-1}(x);\ x \in M\} < k$, then $(\sup \sharp\{f^{-j}(x);\ x \in M\} < k^j,\ \forall j \in N$. Since the pre-images in this case always constitute a finite set, for all $j$, it follows that 2.6 holds. In particular, local diffeomorphisms are backward separated maps.

At section 2.2.1 we present an abstract construction of a nested set. To achieve this, it is necessary that the set $A^*$ given by (2.5) must be nonempty. Next result provides a condition for the existence of hyperbolic nested balls. This condition involves the previous definition. The proof of this result can be found in [24], but we include it here for the completeness of the text.

**Lemma 2.2.13.** *If* $\sum_{n\geq 1} \sigma^{n/2} < 1/4$ *then for every* $0 < r < \delta/2$ *the hyperbolic nested ball* $B_r^*(x)$ *is well defined and also* $B_r^*(x) \supset B_{r/2}(x)$. $\forall x \in M$. *Furthermore, if* $f$ *is backward separated then for each* $x \in M$ *there is* $0 < r_0 < \delta/2$ *such that* $B_r^*(x)$ *is well defined* $\forall 0 < r \leq r_0$ *and, given* $0 < \gamma < 1$, *it is possible find* $0 < r_\gamma < r_0$ *depending only on* $\delta, \alpha, x$ *and* $\gamma$ *such that* $B_r^*(x) \supset B_{\gamma r}(x)$; $\forall\ 0 < r \leq r_\gamma$.

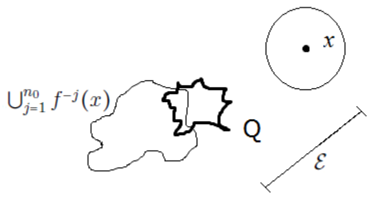



Figure 2.6: If $Q$ is a $\mathcal{E}_{\mathcal{H}}$ pre-image of $B_r(x)$ then $Q \cap B_r(x) = \varnothing$.

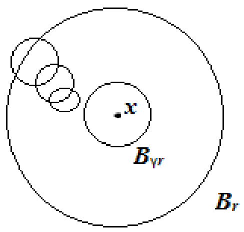



Figure 2.7: Every chain of pre-images in $ch_{\mathcal{E}_{\mathcal{H}}}(B_r(x))$ has diameter small enough in such a way that $\varnothing \neq B_{\gamma r}(x) \subset B_r^*(x)$.

**Proof:**

If $\sum_{n\geq 1} \sigma^{n/2} < 1/4$ and $0 < r < \delta/2$, since the orders of the elements in a chain beginning in $\mathcal{A} = \{B_r(x)\}$, are strictly decreasing (Remark 2.2.11), we have by item (2) of 1.3.2 that if $\mathcal{K} = (P_0, P_1, \cdots, P_k)$ is a chain of $\mathcal{E}_{\mathcal{H}}$−pre-images of $B_r(x)$ with $\mathrm{ord}(P_j) = n_j$ then $\mathrm{diam}(\mathcal{K}) = \sum_{j=1}^{j=k} \mathrm{diam}(P_j) \leq \sum_{j=1}^{j=k} \sigma^{n_j/2}\mathrm{diam}(B_r(x)) \leq \sum_{n\geq 1} \sigma^{n/2}\mathrm{diam}(B_r(x)) < \mathrm{diam}(B_r(x))/4 = r/2$. Thus, by using Corollary 2.2.9, we obtain that $B_r^*(x) \supset B_{r/2}(x)$.

Lets suppose that $f$ is backward separated. Given $0 < \gamma < 1$, since $\sum_{n\geq 1} \sigma^{n/2} < +\infty$

we may obtain $n_0 \in \mathbb{N}$ such that $\sum\limits_{n>n_0} \sigma^{n/2} \cdot r < (1-\gamma)r/2$. Given $x \in M$, let $\epsilon > 0$ be such that $\operatorname{dist}\big(x, \bigcup_{j=1}^{n_0} f^{-j}(x)\backslash\{x\}\big) > \epsilon$, $r_\gamma = \frac{1}{3}\min\{\epsilon,\delta\}$ and $0 < r \le r_\gamma$.

Notice that if $j < n_0$ and $Q$ is an $\mathcal{E}_{\mathcal{H}}$−pre-image of $B_r(x)$ with order $j$ then $B_r(x) \cap Q = \varnothing\ \forall Q \in \mathcal{E}_{\mathcal{Z},j}$. In fact, writing $Q = f^{-V}(B_r(x)) = (f^j|_V)^{-1}(B_r(x))$ for some $V$ in $\mathcal{E}_{\mathcal{H},j}$ we have that $Q \cap (\bigcup\limits_{j=1}^{n_0} f^{-j}(x)) = (f^j|_V)^{-1}(B_r(x)) \cap (\bigcup\limits_{j=1}^{n_0} f^{-j}(x)) \supset (f^j|_V)^{-1}(x) \cap (\bigcup\limits_{j=1}^{n_0} f^{-j}(x)) \neq \varnothing$. Since $\operatorname{dist}\big(x, \bigcup_{j=1}^{n_0} f^{-j}(x)\backslash\{x\}\big) > \epsilon$ and $\operatorname{diam}(Q) < 2r < 2\epsilon/3$ (last inequality is a consequence of the hypothesis that $Q$ is an $\mathcal{E}_{\mathcal{H}}$−pre-image of $B_r(x)$ along with item (2) of 1.3.2), we have that $B_r(x) \cap Q = \varnothing$ (see Figure 2.2.2).

Then, every chain of $\mathcal{E}_{\mathcal{H}}$−pre-images of $B_r(x)$ starts with a pre-image with order bigger than $n_0$. Let $(P) = (P_0, P_1, \cdots, P_k)$ be a chain of $\mathcal{E}_{\mathcal{H}}$−pre-images of $B_r(x)$ with $\operatorname{ord}(P_j) = n_j$. Then, since $\operatorname{diam}(P) = \operatorname{diam}(\bigcup\limits_{j=1}^{k} P_j) \le \sum\limits_{j=1}^{k} \operatorname{diam}\,(P_j)$ and, for each $j \in \{1, \cdots, k\}$, we have $\operatorname{diam}(P_j) = \sup\limits_{x,y\in P_j} \operatorname{dist}(x,y) \le \sup\limits_{x,y\in P_j} \sigma^{n_j/2} \cdot \operatorname{dist}(f^{n_j}(x), f^{n_j}(y)) = \sigma^{n_j/2} \cdot \sup_{x,y\in P_j}(\operatorname{dist}(f^{n_j}(x), f^{n_j}(y))) = \sigma^{n_j/2} \cdot \operatorname{diam}(B_r(x))$, we conclude that:

$$\begin{aligned} \operatorname{diam}(P) \le \sum_{j=1}^{k} \sigma^{n_j/2} \cdot \operatorname{diam}(B_r(x)) &\le \sum_{n>n_0} \sigma^{n/2} \cdot \operatorname{diam}(B_r(x)) \\ &< (1-\gamma)\frac{\operatorname{diam}(B_r(x))}{2} = (1-\gamma)r \end{aligned}$$

and, since any chain intersects the boundary of $B_r(x)$, we can conclude that this chain doesn't intersect $B_{\gamma r}(x)$ (see Figure 2.2.2). Hence, $(B_r(x))^*$ (and also $B_r^*(x)$) contains $B_{\gamma r}(x)$.

□

It is worth to notice that even if for a given $f : M \longrightarrow M$ with almost every point having $(\sigma,\delta)$− hyperbolic times but $\sum\limits_{n\ge1} \sigma^{n/2}$ fails to be smaller than $1/4$ we can still apply Lemma 2.2.13 above replacing $f$ by some iterate $f^\ell$. If $\ell$ is big enough we can ensure that the sum $\sum\limits_{n\ge1} \sigma^{\ell \cdot n/2}$ is smaller than $1/4$.

## 2.3 Full induced Markov map for an expanding map

Fix $\mu \in \mathcal{M}^1_{\mathfrak{exp}}(f)$ and consider $h = (h(x))_{x\in M}$ as the collection of hyperbolic images of points in $M$. Its not hard to show that $h$ is indeed an asymptotically invariant collection and it has positive frequency (see Proposition 1.3.5). Consider a hyperbolic nested set $\Delta \subset M$. Given $x \in \Delta$, let $\Omega(x)$ be the collection of hyperbolic pre-images $V$ of $\Delta$ such that $x \in V$.

**Definition 2.3.1.** *The inducing time on $\Delta$ associated to the "first hyperbolic return to $\Delta$" is the function $R : \Delta \longrightarrow \mathbb{N}$ given by*

$$R(x) = \begin{cases} \min\{\text{ord}(V);\ V \in \Omega(x)\} & se\ \Omega(x) \neq \varnothing \\ 0 & se\ \Omega(x) = \varnothing \end{cases}. \tag{2.7}$$

**Definition 2.3.2.** *The induced map $F$ on $\Delta$ associated to the "first hyperbolic return to $\Delta$" is the map $F : \Delta \longrightarrow \Delta$ given by*

$$F(x) = f^{R(x)}(x),\ \forall x \in \Delta. \tag{2.8}$$

Since the collection of sets $\Omega(x)$ is totally ordered by inclusion, it follows that there exists an unique $I(x) \in \Omega(x)$ such that $\text{ord}(I(x)) = R(x)$, whenever $\Omega(x) \neq \varnothing$.

**Definition 2.3.3.** *The Markov partition associated to the "first hyperbolic return to $\Delta$" is the collection $\mathcal{P}$ of open sets given by*

$$\mathcal{P} = \{I(x);\ x \in \Delta\ e\ \Omega(x) \neq \varnothing\}. \tag{2.9}$$

With these definitions, Pinheiro ensures in [24] that it is possible to construct an induced Markov map for $f$ on an appropriate nested set $\Delta \subset M$: We must to assume that $\Delta$ intersects the hyperbolic attractor associated to $\mu$ (see ).

**Proposition 2.3.4.** *Consider an open nested hyperbolic set $\Delta \subset M$ with diam$(\Delta) < \delta/2$. Suppose in addition that $\Delta \cap \mathcal{A}^f_{hyp} \neq \varnothing$, where $\mathcal{A}^f_{hyp}$ is a compact set such that $\omega_{f,h}(x) = \mathcal{A}^f_{hyp}$ for $\mu$–a.e.p. $x \in M$. Then $(F, \mathcal{P})$ given by 2.8 and 2.9 is a full induced Markov map defined on $\Delta$ with induced time $R : \Delta \longrightarrow \mathbb{N}$ given by 2.7 and $F$ is compatible with $\mu$.*

**Proof:** See Corollary 6.6 and Lemma 6.7 of [24]

□

The next result ensures that we can lift $\mu$ to the level of the induced Markov map $F$ given by 2.3.4.

**Proposition 2.3.5.** *Consider $\mu \in \mathcal{M}_{\mathfrak{erp}}(f)$, an open nested hyperbolic set $\Delta \subset M$ with diam$(\Delta) < \delta/2$ and suppose that $\Delta \cap \mathcal{A}^f_{hyp,+} \neq \varnothing$, where $\mathcal{A}_{hyp,+}$ is a compact set such that $\omega_{f,h,+}(x) = \mathcal{A}^f_{hyp,+}$ for $\mu$–a.e.p. $x \in M$. Suppose in addition that $(F, \mathcal{P})$ is the first hyperbolic return to $\Delta$ with induced time $R : \Delta \longrightarrow \mathbb{N}$ ($F = f^R$). Then, there exists $\xi > 0$ such that $\varphi^{\Delta}_f(x) \geq \xi$ for $\mu$–a.e.p. $x \in M$.*

*Furthermore, there is a finite $F$-invariant measure $\nu \ll \mu$ (indeed, $\nu(Y) \leq \mu(Y)$ for all Borel subset $Y \subset \Delta$) such that $\int R d\nu < \frac{1}{\xi} < +\infty$ and*

$$\mu = \frac{1}{\gamma} \sum_{j=0}^{+\infty} f^j_*(\nu|_{\{R>j\}}),$$

*where $\gamma = \sum_{j=0}^{+\infty} f^j_*(\nu|_{\{R>j\}})(M)$.*

**Proof:** We know that $h = (h(x))_{x\in M}$ is an asymptotically invariant collection. Then the function $\varphi_f^\Delta(x) = \limsup_{n\longrightarrow+\infty} \frac{1}{n} \sharp\left\{1 \le j \le n;\ f^j(x) \in h(x) \cap \Delta\right\}$ is $f$-invariant. By ergodicity of $\mu$ we conclude that there exists $\xi > 0$ such that $\varphi_f^\Delta(x) \ge \xi$ for $\mu$–a.e.p. $x \in \Delta$.

By taking $B = \Delta$, $g = R$, $G_j = H_j(f)$ and using Lemma A.0.2 we obtain that

$$\limsup_{n\longrightarrow+\infty} \frac{1}{n} \sharp\left\{j \ge 0;\ \sum_{k=0}^{j} R \circ F^k(x) \le n\right\} \ge \xi \tag{2.10}$$

for $\mu$–a.e.p. $x \in \Delta$. Since

$$\left\{j \ge 0;\ \sum_{k=0}^{j} R \circ F^k(x) \le n\right\} = \left\{0 \le j < n;\ f^j(x) \in \mathcal{O}_F^+(x)\right\},$$

it follows from 2.10 and Theorem 2.1.8 that there exists a non-trivial $F$–invariant measure such that $\nu(V) \le \mu(V)$ for every Borel set $V \subset \Delta$ (in particular, $\nu \ll \mu$) with $\int R d\nu \le \frac{1}{\xi} < +\infty$.

Now, we use Theorem 2.1.6 to obtain a $f$–invariant finite measure $\eta = \sum_{j=0}^{+\infty} f_*^j(\nu|_{R>j})$ which is absolutely continuous with respect to $\mu$. By the fact that $\mu$ is ergodic, we then obtain:

$$\mu = \frac{1}{\gamma} \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}}),$$

where $\gamma = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})(M)$.

□

**Remark 2.3.6.** *We denote by $\mu(f,\nu)$ as the measure $\frac{1}{\gamma} \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})$, $\gamma = \sum_{j=0}^{+\infty} f_*^j(\nu|_{\{R>j\}})(M)$ obtained from $\nu$.*

## 2.4 Lyapunov exponents

In Dynamical Systems, Lyapunov exponents are largely known in connection with a theorem due to Oseledets (see [22]), which in broad terms say that, under an integrability condition, the tangent space of almost every point splits into a flag in such a way that the iterates of vectors have well-defined rates of exponential growth, in norm, restricted to each subspace of the flag.

More formally, the *Multiplicative Ergodic Theorem*, due to Oseledets, says that if $\mu$ is an ergodic measure such that $\log^+ \|Df(x)\|$ is integrable then there exists numbers

$$\widetilde{\lambda}_1 > \cdots > \widetilde{\lambda}_k$$

such that for $\mu-$almost every point $x \in M$ there is a flag

$$\{0\} = E_1^x \subset E_2^x \subset \cdots E_{k(x)}^x = T_x M$$

which depends measurably on the point $x$ and is invariant by $Df$, such that for $i = 1, 2, \cdots, k(x) - 1$,

$$\lim_{n \longrightarrow +\infty} \frac{1}{n} \log \|Df^n(x)v\| = \widetilde{\lambda}_i, \text{ for all } v \in E_{i+1}^x \backslash E_i^x.$$

It is easy to see that each $\widetilde{\lambda}_i$ is an $f-$invariant function ($\widetilde{\lambda}_i(x) = \widetilde{\lambda}_i(f(x))$). In particular, if $\mu$ is ergodic the functions $\widetilde{\lambda}_i$ are constant almost everywhere. Let

$$\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_d(x)$$

be the numbers $\widetilde{\lambda}_j$ listed in a nondecreasing order and repeated with multiplicity $\dim E_{i+1}^x$ - $\dim E_i^x$. These numbers are called *the Lyapunov exponents* of $f$ at $x$.

In other words, we can say that Lyapunov exponents measure the asymptotic behavior of tangent vectors under iteration. Positive exponents corresponding to exponential growth and negative exponents corresponding to exponential decay of the norm. The uniformly hyperbolic case corresponds to nonzero Lyapunov exponents. The non-uniformly expanding case corresponds to positive Lyapunov exponents.

**Remark 2.4.1.** *It's worth to say that in the one-dimensional case, be non-uniformly expanding is equivalent to have a positive Lyapunov exponent, since:*

$$\limsup_{n \longrightarrow +\infty} \frac{1}{n} \log |(f^n)'(x)| = \limsup_{n \longrightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(f^n(x))|.$$

*In dimension greater than one condition 1.2 is not equivalent to say that $f$ has dim$(M)$ positive Lyapunov exponents at $x \in M$.*

Let $f : M \longrightarrow M$ be a non-flat map and $\mathcal{C} \subset M$ its critical/singular set.

**Definition 2.4.2.** *Let $\mu$ be an $f-$invariant ergodic probability. We say that $\mu$ has all of its Lyapunov exponents finite if* $\limsup_{n \in \mathbb{N}} \frac{1}{n} \log \|(Df^n(x))^{-1}\|^{-1} > -\infty$ *for $\mu-$almost every $x \in M$.*

**Remark 2.4.3.** *Suppose that there are $\Delta \subset M$, a $\mu-$partition $\mathcal{P}$ of $\Delta$, $R : \bigcup_{P \in \mathcal{P}} P \longrightarrow \mathbb{N}$ and an induced Markov map $F = f^R$ on $\Delta$, as in Definitions 2.3.1, 2.9 and 2.8 (Note that in this case we are considering $f$ as a non-uniformly expanding map). Then, by using item a) of 1.3.4 we obtain that $F$ is a piecewise expanding map: There is $0 < \kappa < 1$ such that for $x$ in the interior of the elements $P \in \mathcal{P}$*

$$\|Df(x)^{-1}\| < \kappa.$$

*Additionally, we conclude that all Lyapunov exponents for $F$ are positive on $\Delta$.*

**Lemma 2.4.4.** *If $\lambda$ is a Lyapunov exponent of $F$, then $\lambda/\kappa$ is a Lyapunov exponent of $f$, where $\kappa := \int_\Delta R d\nu$.*

**Proof:** Let $n$ be a positive integer and for each $x \in \Delta$ define $S_n(x) := \sum_{i=0}^{n-1} R(F^i(x))$. By using induction on $n$, the following equation is easily satisfied:

$$F^n(x) = f^{S_n(x)}(x).$$

By construction, we know that $S_n(x) = S_n(y)$ for almost every $y$ near enough of $x$. Thus we can take derivatives of the above equation and conclude that if $v \in T_x M$ then:

$$\frac{1}{S_n(x)} \log \|Df^{S_n(x)}(x) \cdot v\| = \frac{n}{S_n(x)} \frac{1}{n} \log \|DF^n(x) \cdot v\|.$$

Since $\nu$ is an ergodic measure, Birkhoff's ergodic theorem allows us to conclude that

$$\lim_{n \longrightarrow +\infty} \frac{S_n(x)}{n} = \int_\Delta R d\nu = \kappa$$

for $\mu$ almost every $x \in \Delta$.

If $\lambda$ is a Lyapunov exponent of $f$ then

$$\lambda = \lim_{n \longrightarrow +\infty} \frac{1}{n} \log \|DF^n(x) \cdot v\|.$$

Attending to the equations above, we conclude that

$$\frac{\lambda}{\kappa} = \lim_{n \longrightarrow +\infty} \frac{n}{S_n(x)} \frac{1}{n} \log \|DF^n(x) \cdot v\| = \lim_{n \longrightarrow +\infty} \frac{1}{S_n(x)} \log \|Df^{S_n(x)}(x) \cdot v\|$$

is a Lyapunov exponent of $f$ (note that, since $\mu$ is ergodic, the above expression holds for $\mu$−almost every point $x \in M$).

□

The following Lemma can be useful to prove that an ergodic invariant measure $\mu_0$ must be expanding.

**Lemma 2.4.5.** *Let $f : M \longrightarrow M$ be a $C^{1+\alpha}$ map. If $\mu$ is an $f$−invariant ergodic probability with all of its Lyapunov exponents finite (i.e., $\limsup \frac{1}{n} \log \|(Df^n(x))^{-1}\|^{-1} > -\infty$ for $\mu$−a.e.p. $x \in M$) then $\mu$ satisfies the slow approximation condition, that is, for each $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\limsup_{n \longrightarrow +\infty} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \le \varepsilon.,$$

*for $\mu$−a.e.p. $x \in M$.*

**Remark 2.4.6.** *The proof of Lemma 2.4.5 can be found in [24], Lemma* 10.2*. In this case, Pinheiro uses the assumption that the exceptional set $\mathcal{C}$ for $f$ is constituted just by critical points (where the derivative fails to be invertible).*

With Lemma 2.4.5 and by the fact that if $x$ is not a critical point then

$$\limsup_{n\in\mathbb{N}} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\|^{-1} \leq \limsup_{n\in\mathbb{N}} \frac{1}{n} \log \|(Df^n(x))^{-1}\|^{-1},$$

we easily conclude the following Remark.

**Remark 2.4.7.** *Consider $f : M \longrightarrow M$ as a non-flat map. Then an ergodic invariant probability $\mu$ is expanding for $f$ if, and only if, 1.2 holds for $\mu-$almost every point $x \in M$.*

# Chapter 3

# Technical stability conditions

We will start now some technical considerations. Consider a Riemannian compact manifold $M$ and a sequence of $C^{1+\alpha}$ local diffeomorphisms $f_n : M \longrightarrow M$, $n \geq 0$ with $f_n \longrightarrow f_0$ in the $C^1$ topology. Now, consider the following reasonable hypothesis (since we are working in the context of hyperbolic times, these hypothesis make sense): There exists a topological disk $\Delta \subset M$ and for each $n \geq 0$ there is an open set $A_n$ such that $f_n^{-1} : \Delta \longrightarrow A_n$ is a homeomorphism (in fact, a diffeomorphism) that can be extended to a homeomorphism (in fact, a diffeomorphism) from $\partial\Delta \cup \Delta$ onto $\partial A_n \cup A_n$. To be precise, we should write $(f_n|_{A_n})^{-1}$ instead of $f_n^{-1}$, but we will keep this last notation and warn the reader when there is a chance of misunderstanding.

We want to show that $A_n \longrightarrow A_0$ in the sense of Hausdorff distance:

**Definition 3.0.1.** *For each $n \geq 0$ consider a connected open set $V_n \subset M$. We say that $V_n$* converges to $V_0$ *if $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow \partial V_n \subset V_\varepsilon(\partial V_0) = \{x \in M,\ d(x, \partial V_0) < \varepsilon\}$ and $\partial V_0 \subset V_\varepsilon(\partial V_n)$. In this case we denote $V_n \longrightarrow V_0$.*

**Remark 3.0.2.** *Before start next Lemma, we want to remark a matter of notation: If $f_n \longrightarrow f_0$ in the $C^1$ topology this is the same as $d(f_n, f_0) \longrightarrow 0$. In this case, $d(\cdot,\cdot)$ denotes the distance associated to the $C^1$ topology. But $f_n \longrightarrow f_0$ also implies that* $\sup\{d(f_n(x), f_0(x)\} \longrightarrow 0$, *where* sup *in this expression is taken on $M$. In this case, $d(\cdot,\cdot)$ denotes the distance of points in $M$. We will denote both distances by $d(\cdot,\cdot)$ and warn the reader if there is a chance of misunderstanding.*

**Lemma 3.0.3.** *Suppose that $f_n : M \longrightarrow M$, $n \geq 1$, is a convergent sequence in the $C^1$ topology, $A_n$ is an open connected subset of $M$ and, for a fixed open connected set $\Delta \subset M$, the restriction $f_n^{-1} : \Delta \longrightarrow A_n$ is a diffeomorphism from the open set $\Delta$ onto $A_n$ (that can be extended to $\partial\Delta \cup \Delta$). Then there exists a connected open set $A_0 \subset M$ such that*

$$A_n \longrightarrow A_0.$$

*If we denote* $f_0 := \lim\limits_{n\longrightarrow+\infty} f_n$*, then the restriction* $f_0|_{A_0} : A_0 \longrightarrow \Delta$ *is also a diffeomorphism from* $A_0$ *onto* $\Delta$ *(which can be extended to* $\partial A_0 \cup A_0$*).*

**Proof:** Consider $f_i$, for some $i \in \mathbb{N}$ fixed. Since $f_i^{-1} : \Delta \longrightarrow A_i$ is a diffeomorphism, we take some $a \in A_i$ such that $Df_i(a)$ is injective. Defining $c_i := 2\|Df_i(a)^{-1}\|^{-1}$ it is easily seen that $\|Df(a)\cdot v\| \geq 2c\cdot\|v\|$, $\forall v \in T_aM$. It is a well known fact from Calculus that there is $\delta > 0$ such that:

$$d(f_i(x), f_i(y)) \geq c_i d(x,y),\ \forall x,\ y \in B_\delta(a).$$

It is possible to show that $c_i$ and $\delta$ may be taken uniformly for all $a \in A_i$ (since we are taking $f_i|_{A_i}$ as a diffeomorphism on it's image $\Delta$). Therefore, the previous equation becomes:

$$d(f_i(x), f_i(y)) \geq c_i d(x,y),\ \forall x,\ y \in A_i. \tag{3.1}$$

Now, consider $m, n \in \mathbb{N}$ and $x \in \partial\Delta \cup \Delta$. By triangle inequality we know that :

$$\begin{aligned} 0 = d(x,x) &= d(f_m \circ f_m^{-1}(x), f_n \circ f_n^{-1}(x)) \\ &\geq d(f_m \circ f_m^{-1}(x), f_m \circ f_n^{-1}(x)) - d(f_m \circ f_n^{-1}(x), f_n \circ f_n^{-1}(x)) \end{aligned}$$

By using the previous inequality and 3.1, we obtain:

$$\begin{aligned} d(f_m, f_n) &\geq d(f_m \circ f_n^{-1}(x), f_n \circ f_n^{-1}(x)) \\ &\geq d(f_m \circ f_m^{-1}(x), f_m \circ f_n^{-1}(x)) \\ &\geq c_{m,n}\, d(f_m^{-1}(x), f_n^{-1}(x)), \end{aligned}$$

where in the first inequality we used the fact that $d(f_m, f_n) \geq d(f_m(w), f_n(w))$, $\forall w \in M$, in the second we used the triangle inequality and in the third we used 3.1. So, we get:

$$d(f_n^{-1}(x), f_m^{-1}(x)) \leq \frac{1}{c_{n,m}} \cdot d(f_n, f_m), \tag{3.2}$$

for all $x \in \partial\Delta \cup \Delta$.

Note that $\|f_n\| \longrightarrow \|f_0\|$, since $f_n \longrightarrow f_0$ and $\|f_0\| \neq 0$, since $f_0$ is local diffeomorphism. Then the constants $c_{n,m}$ are bounded away from zero and infinity, and so are $\dfrac{1}{c_{n,m}}$. In this way one can find $K > 0$ such that $\dfrac{1}{c_{n,m}} < K$, $\forall n, m \in \mathbb{N}$ and last equation becomes:

$$d(f_n^{-1}(x), f_m^{-1}(x)) \leq K \cdot d(f_n, f_m), \tag{3.3}$$

for all $x \in \partial\Delta \cup \Delta$.

Since $f_n \longrightarrow f_0$, we know that $f_n$ is a Cauchy sequence. By 3.3 we are able to conclude that, for each $x \in \partial\Delta \cup \Delta$, $f_n^{-1}(x)$ is a Cauchy sequence too. Hence, fixed $x \in \partial\Delta \cup \Delta$, the sequence $f_n^{-1}(x)$ is convergent and we can define a set $A_0 \subset M$ such that:

$$\partial A_0 := \{x_0,\ \exists x \in \partial\Delta \text{ with } x_0 = \lim_{n\longrightarrow+\infty} f_n^{-1}(x)\}. \tag{3.4}$$

$$A_0 := \{x_0,\ \exists x \in \Delta \text{ with } x_0 = \lim_{n\longrightarrow+\infty} f_n^{-1}(x)\}. \tag{3.5}$$

We can see evidently by 3.3 that if $x_0 = \lim_{n\longrightarrow+\infty} f_n^{-1}(x)$ then $x_0 = f_0^{-1}(x)$. In this way, 3.3 ensures that $A_n \longrightarrow A_0$ in the sense of Definition 3.0.1: In fact, since $f_n \longrightarrow f_0$, given $\varepsilon > 0$ there is $n_0$ such that $d(f_n, f_0) < \varepsilon/K$ if $n \geq n_0$. If we take $z \in \partial A_n$ for a given $n \geq n_0$ and $x = f_n(z) \in \partial\Delta$, we obtain by 3.3 that $d(f_n^{-1}(x), f_0^{-1}(x)) = d(z, \lim_{n\longrightarrow+\infty} f_n^{-1}(x)) < \varepsilon$, and hence, since $\lim_{n\longrightarrow+\infty} f_n^{-1}(x) \in \partial A_0$, we get $d(z, \partial A_0) < \varepsilon$, that is, $\partial A_n \subset V_\varepsilon(\partial A_0)$. The inclusion $\partial A_0 \subset V_\varepsilon(\partial A_n)$ is immediate.

□

Lemma 3.0.3 can be generalized to the case where there exists $\Delta_n$ for each $n$ and $\partial\Delta_n \longrightarrow \partial\Delta_0$ with $f_n^{-1} : \Delta_n \longrightarrow A_n$ being an inverse branch.

**Lemma 3.0.4.** *Suppose that $f_n : M \longrightarrow M$, $n \geq 1$ is a convergent sequence in the $C^1$ topology and take sets $A_n$ that are open connected subsets of $M$. Suppose in addition that for each $n \geq 0$ there is an open connected set $\Delta_n \subset M$ with $\Delta_n \longrightarrow \Delta_0$ and the restriction $f_n^{-1} : \Delta_n \longrightarrow A_n$ is a diffeomorphism from the open set $\Delta_n$ onto $A_n$ (that can be extended to $\partial\Delta_n \cup \Delta_n$), for $n \geq 1$. Then there exists a connected open set $A_0 \subset M$ such that*

$$A_n \longrightarrow A_0.$$

*If we denote $f_0 := \lim_{n\longrightarrow+\infty} f_n$, then the restriction $f_0|_{A_0} : A_0 \longrightarrow \Delta$ is also a diffeomorphism from $A_0$ onto $\Delta$ (which can be extended to $\partial A_0 \cup A_0$).*

**Proof:** In the first part of the proof we aim to prove a version of 3.3. In order to do that, we will use some of the ideas present in last proof. For each $n \geq 0$, consider $c_n$ given by 3.1. If we take $m, n \in \mathbb{N}$, $x_m \in \partial\Delta_m \cup \Delta_m$ and $x_n \in \partial\Delta_n \cup \Delta_n$, triangle inequality gives us that:

$$\begin{aligned} d(x_m, x_n) &= d(f_m \circ f_m^{-1}(x_m), f_n \circ f_n^{-1}(x_n)) \\ &\geq d(f_m \circ f_m^{-1}(x_m), f_m \circ f_n^{-1}(x_n)) - d(f_m \circ f_n^{-1}(x_n), f_n \circ f_n^{-1}(x_n)) \end{aligned}$$

An analogous procedure to the last proof allows us to find $K > 0$ such that

$$d(f_n^{-1}(x_n), f_m^{-1}(x_m)) \leq K \cdot (d(f_n, f_m) + d(x_n, x_m)), \tag{3.6}$$

for all $x_m \in \partial\Delta_m \cup \Delta_m$ and $x_n \in \partial\Delta_n \cup \Delta_n$.

Now consider $x \in \partial\Delta_0 \cup \Delta_0$. Since $\Delta_n \longrightarrow \Delta_0$, we can arrange that the sequence $x_n$ may be taken in such a way that $d(x_n, x) \longrightarrow 0$ when $n \longrightarrow +\infty$. With this fact and having in mind that $f_n$ is a Cauchy sequence, we obtain that $f_n^{-1}(x_n)$ is also a Cauchy sequence, and so there is $x_0 := \lim_{n\longrightarrow+\infty} f_n^{-1}(x_n)$. So we can define

$$\partial A_0 := \{x_0,\ \exists x_n \in \partial\Delta_n,\ x \in \partial\Delta \text{ with } x_0 = \lim_{n\longrightarrow+\infty} f_n^{-1}(x),\ \text{where } x = \lim_{n\longrightarrow+\infty} x_n\}. \quad (3.7)$$

$$A_0 := \{x_0,\ \exists x_n \in \Delta_n,\ x \in \Delta \text{ with } x_0 = \lim_{n\longrightarrow+\infty} f_n^{-1}(x),\ \text{where } x = \lim_{n\longrightarrow+\infty} x_n\}. \quad (3.8)$$

An analogous procedure to last proof shows that the set $A_0$ defined by 3.7 satisfy the requirements in this Lemma and this concludes the proof.

□

We should notice that Lemmas 3.0.3 and A.0.3 both deal with similar scenarios. But one diference between them is that Lemma A.0.3 deals with continuity of inverse branches, while Lemma 3.0.3 gives us a way to construct the domain of an inverse branch as the limit of the domain of other inverse branches. This will be particularly useful when we deal with the construction of a Markov partition by taking the limit of elements of already known Markov partitions. In order to do that, we need to adapt these results to deal with nested sets and hyperbolic times. So, we will use now Lemmas 3.0.3 and A.0.3 to deal with hyperbolic nested balls (see Definition 2.2.10).

**Lemma 3.0.5.** *Consider $r > 0$, $x \in M$ and the set $B_r(x)$. If $f_n$ is a sequence of non-flat maps converging to $f_0$ in the $C^1$ topology such that for each $n \geq 0$ the $(\sigma,\delta)-$hyperbolic nested ball $\Delta_n = (B_r^{*,f_n})(x)$ is well defined and non-empty, where $0 < r < \delta/2$ and $0 < \sigma < 1$ are real numbers, then $\Delta_n \longrightarrow \Delta_0$.*

**Proof:** We notice that $\delta$ is the same for all $n$. Thus, if $(P_n^0, P_n^1, \cdots, P_n^k)$ is a chain of pre-images of $B_r(x)$ by $f_n$ we conclude by Proposition 1.3.2 that the diameter of this chain is bounded as follows:

$$\sum_{j=0}^{k} \operatorname{diam} P_n^j \leq \sum_{j=0}^{k} \sigma^{\operatorname{ord}(P_n^j)} \leq \sum_{j\geq 0} \sigma^j,\ \forall n \geq 0.$$

We can conclude that given $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that $\sum_{j\geq l} \operatorname{diam} P_n^j < \varepsilon$, for all chain of pre-images $(P_n^0, P_n^1, \cdots, P_n^k)$ of $B_r(x)$ by $f_n$, for every $n \geq 0$.

Fix $\varepsilon > 0$ as above and consider $x \in \partial\Delta_0$. Then there exists a chain $(P_n^0, P_n^1, \cdots, P_n^k)$ of pre-images of $B_r(x)$ by $f_0$ and some $0 \leq j \leq k$ such that $\mathrm{d}(x, \partial P_0^j) < \varepsilon/3$. We have two cases to consider, namely, $j < l$ or $j \geq l$. Suppose that $j < l$ and set $s_j := \operatorname{ord}(P_0^j)$. See that $P_0^j$ is mapped onto $B_r(x)$ diffeomorphically by $f_0^{s_j}$ $(f_0^{-s_j}(B_r(x)) = P_0^j)$. We have by Lemma A.0.3 that every $g : M \longrightarrow M$ sufficiently close to $f_0$ has an inverse branch $g^{-s_j} : B_r(x) \longrightarrow P_g$, with $P_g \subset M$, that is close to the inverse branch $f_0^{-s_j} : B_r(x) \longrightarrow P_0^j$ of $f_0$ at $B_r(x)$. Then, since $f_n \longrightarrow f_0$, there is $n_1 \in \mathbb{N}$ such that $n \geq n_1 \Rightarrow \exists P_n^j \subset M$ such that $f_n^{-j} : B_r(x) \longrightarrow P_n^j$ is an inverse branch with $V_\varepsilon(\partial P_n^j) \subset \partial P_0^j$ and $V_\varepsilon(\partial P_0^j) \subset \partial P_n^j$, and so we have constructed a sequence of subsets $P_n^j \subset M$ such that $P_n^j \longrightarrow P_0^j$ when $n \longrightarrow +\infty$.

Thus we can conclude that, since $d(x, \partial P_0^j) < \varepsilon/3$, $x \in V_\varepsilon(\partial P_n^j)$ for all $n \geq n_1$. Since $P_0^j$ belongs to a chain of pre-images of $B_r(x)$ by $f_0$ and $P_n^j \longrightarrow P_0^j$, we obtain that there is $n_2 \in \mathbb{N}$ such that the set $P_n^j$ must belong to some chain of pre-images of $B_r(x)$ by $f_n$, for $n \geq n_2$. In fact, if $j = 0$ we know that $P_0^j$ and $B_r(x)$ are linked sets. Since $P_n^j \longrightarrow P_0^j$ we must have in addition that $P_n^j$ and $B_r(x)$ are linked sets for $n$ large enough. If we have that $(P_n^0, P_n^1, \cdots, P_n^{j-1})$ is a chain of pre images of $B_r(x)$ by $f_n$ is already constructed, and $P_0^{j-1}$ and $P_0^j$ are linked sets then in the same way as above we can construct sets $P_n^j$ such that $P_n^j$ and $P_n^{j-1}$ are linked sets for $n$ latge enough. Thus $(P_n^0, P_n^1, \cdots, P_n^j)$ is also a chain of pre images for $n$ large enough. Taking $n_0 = \max\{n_1, n_2\}$ we obtain that $x \in V_\varepsilon(\partial \Delta_n)$ for $n \geq n_0$. If $j > \ell$, since $\sum_{j \geq l} \operatorname{diam} P_0^j < \varepsilon$ its easy to see that, by using the chain previously constructed, $x \in V_\varepsilon(\partial \Delta_n)$ for $n$ big enough.

□

**Lemma 3.0.6.** *Consider a sequence of open sets $A_0, A_1, \cdots, A_n, \cdots$ such that $A_n \longrightarrow A_0$ when $n \longrightarrow +\infty$ and a sequence of probability measures $\nu_0, \nu_1, \cdots, \nu_n, \cdots$ with $\nu_n \longrightarrow \nu_0$ when $n \longrightarrow +\infty$ in the weak$-*$ topology and such that $\nu_0(\partial A_0) = 0$. If there is $C > 0$ such that $\nu_n(A_n) \leq C, \forall n \geq 1$ then $\nu_0(A_0) \leq C$.*

**Proof:** First we will show that $|\nu_n(A_n) - \nu_0(A_0)| \longrightarrow 0$ when $n \longrightarrow +\infty$. Consider $\varepsilon > 0$. Since $\nu_0(A_0) = 0$, take a neighborhood $V = V_\gamma(\partial A_0)$ of $\partial A_0$ with $\gamma > 0$ small in such a way that $\nu_0(V) < \varepsilon/6$. Since $\nu_n \longrightarrow \nu_0$, there is $n_1 \geq 1$ such that $n > n_1 \Rightarrow |\nu_n(V) - \nu_0(V)| < \varepsilon/6$ and $|\nu_n(A_0) - \nu_0(A_0)| < \varepsilon/3$. Since $A_n \longrightarrow A_0$, there is $n_2 \geq 1$ such that $n > n_2 \Rightarrow \partial A_n \subset V$ and $\partial A_0 \subset V_\gamma(\partial A_n)$. In particular, for $n > n_2$ we have that both $A_n \backslash A_0$ and $A_0 \backslash A_n$ are subsets of $V$. Then we can see that $|\nu_n(A_n) - \nu_n(A_0)| = |\nu_n(A_n \backslash A_0) - \nu_n(A_0 \backslash A_n)| \leq 2\nu_n(V)$. We conclude that, taking $n_0 = \max\{n_1, n_2\}$, if $n > n_0$ then:

$$
\begin{aligned}
|\nu_n(A_n) - \nu_0(A_0)| &= |\nu_n(A_n) - \nu_n(A_0)| + |\nu_n(A_0) - \nu_0(A_0)| \\
&\leq 2\nu_n(V) + |\nu_n(A_0) - \nu_0(A_0)| \\
&\leq 2\nu_0(V) + 2|\nu_n(V) - \nu_0(V)| + |\nu_n(A_0) - \nu_0(A_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\end{aligned}
$$

and so

$$
|\nu_n(A_n) - \nu_0(A_0)| \longrightarrow 0, \text{ when } n \longrightarrow +\infty. \tag{3.9}
$$

Since $|\nu_0(A_0)| \leq |\nu_n(A_n)| + |\nu_n(A_n) - \nu_0(A_0)| \leq C + |\nu_n(A_n) - \nu_0(A_0)|$ and the second term in last sum converges to zero when $n \longrightarrow +\infty$ we conclude that $\nu_0(A_0) \leq C$.

□

It is clear that Lemma 3.0.6 remains true if we replace $\leq$ by $\geq$ in both inequalities of its statement.

**Lemma 3.0.7.** *Now we consider $f_n \longrightarrow f_0$ and for each $n \geq 0$ a set $\Delta_n$ with $\partial\Delta_n \longrightarrow \partial\Delta_0$ and an induced Markov map $F_n : \Delta_n \longrightarrow \Delta_n$ with partition $\mathcal{P}_n$ of $\Delta_n$ and return time $R_n : \Delta_n \longrightarrow \mathbb{N}$. For each $n \geq 0$ consider a $F_n$−invariant measure $\nu_n$ compatible with $F_n$ (see Definition 2.1.4) such that $\nu_n \longrightarrow \nu_0$ in the weak−$\star$ topology.*

*Suppose in addition that there is a function $\varphi : \mathbb{N} \longrightarrow \mathbb{R}$ in such a way that*

$$\nu_n(\{R_n \geq k\}) \leq \varphi(k), \ \forall n \in \mathbb{N}.$$

*Then*

$$\nu_0(\{R_0 \geq k\}) \leq \varphi(k), \ \forall n \in \mathbb{N}. \tag{3.10}$$

**Proof:** Fix $k \in \mathbb{N}$. Define $A_n := \{R_n < k\}$ for each $n \in \mathbb{N}$ and $C := 1 - \varphi(k)$. By hypothesis $\nu_n(A_n) \geq C$. So this result is a direct consequence of Lemma 3.0.6.

□

**Remark 3.0.8.** *If $\mathcal{P}$ is a Markov partition on $\Delta \subset M$, $R : \Delta \longrightarrow \mathbb{N}$ is an inducing time function on $\Delta$ (which is constant on each element $P \in \mathcal{P}$) and $\nu$ is a probability on $\Delta$, then $\int R d\nu = \sum_{j\geq 1} \nu(R \geq j)$. In fact:*

$$\begin{aligned}
\sum_{j\geq 1} \nu(R \geq j) &= \nu(R \geq 1) + \nu(R \geq 2) + \nu(R \geq 3) + \cdots + \nu(R \geq j) + \cdots \\
&= \nu(R = 1) + \nu(R = 2) + \nu(R = 3) + \cdots + \nu(R = j) + \cdots \\
&\qquad + \nu(R = 2) + \nu(R = 3) + \cdots + \nu(R = j) + \cdots \\
&\qquad\qquad + \nu(R = 3) + \cdots + \nu(R = j) + \cdots \\
&\qquad\qquad\qquad + \cdots \\
&= \textstyle\sum_{j\geq 1} j \cdot \nu(R = j) = \int R d\nu
\end{aligned}$$

**Lemma 3.0.9.** *Let $g_n : M \longrightarrow M$ be a sequence of dynamics that converges in the $C^1$ topology to a map $g_0 : M \longrightarrow M$. Let $\nu_n$ be a sequence of invariant measures w.r.t. $g_n : M \longrightarrow M$ which converge to a measure $\nu_0$ in the weak-* topology. Then $\nu_0$ is an invariant measure w.r.t. $g_0 : M \longrightarrow M$.*

**Proof:** Let $\mathcal{F}$ denote the space of all continuous functions $\varphi : M \longrightarrow \mathbb{R}$ endowed with the sup norm $\|\cdot\|_0$. We know that a given measure $\mu$ in $M$ is invariant with respect to some mensurable transformation $f : M \longrightarrow M$ if, and only if, $\int \varphi d\mu = \int \varphi \circ g d\mu$, $\forall \varphi \in \mathcal{F}$. Also, if we suppose that $\nu_n$ converges to $\nu_0$ in the weak-* topology, as in the hypothesis, this must be equivalent to the sequence of real numbers $\left|\int \varphi d\nu_n - \int \varphi d\nu_0\right|$ converge to zero, for all $\varphi \in \mathcal{F}$. Therefore, assuming that $\nu_n$ is $g_n$-invariant for all $n \in \mathbb{N}$ is equivalent to say that $\int \varphi d\nu_n = \int \varphi \circ g_n d\nu_n$, $\forall \varphi \in \mathcal{F}$, $\forall n \in \mathbb{N}$, and if we show that $\int \varphi d\nu_0 = \int \varphi \circ g_0 d\nu_0$, $\forall \varphi \in \mathcal{F}$, we are done.

Take $\varphi \in \mathcal{F}$. See that

$$\begin{aligned}
& |\varphi \circ g_n d\nu_n - \int \varphi \circ g_0 d\nu_0| \\
\leq \ & |\int \varphi \circ g_n d\nu_n - \int \varphi \circ g_0 d\nu_n| + |\int \varphi \circ g_0 d\nu_n - \int \varphi \circ g_0 d\nu_0| \\
\leq \ & ||\varphi||_0 \cdot ||g_n - g_0||_0 + |\int \varphi \circ g_0 d\nu_n - \int \varphi \circ g_0 d\nu_0| ,
\end{aligned}$$

and the last term converges to zero, since $g_n$ converges to $g_0$ and $\nu_n$ converges to $\nu_0$. This concludes the proof.

□

**Remark 3.0.10.** *Consider $f_n \longrightarrow f_0$ a convergent sequence of dynamics in the $C^1-$topology and suppose that for all $n \geq 1$ there is an open disk $\Delta_n \subset M$ and an induced Markov map $F_n : \Delta_n \longrightarrow \Delta_n$ as in Definition 2.8. By Lemmas 3.0.5 and 3.0.4 there exists $\Delta_0 \subset M$ where we can define a map $F_0 : \Delta_0 \longrightarrow \Delta_0$ in the following way: Consider, for each $k > 0$ and $n > 0$ the set $A_n^k \in \mathcal{P}_n$ such that $R_n(A_n^k) = k$. We can apply Lemma 3.0.3 to each connected component of $A_n^k$ and thus obtain a set $A_0^k \subset \Delta_0$ with $A_n^k \longrightarrow A_0^k$. Defining $\mathcal{P}_0 := \bigcup_{k>0} A_0^k$ and $R_0 : \Delta_0 \longrightarrow \mathbb{N}$ as $R_0(A_0^k) = k$ and $R_0\left(\Delta_0 \backslash \bigcup_{k>0} A_0^k\right) = 0$ we can set $F : \Delta_0 \longrightarrow \Delta_0$ as $F_0(x) = f^{R_0(x)}(x)$. Its clear that with the current notation, $A_n^k = \{R_n = k\}$, for all $n, k \in \mathbb{N}$.*

In next Lemma we assume the context of Remark 3.0.10.

**Lemma 3.0.11.** *Consider, for each $n \geq 0$, a sequence of probability measures $\nu_n$ on $\Delta_n \subset M$ such that $\nu_n \longrightarrow \nu_0$ in the weak$-\star$ topology. Suppose that there is $C > 0$ such that $\int R_n d\nu_n \leq C$ for all $n \geq 1$, where $R_n$ is the return time map associated to $F_n$ defined on $\Delta_n$, as in Remark 3.0.10. Then*

$$\int R_0 d\nu_0 \leq C.$$

**Proof:** Denote $a_{n,k} := \nu_n(\{R_n = k\})$ and $b_{n,k} := \nu_n(\{R_n \geq k\})$. By Remark 3.0.8 we know that, for each $n \geq 0$,

$$\int R_n d\nu_n = \sum_{k \geq 1} k \cdot a_{n,k} = \sum_{k \geq 1} b_{n,k}.$$

Since each $\nu_n$ is a probability measure on $\Delta_n$, we have that for each $n$:

$$\sum_{k \geq 1} a_{n,k} = 1. \tag{3.11}$$

We know by 3.9 that for each $k \geq 1$

$$a_{n,k} \longrightarrow a_{0,k}, \text{ when } n \longrightarrow +\infty. \tag{3.12}$$

In order to show that $\sum_{k \geq 1} b_{0,k} \leq C$, we proceed with the following inductive construction: Fix $k$ and consider $\varepsilon_k > 0$ (Note that $\varepsilon_k$ may be taken arbitrarily. However in what follows we will consider a suitable choice for $\varepsilon_k$ to be established later). By 3.12 we know that $\sum_{s=1}^{k} a_{n,s}$ is a convergent sequence. Then, consider $n_0^k \in \mathbb{N}$ such that

$$m, n \geq n_0^k \Rightarrow \left| \sum_{s=1}^{k} a_{s,n} - \sum_{s=1}^{k} a_{s,m} \right| < \varepsilon_k.$$

Set $0 < \gamma_k < 1$ such that $\sum_{s=1}^{k} a_{s,n_0^k} = 1 - \gamma_k$ (see 3.11). Therefore we have that $\sum_{s=1}^{k} a_{n,s} \in (1 - \gamma_k - \varepsilon_k, 1 - \gamma_k + \varepsilon_k) \forall n \geq n_0^k$, that is,

$$\sum_{s=k+1}^{+\infty} a_{n,s} \in (\gamma_k - \varepsilon_k, \gamma_k + \varepsilon_k), \forall n \geq n_0^k. \tag{3.13}$$

By definition of $b_{n,k}$, we may write previous statement as:

$$b_{n,k} \in (\gamma_k - \varepsilon_k, \gamma_k + \varepsilon_k), \forall n \geq n_0^k. \tag{3.14}$$

Setting $\varphi(k) := \gamma_k + \varepsilon_k$ we may apply Lemma 3.0.7 and obtain that

$$b_{0,k} < \gamma_k + \varepsilon_k.$$

To conclude this proof we just need to ensure that $\sum_{k \geq 1} b_{0,k} \leq C$. In fact, it's not difficult to see that by construction we have that $\sum_{k \geq 1} \gamma_k \leq C$. If we choose $\varepsilon_k$ as the general term of a convergent series (for instance, $\varepsilon_k = \frac{1}{k^2}$) then we obtain that

$$\sum_{k \geq 1} b_{0,k} \leq \sum_{k \geq 1} \gamma_k + \varepsilon_k \leq C + \sum_{k \geq 1} \varepsilon_k < +\infty. \tag{3.15}$$

Since the choice of $\varepsilon_k$ is arbitrary, we may take $\sum_{k \geq 1} \varepsilon_k$ as small as we want and so 3.15 gives us that $\sum_{k \geq 1} b_{0,k} \leq C$.

□

**Lemma 3.0.12.** *Consider $f_n \longrightarrow f_0$ as a convergent sequence of dynamics in the $C^1$−topology. Suppose that for all $n \geq 1$ there is an open disk $\Delta_n \subset M$, $0 < \sigma < 1$, $\delta > 0$ and an induced Markov map $F_n : \Delta_n \longrightarrow \Delta_n$ as in Definition 2.3.2 with $(\sigma, \delta)$−hyperbolic return time $R_n : \Delta_n \longrightarrow \mathbb{N}$ as in Definition 2.3.1. Suppose that there is $\Delta_0 \subset M$ such that $\Delta_0 = \lim_{n \longrightarrow +\infty} \Delta_n$ and set $F_0 : \Delta_0 \longrightarrow \Delta_0$ as in Remark 3.0.10. Then $\{j \in \mathbb{N};\ f_0^j(x) \in \mathcal{O}_{F_0}^+(x)\}$ is a subset of natural numbers that satisfy the geometric properties of hyperbolic times given by Proposition 1.3.2.*

**Proof:** Let $F_0 : \Delta_0 \longrightarrow \Delta_0$ be given by Lemma 3.0.4. We will utilize the notation of Remark 3.0.10. For each $n \geq 1$ and $F_n : \Delta_n \longrightarrow \Delta_n$, there is a partition $\mathcal{P}_n$ build up with sets $A_n^k$ such that all points $y, z \in A_n^k$ satisfy

$$d(f_n^{k-j}(y), f_n^{k-j}(z)) \leq \sigma^{j/2} d(f_n^k(y), f_n^k(z));\ \forall 1 \leq j < k, \tag{3.16}$$

as we can see by Proposition 1.3.2.

Now, consider $y, z$ in a connected component of $A_0^k$. Since $f_0$ is continuous in a compact set, given $\varepsilon > 0$ there exists $\gamma > 0$ such that $\forall x, w \in M$, $d(x,w) < \gamma \Rightarrow d(f_0(x), f_0(w)) < \varepsilon/8$. We may take $\gamma$ small enough in such a way that $d(f_0^j(x), f_0^j(w)) < \varepsilon/8$, $\forall 0 \leq j \leq k$. By Lemma 3.0.4 $A_n^k \longrightarrow A_0^k$. Then there is $n_0 \in \mathbb{N}$ such that if $n > n_0$ there are $y_n, z_n$ belonging to a connected component of $A_n^k$ with $d(y_n, y) < \gamma$ and $d(z_n, z) < \gamma$. By triangular inequality follows that

$$\begin{aligned} d(f_0^{k-j}(y), f_0^{k-j}(z)) \leq \;& d(f_0^{k-j}(y), f_0^{k-j}(y_n)) + d(f_0^{k-j}(y_n), f_n^{k-j}(y_n)) + \\ & d(f_n^{k-j}(y_n), f_n^{k-j}(z_n)) + d(f_n^{k-j}(z_n), f_0^{k-j}(z_n)) + d(f_0^{k-j}(z_n), f_0^{k-j}(z)). \end{aligned}$$

By construction, we have that $d(f_0^{k-j}(y), f_0^{k-j}(y_n))$ and $d(f_0^{k-j}(z_n), f_0^{k-j}(z))$ are both smaller than $\varepsilon/8$. Since $f_n \longrightarrow f_0$ in the $C^1$–topology, there is $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow d(f_n, f_0) < \varepsilon/8$. Then, for $n > n_1$ we have that $d(f_0^{k-j}(y_n), f_n^{k-j}(y_n))$ and $d(f_n^{k-j}(z_n), f_0^{k-j}(z_n))$ are both smaller than $\varepsilon/8$. 3.16 yields that $d(f_n^{k-j}(y_n), f_n^{k-j}(z_n)) < \sigma^{j/2} d(f_n^k(y_n), f_n^k(z_n))$. We conclude that $d(f_0^{k-j}(y), f_0^{k-j}(z)) < \varepsilon/2 + \sigma^{j/2} d(f_n^k(y_n), f_n^k(z_n))$. But again:

$$\begin{aligned} d(f_n^k(y_n), f_n^k(z_n)) \leq \;& d(f_n^k(y_n), f_0^k(y_n)) + d(f_0^k(y_n), f_0^k(y)) + \\ & d(f_0^k(y), f_0^k(z)) + d(f_0^k(z), f_0^k(z_n)) + d(f_0^k(z_n), f_n^k(z_n)). \end{aligned}$$

With a similar argument we obtain that $d(f_n^k(y_n), f_n^k(z_n)) < \varepsilon/2 + d(f_0^k(y), f_0^k(z))$, and so:

$$d(f_0^{k-j}(y), f_0^{k-j}(z)) < \varepsilon/2 + \varepsilon/(2\sigma) + \sigma^{j/2} d(f_0^k(y), f_0^k(z)).$$

With this we obtain that all points $y, z$ in a connected component of $A_n^k$ satisfy

$$d(f_0^{k-j}(y), f_0^{k-j}(z)) \leq \sigma^{j/2} d(f_0^k(y), f_0^k(z)); \; \forall 1 \leq j < k. \tag{3.17}$$

This ends the proof.

□

We know by Proposition 1.3.5 that if $f : M \longrightarrow M$ is a non-uniformly expanding map (with respect to a given probability $\mu$) then almost every point $x$ has infinitely many hyperbolic times. In the next Lemma we will pursue the converse of this fact. We obtain a sufficient condition for $f$ being non-uniformly expanding. As we shall see later, this condition is connected with the integrability of return times.

**Lemma 3.0.13.** *Let $(F_0, \mathcal{P}_0)$ be an induced full Markov map for $f_0$ defined on an open set $\Delta_0 \subset M$, $R_0$ be the induced time of $F_0$ and $\mu_0$ be an ergodic $f_0-$invariant probability such that $\mu(\{R_0 = 0\}) = 0$. If there is an $F_0-$invariant finite measure $\nu_0 \ll \mu_0$ and $\theta > 0$ such that $0 < \int R_0 d\nu_0 < \frac{1}{\theta} < +\infty$ then, for $\mu_0$ almost every $x \in \Delta_0$, $\limsup_{n \longrightarrow +\infty} \frac{1}{n} \sharp\{0 \leq j < n;\ f_0^j(x) \in \mathcal{O}^+_{F_0}(x)\} > \theta > 0$.*

**Proof:** Since

$$\sharp\{j \geq 0;\ \sum_{k=0}^{j} R_0 \circ F_0^k(x) < n\} = \sharp\{0 \leq j < n;\ f_0^j(x) \in \mathcal{O}^+_{F_0}(x)\} \tag{3.18}$$

and

$$\sharp\{j \geq 0;\ \sum_{k=0}^{j} R_0 \circ F_0^k(x) < n\} = \sup\{j \geq 0;\ \sum_{k=0}^{j} R_0 \circ F_0^k(x) < n\}, \tag{3.19}$$

if we show that $\limsup_{n \longrightarrow +\infty} \frac{1}{n} \sup\{j \geq 0;\ \sum_{k=0}^{j} R_0 \circ F_0^k(x) < n\} > 0$ then we are done. Denote $v_n(x) := \sup\{j \geq 0;\ \sum_{k=0}^{j} R_0 \circ F_0^k(x) < n\}$. We know that $\nu_0$ is ergodic, since $\mu_0$ is ergodic. Then, by Birkhoff's theorem we have that

$$0 < \lim_{k \longrightarrow +\infty} \frac{1}{k} \sum_{s=0}^{k-1} R_0 \circ F_0^s(x) = \int R_0 d\nu_0 < \frac{1}{\theta} < +\infty$$

for $\nu_0-$almost every point $x \in \Delta_0$. In particular, $0 < \limsup_{n \longrightarrow +\infty} \frac{1}{v_n(x) + 2} \sum_{s=0}^{v_n(x)+1} R_0 \circ F_0^s(x) = \int R_0 d\nu_0 < +\infty$. By construction, we have that

$$\frac{1}{v_n(x) + 2} \sum_{s=0}^{v_n(x)+1} R_0 \circ F_0^s(x) \geq \frac{n}{v_n(x) + 2} = \frac{n}{v_n(x)} \cdot \frac{v_n(x)}{v_n(x) + 2},$$

what gives us that $\limsup_{n \longrightarrow +\infty} \frac{n}{v_n(x)} \leq \limsup_{n \longrightarrow +\infty} \frac{1}{v_n(x) + 2} \sum_{s=0}^{v_n(x)+1} R_0 \circ F_0^s(x) < +\infty$ and hence

$$\limsup_{n \longrightarrow +\infty} \frac{v_n(x)}{n} \geq \frac{1}{\int R_0 d\nu_0} > \theta > 0.$$

□

The following Lemma is a straightforward fact, which we will present without proof.

**Lemma 3.0.14.** *Suppose that $f_n \longrightarrow f_0$ in the $C^1$ topology and that, for each $n \geq 0$, $\mu_n(f_n, \nu_n)$ is the measure obtained from $\nu_n$ and $f_n$ as in Remark 2.3.6. If $\nu_n \longrightarrow \nu_0$ in the weak$-\star$ topology, then $\mu_n(f_n, \nu_n) \longrightarrow \mu_0(f_0, \nu_0)$ in the weak$-\star$ topology.*

**Remark 3.0.15.** *The strategy to prove Theorem A is go back to the approach of induced maps and lift each $\mu_n$ to an absolutely continuous measure $\nu_n$ which is invariant with respect to an induced map $F_n$ associated to $f_n$ in a suitable set $\Delta_n$, (which need not to be the same for all $n \geq 0$). It's not hard to prove that since $f_n \longrightarrow f_0$, there is partition $\mathcal{P}_0$ of $\Delta_0$ which is the limit of the partitions $\mathcal{P}_n$ (the meaning of this last assertion will be properly defined in the sequel) and a map $F_0$ also defined in $\Delta$ such that $F_n \longrightarrow F_0$. Formal statements will appear latter. With Lemma 3.0.9, we can obtain a $F_0-$invariant measure $\nu_0$ which is the limit of $\nu_n$ in the topology weak-*. In this scenario we take into account the property of positive frequency of hyperbolic times to obtain some bounds to the measure of the tail of return times, both for $\nu_n$ and $\nu_0$. This will ensure the integrability of the return time function $R_0$ with respect to $\nu_0$. Now, if we take $\mu_0$ as the normalization of the projection of $\nu_0$, we obtain by Lemmas 2.4.4, 2.4.5 and 3.0.13 that $\mu_0$ must be an expanding measure for $f_0$, that is, $\mu_0 \in \mathcal{M}^1_{\mathfrak{exp}}(f_0)$.*

# Chapter 4

# Proof of Theorems A and B

## 4.1 Proof of Theorem A

In the next theorem we utilize some results of Chapter 3 to obtain a model result which we believe that can be used in many other situations besides the present context. Essentially it says that if we have suitable Markov partitions for a sequence of dynamics $f_n$ and $f_n \longrightarrow f_0$ in the $C^1$-topology, then one can construct a suitable Markov partition for $f_0$.

Before we enounce the result, we will make clear the meaning of suitable Markov partition we quote before. We know by Proposition 2.3.5 that if $\mu$ is an expanding measure for $f$ and $\Delta \subset M$ is an open hyperbolic nested set small enough that intersects the statistical hyperbolic attractor $\mathcal{A}^f_{hyp,+}$ (see Remark 1.3.15) then one can construct a Markov partition in $\Delta$ and Markov map $F : \Delta \longrightarrow \Delta$ which is based on hyperbolic returns $R(x)$ of $x$ to $\Delta$. Even more, there exists an $F$-invariant measure $\nu \ll \mu$ with respect to which the return time $R$ is integrable. The value of this integral is related to the frequency $\theta_f > 0$ of hyperbolic returns. In the opposite direction, if we begin with the hypothesis that we already built an induced Markov map $F$ on some set $\Delta \subset M$ and the inducing time is in fact the hyperbolic return time to $\Delta$, which is integrable with respect to some $F$-invariant measure $\nu$, then we will observe positive frequency of hyperbolic times for almost every point in $M$ from the point of view of the projection $\mu$ of $\nu$. This projection will be, in fact, an expanding measure (see Theorem 2.1.6 and Lemma 3.0.13).

As we can see, the integrability of the return time plays a key role in the process of obtain an expanding measure. So, what we mean with the term "suitable Markov map" quoted before is a Markov map for which there exists an invariant measure and the return time is integrable with respect to this measure. In the next result we show that if there are suitable Markov maps for each $f_n$ then there exists also a suitable Markov map for $f_0$.

**Theorem 4.1.1.** *Suppose that for each $n \geq 1$ there exists an induced Markov map $F_n$ :*

*$\Delta_n \longrightarrow \Delta_n$ defined in some topological disk $\Delta_n$ with return time $R_n : \Delta_n \longrightarrow \mathbb{R}$ and a Markov partition $\mathcal{P}_n = (P_n^k)_{k\in\mathbb{N}}$ on $\Delta_n$. Suppose in addition that there is a $F_n-$invariant measure $\nu_n$. If there is a set $\Delta_0$ such that $\Delta_n \longrightarrow \Delta_0$, there are sets $P_0^k \subset \Delta_0, k \in \mathbb{N}$, such that for each $k$ we have $P_n^k \longrightarrow P_0^k$ and there exists $C > 0$ such that*

$$\int R_n d\nu_n \leq C, \ \forall n \geq 1,$$

*then $\mathcal{P}_0 := (P_0^k)_{k\in\mathbb{N}}$ is a Markov partition on $\Delta_0$, there is an induced Markov map $F_0 : \Delta_0 \longrightarrow \Delta_0$ with inducing time $R_0 : \Delta_0 \longrightarrow \mathbb{N}$ given by $R_0(P_0^k) = k$ and there exists a measure $\nu_0$ on $\Delta_0$ such that $\int R_0 d\nu_0 \leq C$.*

**Proof:** We will see that $\mathcal{P}_0 := (P_0^k)_{k\in\mathbb{N}}$ is in fact a Markov partition on $\Delta_0$ and that $(F_0, \mathcal{P}_0)$ is a Markov map induced by $f_0$ with return time $R_0$.

By construction, the elements of $\mathcal{P}_0$ are open sets. See that if $k_1 \neq k_2$ then $A_0^{k_1} \cap A_0^{k_2} = \emptyset$. In fact, if by contradiction we suppose that some connected component $\tilde{A}_0^{k_i}$ of $A_0^{k_i}$, for $i = 1,\ 2$ is such that $\tilde{A}_0^{k_1} \cap \tilde{A}_0^{k_2} \neq \emptyset$ it means that for all $\varepsilon > 0$ we have $\partial\tilde{A}_0^{k_1} \subset V_\varepsilon(\tilde{A}_0^{k_2})$ or $\partial\tilde{A}_0^{k_2} \subset V_\varepsilon(\tilde{A}_0^{k_1})$. By construction, there is a sequence of sets $\tilde{A}_n^{k_i} \in \mathcal{P}_n$ such that $\tilde{A}_n^{k_i} \longrightarrow \tilde{A}_0^{k_i}$ when $n \longrightarrow +\infty$. With this and using that $\tilde{A}_0^{k_1} \cap \tilde{A}_0^{k_2} \neq \emptyset$, we can easily conclude that given $\varepsilon > 0$ we have $\partial\tilde{A}_n^{k_1} \subset V_\varepsilon(\tilde{A}_n^{k_2})$ and $\partial\tilde{A}_n^{k_2} \subset V_\varepsilon(\tilde{A}_n^{k_1})$ for $n$ big enough, and hence $\tilde{A}_n^{k_1} \cap \tilde{A}_n^{k_2} \neq \emptyset$ for $n$ big enough, what is an absurd since $\mathcal{P}_n$ is a partition of $\Delta_n$ $\forall n \geq 1$. Then we know that $\mathcal{P}_0$ satisfies the first condition in Definition 2.1.1.

By construction we know that given $P \in \mathcal{P}_0$ then $F_0(P) = \Delta_0$ and also $P$ is sent diffeomorphically onto $\Delta_0$ by $F_0$ (and $F_0|_P$ can be extended to a diffeomorphism between $\overline{P}$ and $\overline{\Delta_0}$), that is, $\mathcal{P}_0$ satisfies second and third conditions of a full Markov partition. It is easy to see that if $x \in \Delta_0$ and $\mathcal{C}_j^0(x)$ denotes the $j-$cylinder containing $x$ with respect to $\mathcal{P}_0$, then $\lim\limits_j \operatorname{diam}(\mathcal{C}_j^0(x)) = 0$. In fact, given $x \in \cap_{n\geq 0} F^{-n}(\cup_{P\in\mathcal{P}_0})$, set $P_j = \mathcal{P}_0(F^j(x))$. As $\operatorname{diam}(\mathcal{P}_{0,n}(x)) = \operatorname{diam}((F_0|_{P_1})^{-1} \circ (F_0|_{P_2})^{-1} \circ \cdots \circ (F_0|_{P_n})^{-1}(\Delta_0))$. Then we can conclude that $\mathcal{P}_0$ is in fact a full Markov partition of $\Delta_0$ with respect to $F_0$ and that $(F_0, \mathcal{P}_0)$ is a full induced Markov map defined on $\Delta_0$.

Define a function $R_0 : \Delta_0 \longrightarrow \mathbb{N}$ by setting $R_0(x) = k$, if $x \in A_0^k$ and $R_0(x) = 0$ otherwise. Also, define a map $F_0 : \Delta_0 \longrightarrow \Delta_0$ by $F_0(x) = f_0^{R_0(x)}(x)$.

Since for all $n$ the measures $\nu_n$ satisfy $\int R_n d\nu_n \leq C$, by utilizing 3.0.11 we can ensure that:

$$0 < \int R_0 d\nu_0 \leq C.$$

□

See that in Theorem 4.1.1 we require that all the integrals are uniformly bounded by the same constant $C > 0$. It is not clear yet if this condition can be weakened to, for instance, $\int R_n d\nu_n < C_n$, with $C_n > 0$ or any other weaker form.

**Remark 4.1.2.** *We will prove Theorem A in the case where the contraction rate $\sigma$ is the same for every dynamics. However it is not difficult to see that the Theorem remains valid when we consider each $f_n$ with a contraction rate $\sigma_n$ with $\sigma_n \longrightarrow \sigma_0$. In fact, Lemma 3.0.12 remains valid if each $R_n$ is a $(\sigma_n, \delta)-$hyperbolic return with $\sigma_n \longrightarrow \sigma_0$.*

**Proof of Theorem A:** Consider a sequence $f_n$ of non-flat maps converging to $f_0$ in the $C^1$ topology and suppose that there is $\mu_n \in \mathcal{M}_{\mathfrak{erp}}(\sigma^{1/2}, \ell, \delta, \theta, f_n)$ for $n \geq 1$ where $\lambda, \delta, \theta > 0$ and $\ell \in \mathbb{N}$ are fixed. We may consider without loss of generality $\ell = 1$, the other cases are treated analogously. Proposition 1.3.13 gives us, for each $n \geq 1$, the existence of a statistical ergodic attractor $\mathcal{A}^n_{hyp,+}$ such that $\omega_{f_n,h,+}(x) = \mathcal{A}^n_{hyp,+}$ for $\mu_n$–a.e.p. $x \in M$ (see Remark 1.3.15).

Consider $\mathcal{A}^0_{hyp,+} = \lim\limits_{n \longrightarrow +\infty} \mathcal{A}^n_{hyp,+}$ as the limit set given by Definition 2.1.9 and take $x \in \mathcal{A}^0_{hyp,+}$. We know by construction that if $V \ni x$ is a neighborhood of $x$ then $V \cap \mathcal{A}^n_{hyp,+} \neq \varnothing$ for infinitely many $n \in \mathbb{N}$. Restricting ourselves to a subsequence if necessary we may assume that $V \cap \mathcal{A}^n_{hyp,+} \neq \varnothing$ for all $n$.

Fix $n \geq 1$ and consider the hyperbolic nested set $\Delta_n := B^{*,f_n}_r(x)$ associated to $f_n$ as given by Definition 2.2.10. By Proposition 2.2.13 we may take $\Delta_n$ as some hyperbolic nested ball $B^*_r(x)$ such that $\operatorname{diam}(\Delta_n) < \delta/2$.

Taking $V = \Delta_n$ we know that $\Delta_n \cap \mathcal{A}^n_{hyp,+} \neq \varnothing$. Then Proposition 2.3.4 gives us that there exists a full induced Markov map $(F_n, \mathcal{P}_n)$ on $\Delta_n$, where $\mathcal{P}_n$ is a Markov partition on $\Delta_n$ with induced time $R_n : \Delta_n \longrightarrow \mathbb{N}$. $F_n$, $R_n$ and $\mathcal{P}_n$ are given, respectively by 2.8, 2.3.1 and 2.9.

Also, since $\Delta_n \cap \mathcal{A}^n_{hyp,+} \neq \varnothing$, Proposition 2.3.5 gives us the existence of a $F_n$–invariant measure $\nu_n \ll \mu_n$ compatible with $F_n$ and $\xi_n > 0$ such that $\varphi^{\Delta_n,f_n}_h(x) \geq \xi_n$ for $\mu_n$–a.e.p. $x \in M$ (where $\varphi^{\Delta_n,f_n}_h(x)$ denotes the frequency of hyperbolic returns to $\Delta_n$, see 1.14) and also

$$\int R_n d\nu_n < \frac{1}{\xi_n}, \tag{4.1}$$

for all $n \geq 1$. Hence, by Theorem 2.1.7 we get

$$\mu_n = \frac{1}{\gamma_n} \sum_{j=0}^{+\infty} (f_n)^j_* (\nu_n|_{\{R_n > j\}}),$$

where $\gamma_n = \sum\limits_{j=0}^{+\infty} (f_n)^j_* (\nu_n|_{\{R_n > j\}})(M)$.

We know that $\mathcal{P}_n$ is a Markov partition of $\Delta_n$ which is composed by a family of sets $A^k_n$, $k \geq 1$, where $A^k_n$ is a union of connected open sets (the elements of the partition $\mathcal{P}_n$ with induced time $k$) which are mapped diffeomorphically onto $\Delta_n$ by $F_n$ (see that, restricted to these sets, $F_n = f^k_n$). Take an open connected component of $A^k_n$, which we will denote by $\tilde{A}^k_n$.

Until now we have maintained $n$ fixed. By varying $n$ we may consider the sets $\Delta_n$ as defined above and also define the set $\Delta_0 := B_r^{*,f_0}(x)$. By Lemma 3.0.5 we know that $\Delta_n \longrightarrow \Delta_0$. Applying Lemma 3.0.4 to $\tilde{A}_n^k$ and $\Delta_n$ we obtain that, since $F_n = f_n^k \longrightarrow f_0^k$ (restricted to $\tilde{A}_n^k$), there exists an open connected set $\tilde{A}_0^k \subset \Delta_0$ such that the restriction $f_0^k|_{\tilde{A}_0^k} : \tilde{A}_0^k \longrightarrow \Delta_0$ is a diffeomorphism. We denote by $A_0^k$ to the union of all connected open sets $\tilde{A}_0^k$ obtained by this construction applied to all connected components of $A_n^k$ and we define $\mathcal{P}_0 := \bigcup_{k\geq 1} A_0^k$.

Take a finite cover $\mathcal{B} = \{B_1, B_2, \cdots, B_N\}$ of $M$ with balls with radius $r > 0$ such that $2r < \delta/2$. By hypothesis, $\theta_{f_n} \geq \theta$ for all $n \geq 1$. So, for each $n \geq 1$ there is a ball $B_j \in \mathcal{B}$ such that the frequency of hyperbolic returns to $B_j$ satisfies $\varphi_h^{B_j,f_n} \geq \theta$ (otherwise, if $\varphi_h^{B_j,f_n} < \theta$ for every $B_j \in \mathcal{B}$, we would have, by definition, that $\theta_f < \theta$, contradicting our hypothesis). Passing to a subsequence, if necessary, we may assume that $B_j$ is the same for all $n \geq 1$. It is a straightforward fact that this set satisfies:

$$B_j \cap \mathcal{A}_{hyp,+}^0 \neq \varnothing.$$

Let $B_j = B_r(x)$, for some $x \in M$. Thus, there is no harm in assume that the sets $\Delta_n$ can be taken as $\Delta_n = B_{r'}^*(x)$, with $0 < 2r < 2r' < \delta/2$. We have just assured that we can take $B_j = B_r(x) \subset \Delta_n = B_{r'}^{*,f_n}(x)$, with $\operatorname{diam}(\Delta_n) < \delta/2$.

Since $B_j \subset \Delta_n$ and $\varphi_h^{B_j,f_n}(y) \geq \theta$, for $\mu_n$–almost every point $y \in B_j$, we have a fortiori that $\varphi_h^{\Delta_n,f_n}(y) \geq \theta$, for $\mu_n$–almost every point $y \in \Delta_n$. Therefore, 4.1 may be rewritten as:

$$\int R_n d\nu_n < \frac{1}{\theta}, \tag{4.2}$$

Since for all $n$ the measures $\nu_n$ satisfy 4.2, setting $C := \dfrac{1}{\theta}$ and utilizing Theorem 4.1.1 we can ensure that $\mathcal{P}_0 = (A_0^k)_{k\in\mathbb{N}}$ is a Markov partition on $\Delta_0$ and there is an induced Markov map $F_0 : \Delta_0 \longrightarrow \Delta_0$ for $f_0$ with inducing time $R_0$. Furthermore:

$$0 < \int R_0 d\nu_0 \leq \frac{1}{\theta}.$$

We have just concluded that $R_0$ is integrable with respect to $\nu_0$. Then, by using Theorem 2.1.6 we obtain an ergodic measure $\mu_0$ on $M$ such that $\nu_0 \ll \mu_0$. By Lemma 3.0.14 we obtain that in fact, passing to a subsequence if necessary, $\mu_n \longrightarrow \mu_0$ when $n \longrightarrow +\infty$. Lemma 3.0.9 gives us a measure that $\mu_0$ which is in fact an $f_0$–invariant measure.

**Claim. 4.1.3.** *$\mu_0$ is an expanding measure for $f_0$.*

**Proof:** By construction, there is a region $\Delta \subset M$ such that for $\mu$−almost every point $x \in \Delta$ belongs to a hyperbolic pre-ball where the geometric properties of hyperbolic times (items 1. and 2. of Proposition 1.3.2) holds in a contraction rate given by $\sigma^{1/2}$. Since $\mu$−almost every point $x \in \Delta$ satisfies the geometric properties of hyperbolic times, item *b*) of Remark 1.3.4 gives us that $\|DF(x)^{-1}\| > \sigma^{-1/2}$ for $\nu$−almost every point $x \in \Delta$. So $F$ has all of its Lyapunov exponents bounded by $c := -\log \sigma^{1/2} > 0$. Setting $\kappa := \int R d\nu < +\infty$, we can use Lemma 2.4.4 to conclude that $f$ has all of its Lyapunov exponents bounded by $c$. By Theorem 5.2.1 we conclude that there exists $\lambda > 0$ and $\ell \in \mathbb{N}$ such that $\mu$ is a $(\lambda, \ell)$−expanding measure for $f$.

□

In this way, we have just shown that $\Delta_0$ is a region satisfying $\varphi^{\Delta_0}_{f_0,hyp} \geq \theta$ $\mu_0$−almost every point (and so, $\theta_{f_0} \geq \theta$). Also, by Lemma 3.0.12 we conclude that $\mu_0$−almost every point $x \in \Delta_0$ admits a pre-ball that expands with respect to $f$ in a rate controlled by $\sigma^{1/2}$ and the size of hyperbolic balls is $\delta$, which can be extended to $\mu_0$−almost every point of $M$ by ergodicity of $\mu_0$, that is, $\mu_0 \in \mathcal{M}_{\mathfrak{erp}}(\sigma^{1/2}, \ell, \theta, \delta, f_0)$, as we wanted.

□

**Remark 4.1.4.** *We can see that te geometric expanding behavior is obtained in Theorem A via continuity arguments. If we want to go further and recover NUE behavior (as in the analytical definition) for a measure that has bounded parameters we need to assume that the exceptional set $\mathcal{C}$ of the dynamics involved is constituted only by critical points (where the derivative fails to be invertible) or only by singular points. This restriction is necessary because to recover slow recurrence to the exceptional set (in the proof of Theorem 5.2.1), Oliveira uses Lemma 2.4.5 (see Remark 2.4.6).*

## 4.2 Proof of Theorem B

The objective of this section is to prove the following result.

**Theorem 4.2.1.** *Given $\varepsilon > 0$, $f : M \longrightarrow M$ and a measure $\mu \in \mathcal{M}_{\mathfrak{erp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$, $\exists \gamma > 0$ and $\sigma'^{1/2} > 0$ with $|\sigma' - \sigma| < \varepsilon$ such that if $d(f, g) < \gamma$ then there exists $\nu \in \mathcal{M}_{\mathfrak{erp}}(\sigma'^{1/2}, \ell, \delta, \theta, g)$ with $d_*(\mu, \nu) < \varepsilon$.*

Theorem B is a direct consequence of Theorem 4.2.1.

**Lemma 4.2.2.** *Let $x \in M$ and consider $\omega := \delta_x$ as the Dirac measure supported on $x$. Then, given $\varepsilon > 0$, $\exists \zeta > 0$ such that $y \in M$, $d(x, y) < \zeta \Rightarrow d_*(\omega, \nu) < \varepsilon/3$, where $\nu := \delta_y$.*

**Proof:** In fact, given $\varepsilon > 0$, consider $i_0 > 0$ such that $\displaystyle\sum_{i=i_0}^{+\infty} \frac{1}{2^i} < \varepsilon/12$ (because $\displaystyle\sum_{i=1}^{+\infty} \frac{1}{2^i}$ is a convergent series) and $\zeta > 0$ suitable to the uniform continuity of each $\varphi_i$, $i \in \{1, \cdots, i_0\}$. Precisely, $\zeta > 0$ is such that

$$\forall x_1, x_2 \in M,\ d(x_1,x_2) < \zeta \Rightarrow |\varphi_i(x_1) - \varphi_i(x_2)| < \varepsilon / \left(6 \cdot \sum_{k=1}^{i_0-1} \frac{1}{2^k}\right), \forall i \in \{1, \cdots, i_0 - 1\}, \quad (4.3)$$

which is possible since we have a finite number of $\varphi_i's$ and each $\varphi_i$ is uniformly continuous on the compact $M$).

Thus, if $y \in M$ is such that $d(x,y) < \zeta$ and $\nu = \delta_y$, we have that

$$\begin{aligned} d_*(\omega,\nu) &= \sum_{i=1}^{+\infty} \frac{1}{2^i} \left| \int_M \varphi_i d\omega - \int_M \varphi_i d\nu \right| \\ &= \sum_{i=1}^{+\infty} \frac{1}{2^i} |\varphi_i(x) - \varphi_i(y)| \\ &= \sum_{i=1}^{i_0-1} \frac{1}{2^i} |\varphi_i(x) - \varphi_i(y)| + \sum_{i=i_0}^{+\infty} \frac{1}{2^i} |\varphi_i(x) - \varphi_i(y)| \\ &< \varepsilon/6 + 2 \cdot \sum_{i=i_0}^{+\infty} \frac{1}{2^i} < \varepsilon/6 + 2 \cdot \varepsilon/12 = \varepsilon/3. \end{aligned}$$

In the last inequalities we used the fact that $|\varphi_i(x_1) - \varphi_i(x_2)| < 2,\ \forall x_1,\ x_2 \in M,\ \forall i \in \mathbb{N}$, because the functions $\varphi_i$ are taken in the unit ball $B^1$.

□

Lemma 4.2.2 above can be generalized in such a way that one can obtain the following:

**Remark 4.2.3.** *For each $\varepsilon > 0$ and $n \in \mathbb{N}$ there is $\zeta > 0$ such that $d(y,x) < \zeta$ then*

$$d_*\left(\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \omega, \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu\right) < \varepsilon/3.$$

*To do this it is enough to require that $\zeta$ satisfies 4.3 and the points $y$ are in a $\zeta$−neigborhood of $x$ in such a way that $d(f^j(x), f^j(y)) < \zeta,\ \forall j \in \{0, \cdots, n-1\}$ (that is, $y$ is a $\zeta$−shadow for $x$ until the $n^{th}$−iterate). In this way we have:*

$$\begin{aligned} & d_*\left(\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \omega, \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu\right) \\ = & \sum_{i\in\mathbb{N}} \frac{1}{2^i} \left| \int \varphi_i d\left(\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \omega\right) - \int \varphi_i d\left(\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu\right) \right| \\ \le & \sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{1}{n}\sum_{j=0}^{n-1} \left| \int \varphi_i \circ f^j d\omega - \int \varphi_i \circ f^j d\nu \right| \\ = & \sum_{i\in\mathbb{N}} \frac{1}{2^i}\frac{1}{n}\sum_{j=0}^{n-1} \left| \varphi_i \circ f^j(x) - \varphi_i \circ f^j(y) \right| \\ = & \sum_{i=1}^{i_0-1} \frac{1}{2^i}\frac{1}{n}\sum_{j=0}^{n-1} \left| \varphi_i \circ f^j(x) - \varphi_i \circ f^j(y) \right| + \sum_{i=i_0}^{+\infty} \frac{1}{2^i}\frac{1}{n}\sum_{j=0}^{n-1} \left| \varphi_i \circ f^j(x) - \varphi_i \circ f^j(y) \right| \\ < & \varepsilon/4 + 2 \cdot \sum_{i=i_0}^{+\infty} \frac{1}{2^i} < \varepsilon/6 + 2 \cdot \varepsilon/12 = \varepsilon/3. \end{aligned}$$

**Remark 4.2.4.** *Consider $\delta > 0$, $0 < \sigma < 1$ and suppose that $p$ is a periodic point of period $n$ such that $n$ is also a $(\sigma, \delta)$–hyperbolic time for $p$. By definition of hyperbolic time (see Definition 1.3.1), we have that $\prod_{j=0}^{n-1} \|(Df \circ f^j(x))^{-1}\| \le \sigma^n$ what, by Chain Rule, gives that $\|Df^n(p)^{-1}\| \le \sigma^n < 1$, that is, $p$ is a periodic repeller of $f$ (see item $a$) of Remark 1.3.4.*

The orbit of periodic repellers is a particular example of uniformly hyperbolic sets.

We know that hyperbolic fixed points have the following property:

**Fact 4.2.5.** *Let $f \in Diff^r(M)$ and $p$ be a hyperbolic fixed point of $f$. Then, there are neighborhoods $\mathcal{N}$ of $f$ in $Diff^r(M)$ and $U$ of $p$ in $M$, and a continuous map $\rho : \mathcal{N} \longrightarrow U$ which associates to each $g \in \mathcal{N}$ the only fixed point of $g$ in $U$, and that fixed point is hyperbolic.*

We may now prove Theorem 4.2.1. The strategy to prove this theorem is to use hyperbolic continuation (Fact 4.2.5) in order to obtain an invariant probability measure for every dynamic $g$ close enough to $f$ and then ensure that the measure obtained is indeed an expanding measure for $g$.

**Proof of Theorem 4.2.1:** Consider $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f_0)$ and a typical point $x$ of $\mu$, that is, a point such that $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \longrightarrow \mu$ in the weak–$*$ topology when $n \longrightarrow \infty$. By simplicity we consider $\ell = 1$. The case where $\ell > 1$ is treated analogously.

Let $\varepsilon > 0$ and consider $n_0 \in \mathbb{N}$ such that

$$d_* (\nu_x, \mu) < \varepsilon/3, \tag{4.4}$$

where $\nu_x := \frac{1}{n_0}\sum_{j=0}^{n_0-1} \delta_{f^j(x)}$.

Consider $\Delta \subset M$ with $\operatorname{diam}(\Delta) < \delta/2$ and suppose that $\Delta \cap \mathcal{A}_{hyp,+} \neq \emptyset$, where $\mathcal{A}_{hyp,+}$ is a compact set such that $\omega_{f,+,h}(x) = \mathcal{A}_{hyp,+}$ for $\mu$–a.e.p. $x \in M$. Let $F : \Delta \longrightarrow \Delta$ be a map with return time $R : \Delta \longrightarrow \mathbb{N}$, as in Definitions 2.3.2 and 2.3.1 (we know by Proposition 2.3.4 that $F$ is indeed a full induced Markov map with respect to $f$). Without loss of generality, we can take $x \in \Delta$. For example, if $\sum_{n\le 1} \sigma^{n/2} < \frac{1}{4}$ or if $f$ is backward separated we can take $\Delta$ as some hyperbolic nested ball $B_r^*(y)$, with $0 < r < \delta/4$ and $x \in B_r^*(y)$, given by Lemma 2.2.13.

Let $P$ be an element of the Markov partition $\mathcal{P}$ in $\Delta$ with $R(P) = k \in \mathbb{N}$. We know that $P$ is mapped (with expanding behavior) onto $\Delta$ by $F$, that is, $\|DF(x)^{-1}\| < \sigma^k < 1$ for $x \in \Delta$ (see item $a$) of Remark 1.3.4). Thus $(F|_P)^{-1} : \Delta \longrightarrow P$ is a contraction map and so $F$ admits a fixed point in $P$. Since $\mathcal{P}$ has countable many elements, we may write $\mathcal{P} = \{P_1, P_2, \cdots, P_k, \cdots\}$. We can associate to a given point $y \in \Delta$ it's itinerary in the elements of the partition $\mathcal{P}$ by defining $i_j(y) := k$ se $f^j(y) \in P_k$. Define the cylinder $C(j_0,\ j_1,\ \cdots j_n)$ associated to the map $F$ as the set of points $y \in \Delta$ such that $F^s(y) \in P_{j_s},\ \forall s \in \{0, \cdots, n\}$. It is not difficult to see that the inverse of $F$ restricted to this cylinder is also a contraction map. So each cylinder $C(j_0,\ j_1,\ \cdots j_n)$ contains a periodic point of $F$ with period $n+1$.

Consider a cylinder $C(j_0,\ j_1,\ \cdots j_{m_0-1})$ containing $x$. Since $\mathcal{P}$ is a Markov partition, we have that $\operatorname{diam}(C(j_0,\ j_1,\ \cdots j_{m_0-1})) \longrightarrow 0$ when $m_0 \longrightarrow +\infty$. Then given $\zeta > 0$ we may take $m_0$ large enough such that $\operatorname{diam}(C(j_0,\ j_1,\ \cdots j_{m_0-1})) < \zeta$, where $\zeta$ is as obtained on Remark 4.2.3 (it's clear that we may always take $m_0 \geq n_0$). In this way, we ensure that $d(f^j(x), f^j(x_f)) < \zeta,\ \forall j \in \{0, 1, \cdots, m_0\}$ and also that

$$d_*\left(\frac{1}{m_0}\sum_{j=0}^{m_0-1}\delta_{f^j(x)}, \nu_{x_f}\right) < \varepsilon/3, \tag{4.5}$$

where $\nu_{x_f} := \dfrac{1}{m_0}\sum_{j=0}^{m_0-1}\delta_{f^j(x_f)}$.

By Fact 4.2.5 we conclude that there is $\gamma_1 > 0$ such that if $d_1(g, f) < \gamma_1$ then $g$ admits a hyperbolic periodic repeller $x_g$ close enough to $x_f$: since the map $\rho$ above is continuous we conclude that $x_g$ may be taken in such a way that $d(f^j(x_g), f^j(x_f)) < \zeta,\ \forall 0 \leq j \leq m_0$, where $\zeta$ is taken as in 4.3. Thus we can ensure by Remark 4.2.3 that

$$d_*(\nu_{x_f}, \nu_{x_g}) < \varepsilon/3, \tag{4.6}$$

where $\nu_{x_g} := \dfrac{1}{m_0}\sum_{j=0}^{m_0-1}\delta_{g^j(x_g)}$.

By construction, $m_0 \geq n_0$ yields that $d_*\left(\dfrac{1}{m_0}\sum_{j=0}^{m_0-1}\delta_{f^j(x)}, \mu\right) \leq d_*(\nu_x, \mu)$. So 4.4, 4.5 and 4.6 yields that $d_*(\nu_g, \mu) \leq d_*(\nu_{x_g}, \nu_{x_f}) + d_*(\nu_{x_f}, \nu_x) + d_*(\nu_x, \mu) < \varepsilon$.

Define $\nu := \nu_{x_g}$, which is an ergodic invariant measure, since it is supported in a periodic orbit. For the same reason, we can see that, if $g$ is taken sufficiently close to $f$, points in the orbit of $x_g$ that are close to the respective points in the orbit of $x_f$ will have the same hyperbolic times (although the contraction rate for points in the orbit of $x_g$ may be different, lets say, bounded by some $0 < \sigma' < 1$, which may be taken such that $|\sigma - \sigma'| < \varepsilon$ since $x_g$ is obtained via hyperbolic continuation). So, a fortiori we have that the frequency of hyperbolic times for points in the orbit of $x_f$ and $x_g$ will be the same. Also the radius of hyperbolic balls will be bounded the same constant $\delta > 0$. We conclude that $\nu \in \mathcal{M}_{\mathfrak{erp}}(\sigma'^{1/2}, \ell, \delta, \theta, g)$.

$\square$

# Chapter 5

# Proof of Main Theorem

## 5.1 Integrability of the first hyperbolic time map

In this section we relate the integrability of the first hyperbolic time map with the frequency that hyperbolic times appear for the majority of points, and hence, with the existence of expanding measures related to sequences of dynamics. This kind of situation was studied by Alves, Araújo in [5] in the case of one fixed dynamic, where they showed that integrability of the first hyperbolic time map is sufficient condition for the existence of positive frequency of hyperbolic times for points in a set of full measure. The idea behind their proofs is to ensure that integrability of the first hyperbolic time map implies that the system is non-uniformly expanding and so (by using Proposition 1.3.5, for instance) obtain that almost every point has positive frequency of hyperbolic times. Since we are dealing with invariant reference measures, our approach is different from theirs.

Consider a non-flat map $f$ and suppose that there exists $(\sigma, \delta)-$hyperbolic times for almost every point with respect to a given $f-$invariant ergodic reference measure $\mu$. As we saw in Section 1.3, this allows us to introduce a map $h : M \longrightarrow \mathbb{Z}^{+}$ defined $\mu-$almost every where which assigns to $x \in M$ its first $(\sigma, \delta)-$hyperbolic time (in other words, $h(x) := \min\{n \in \mathbb{N};\ x \in H_n(\sigma, \delta, f)\}$). Observe that by definition of hyperbolic time, if $n$ is a $\sigma-$hyperbolic time for $x \in M$ and $\ell$ is a $\sigma-$hyperbolic time for $f^n(x)$ then $n + \ell$ is a $\sigma-$hyperbolic time for $x$. Moreover, since $h$ is well defined $\mu-$almost everywhere and we are working with $f-$ non-singular measures ($f$ preserves sets of measure $\mu$ zero), then $\mu-$almost every points must have infinitely many hyperbolic times.

The case we are mainly interested is a sequence of dynamics $f_n$ converging to $f_0$ where for each $n \in \mathbb{N}$ there is an $f_n-$invariant ergodic measure $\mu_n$ and the first $(\sigma, \delta)-$hyperbolic time map for $f_n$, which we denote by $h_n$, is $\mu_n-$integrable (note that $\sigma$ and $\delta$ are fixed).

**Theorem 5.1.1.** *Let $f_n : M \longrightarrow M$ be a sequence of non-flat maps which converge in* *the $C^1$–topology to a non-flat map $f_0$. Fix $0 < \sigma < 1$ and $\delta > 0$ and suppose that the first $(\sigma, \delta)$–hyperbolic time map $h_n$, associated with $f_n$, is integrable with respect to an $f_n$–invariant ergodic probability $\mu_n$ with the same bound $K > 0$ for all $n \geq 1$ (i.e., there is $K > 0$ such that $\int h_n d\mu_n < K$, $\forall n \geq 1$). Then there is $\theta > 0$ such that for all accumulation point $\mu_0$ of $\mu_n$ the frequency of $(\sigma, \delta)$–hyperbolic times with respect to $f_0$ is bounded from below by $\theta$ for $\mu_0$–almost every point $x \in M$.*

**Proof:** Fix $s \geq 1$. We will show that $\mu_s$–almost every point on $M$ has infinite $(\sigma, \delta)$–hyperbolic times and then ensure that these hyperbolic times occur with frequency higher than $\theta' := \dfrac{1}{K}$. Since $s$ is taken arbitrarily, by applying previous results we conclude that every accumulation point of the sequence $\mu_n$ must belong to $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f_0)$.

Let $Y_s \subset M$ be the set of points where $h_s$ is defined. By construction, $\mu_s(Y_s) = 1$. Since $\mu_s$ is $f_s$–invariant, $\mu_s\left(\bigcap_{i\geq 0}(f_s^{-1})^i(Y_s)\right) = 1$ and so we conclude that $\mu_s$–almost every point is such that $h_s$ is defined in the entire orbit of this point by $f_s$. We know that if $n$ is a $\sigma$–hyperbolic time for $x \in M$ and $\ell$ is a $\sigma$–hyperbolic time for $f^n(x)$ then $n + \ell$ is a $\sigma$–hyperbolic time for $x$. Thus, if we take $x \in M$ such that $h_s$ is defined in every point of $\mathcal{O}_{f_s}(x)$, we easily conclude that $x$ has infinitely many hyperbolic times, and hence $\mu_s$–almost every point $x$ in $M$ has infinitely many hyperbolic times.

To conclude the second part, we use Lemma A.0.5 applied to $f_s, F_s$ and $h_s$ and obtain that in fact there are $\lambda$, $\theta'$ and $\ell$ which do not depend on $s$ such that $\mu_s$ is a $(\lambda, \ell)$–expanding measure with frequency of hyperbolic times higher than $\theta'$. In particular, by using Lemma 1.3.16 and equation 1.15, we can find $\theta > 0$ such that the $\mu_s$–frequency of hyperbolic returns satisfies $\theta_{f_s} \geq \theta$. It is clear that if we take some open region $\Delta_s$ intersecting the statistical hyperbolic attractor $\mathcal{A}_{h,+}^{f_s}$ then $\mu_s$–almost every point in this region is contained in hyperbolic pre-balls where the contraction rate is given by $\sigma^{1/2}$, where $\sigma = e^{-\lambda/4}$. We conclude that $\mu_s \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f_s)$. Since $s$ was taken arbitrarily, we are allowed to use Theorem A to conclude that every accumulation point $\mu_0$ of the sequence $\mu_s$ belongs to $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f_0)$. In particular, it satisfies: $\mu_0$–almost every point has $\mu_0$–frequency of hyperbolic returns $\theta_{f_0}$ higher than $\theta$. By Claim 4.1.3 we know that $\mu_0$ is in fact an expanding measure for $f_0$. By Proposition 1.3.5 we conclude that there is $\tilde{\theta} > 0$ which is a lower bound for the frequency of hyperbolic times of $\mu_0$–almost every point $x \in M$. Since the frequency of hyperbolic times is higher than or equal to the frequency of hyperbolic returns, the proof is complete.

$\square$

## 5.2 Proof of the Main Theorem

In this section we prove our main result and we take into account the influence of Lyapunov exponents on the dynamic (see Definition 1.1.3).

We saw previously by Lemma 1.3.5 that if $\mu$ is a $(\lambda,\ell)$−expanding probability for a non-flat map $f : M \longrightarrow M$ with non flat critical/singular set $\mathcal{C}$ then there exists $\sigma,\delta,\theta' > 0$ such that $\mu$−almost every point $x \in M$ has $(\sigma,\delta)$−hyperbolic times with frequency greater than $\theta'$. In [21] Oliveira shows that if a measure has all of its Lyapunov exponents positive, this is a sufficient condition for the existence of hyperbolic times with positive frequency, as we can see in the following result.

**Theorem 5.2.1.** *Suppose that $f : M \longrightarrow M$ is a $C^{1+\alpha}$ map with non-degenerated critical set $\mathcal{C}$ which preserves an ergodic invariant probability measure $\mu$. Suppose in addition that $\log^+\|Df\|$ is $\mu$−integrable and that $\mu$ has all its Lyapunov exponents positive. Then there exists $\ell \in \mathbb{N}$ and a real number $\lambda > 0$ such that $\mu$ is a $(\lambda,\ell)$−expanding measure for $f$.*

In particular, by Proposition 1.3.5 we conclude that $\mu$−almost every point admits positive frequency of hyperbolic times.

**Remark 5.2.2.** *Theorem 5.2.1 is a restatement of Lemma* 3.5 *of [21]. Its worth to note that in this Lemma Oliveira uses the hypothesis of strong transitivity on the dynamics. However, this hypothesis is utilized to prove the existence of periodic points in dynamic balls, and not to estimate the frequency of hyperbolic times.*

By chain rule we may easily conclude that

$$\limsup_{n\longrightarrow\infty} \frac{1}{n}\log\|Df^n(x)^{-1}\|^{-1} \geq \limsup_{n\longrightarrow\infty} \frac{1}{n}\sum_{i=0}^{n-1}\log\|Df(f^i(x))^{-1}\|^{-1}.$$

Then we see that expanding measures have all Lyapunov exponents positive. Conversely, Theorem 5.2.1 shows us that positiveness of Lyapunov exponents implies NUE behavior (also see Claim 4.1.3). So

$$\mathcal{M}^1_{\mathfrak{exp}}(f) = \mathcal{M}^1_+(f).$$

In this way, we can prove our Main Theorem in the context of expanding measures, and the same conclusions will hold when we assume positiveness of all Lyapunov exponents. We will use Theorems A and B in the proof of the Main Theorem. Most of the work is already done, we only need to consider some adjustments in order to obtain a suitable decomposition as in item 1 of Definition 1.4

Lemmas 5.2.3 and 5.2.4 below provide another point of view for Theorems A and B, respectively: the statement of those theorems implies the statement of these lemmas (actually, they are equivalent). We stress that the distance $d(f,g)$ is considered associated to the $C^1$–topology.

**Lemma 5.2.3.** *Given $\varepsilon > 0$ and a non-flat map $f : M \longrightarrow M$ there exists $\gamma > 0$ such that if $d(f,g) < \gamma$ then for all $\nu \in \mathcal{M}_{\mathfrak{exp}}(\sigma'^{1/2}, \ell, \delta, \theta, g)$ there exists a measure $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$, with $|\sigma' - \sigma| < \varepsilon$ such that $d_*(\mu, \nu) < \varepsilon$.*

**Proof:** We argue by contradiction. Suppose that there exists $\varepsilon > 0$ and a non-flat map $f : M \longrightarrow M$ such that for each $\gamma > 0$ there exists a non-flat map $g : M \longrightarrow M$ with $d(f,g) < \gamma$ satisfying: Every $\nu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, g)$ is such that $d_*(\mu,\nu) \geq \varepsilon$, for every $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$. Setting $\gamma_n = \frac{1}{n}$, we obtain, for each $n \geq 1$ a dynamics $g_n$ and a measure $\nu_n \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, g_n)$ such that $g_n$ converges to $f$ and $d_*(\mu, \nu_n) \geq \varepsilon$, for every $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$ (we can ensure that $\nu_n$ exists by using Theorem B). But this is a contradiction to Theorem A.

□

**Lemma 5.2.4.** *Given $\varepsilon > 0$, a non-flat map $f : M \longrightarrow M$ and a measure $\mu \in \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$ , $\exists \gamma > 0$ and $\sigma'^{1/2} > 0$ such that if $d(f,g) < \gamma$ then there exists $\nu \in \mathcal{M}_{\mathfrak{exp}}(\sigma'^{1/2}, \ell, \delta, \theta, g)$ with $d_*(\mu,\nu) < \varepsilon$.*

This Lemma has the same statement as Theorem 4.2.1. We have rewrote it here again for the fluency of the text to the reader. Before we prove our main result, we ensure that in fact measures with bounded parameters are in fact expanding measures.

Now we prove our main result.

**Proof of Main Theorem:** Fix $\sigma, \ell, \delta, \theta > 0$ and consider $\sigma', \delta', \theta' > 0$ such that $\sigma' \geq \sigma$, $\delta' \geq \delta$ and $\theta' \geq \theta$. It is a straightforward fact that $\mathcal{M}_{\mathfrak{exp}}(\sigma'^{1/2}, \ell, \delta, \theta, f) \supset \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f)$, $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f) \supset \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta', \theta, f)$ and $\mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta, f) \supset \mathcal{M}_{\mathfrak{exp}}(\sigma^{1/2}, \ell, \delta, \theta', f)$.

Consider the family of sets given by $\mathcal{M}_{f,\imath} := \bigcup_{s=1}^{\imath} \mathcal{M}_{\mathfrak{exp}}(\imath/(\imath+1), s, 1/\imath, 1/\imath, f), \imath \geq 1$. By the inclusions in last paragraph, we have that $\mathcal{M}_{f,1} \subset \mathcal{M}_{f,2} \subset \cdots \subset \mathcal{M}_{f,\imath} \subset \cdots$. By construction we know that each element of $\mathcal{M}_{f,\imath}$ is in fact an expanding measure. Also, its clear that each element of $\mathcal{M}^1_+(f)$ belongs to $\mathcal{M}_{f,\imath}$, for some $\imath \in \mathbb{N}$. Then we can write

$$\mathcal{M}^1_+(f) = \bigcup_{\imath \geq 1} \mathcal{M}_{f,\imath}.$$

Also, for each $g$ in a neighborhood small enough of $f$, define

$$\mathcal{M}_{g,\imath} := \bigcup_{s=1}^{\imath} \mathcal{M}_{\mathfrak{exp}}(\imath/(\imath+1), s, 1/\imath, 1/\imath, g) \bigcup \bigcup_{s=1}^{\imath} \mathcal{M}_{\mathfrak{exp}}((\imath+1)/(\imath+2), s, 1/\imath, 1/\imath, g), \imath \geq 1.$$

We know by Theorem C that $\mathcal{M}_{\mathfrak{exp}}(1/\imath, s, 1/\imath, 1/\imath, f)$ is a compact set, $\forall \imath, s \in \mathbb{N}$. So, each set $\mathcal{M}_{f,\imath}$ is in fact a compact set, since it is a finite union of compact set.

Therefore, $\mathcal{M}_+^1(f)$ is $\sigma-$compact in the weak$-*$ topology. Analogously we obtain that $\mathcal{M}_+^1(g)$ is $\sigma-$compact in the weak$-*$ topology.

In order to conclude that for each $\imath \geq 1$ the function $g \mapsto \mathcal{M}_{g,\imath}$ is continuous at $f$ it is enough to see that given $\varepsilon > 0$, there is $\gamma > 0$ such that if $d(g, f) < \gamma$ then $d(\mathcal{M}_{f,\imath}, \mathcal{M}_{g,\imath}) < \varepsilon$ (in the Hausdorff topology), that is:

1 For each $\mu \in \mathcal{M}_{f,\imath}$ there exists $\nu \in \mathcal{M}_{g,\imath}$ with $d_*(\mu, \nu) < \varepsilon$ and

2 For each $\nu \in \mathcal{M}_{g,\imath}$ there exists $\nu \in \mathcal{M}_{f,\imath}$ with $d_*(\mu, \nu) < \varepsilon$.

But items 1 and 2 above are achieved by applying Lemmas 5.2.3 and 5.2.4 to $\mu$ and $\nu$, respectively, and we are done.

□

# Chapter 6

# Future perspectives

## 6.1 Partially hyperbolic diffeomorphisms

In [30] K. Rocha extends the construction of induced Markov maps (built in hyperbolic times) proposed by Pinheiro to partially hyperbolic diffeomorphisms whose central-stable direction is uniformly contractive and central-unstable direction is non-uniformly expanding and also obtained a lifted measures for hyperbolic measures associated to the partially hyperbolic diffeomorphism (see Theorems A and B of [30]). So a natural question is:

**Question 6.1.1.** *Can we extend our notion of continuous variation for partially hyperbolic diffeomorphisms whose central-stable direction is uniformly contractive and central-unstable direction is non-uniformly expanding?*

## 6.2 Non-hyperbolic flows

**Question 6.2.1.** *Is it possible to obtain results about continuous variation of expanding measures for semi-flows in the non-uniformly expanding context?*

## 6.3 Iterated Functions System (IFS)

We consider an *Iterated Function System*, or *IFS*, as a finite collection $G = (g_0, \cdots, g_{\ell-1})$ of diffeomorphisms of a compact connected manifold $M$. Consider now the semigroup generated by these transformations. An IFS can be embedded in a single dynamical system, the $1-$*step skew-product* $\varphi_G : \ell^{\mathbb{Z}} \times M \longrightarrow \ell^{\mathbb{Z}} \times M$ over the full shift $\sigma$ on $\ell^{\mathbb{Z}} = \{0, \cdots, \ell - 1\}^{\mathbb{Z}}$, which is defined by $\varphi_G(\omega, x) = (\sigma(\omega), g_{\omega_0}(x))$.

In this scenario, for any ergodic $\varphi_G-$invariant measure $\mu$, Oseledets theorem associates its *fibered Lyapunov exponents*, which are the values that can occur as limits

$$\lim_{n \longrightarrow +\infty} \frac{1}{n} \log \|D(g_{\omega_{n-1}} \circ \cdots \circ g_{\omega_0})(x) \cdot v\|, (\text{where } v \in T_x M \backslash \{0\}$$

for a positive measure subset of points $((\omega_n), x) \in \ell^{\mathbb{Z}} \times M$.

In [12] Bochi et al. obtained robustness for IFS exhibiting vanishing Lyapunov exponents. In an opposite direction, we would like to understand:

**Question 6.3.1.** *Under what conditions there is some kind of continuous variation for invariant probability measures of an IFS with non-zero Lyapunov exponents?*

## 6.4 Stability of equilibrium states for partially hyperbolic skew-products

This application refers to the theory of equilibrium states. In the classical setting, given a continuous map $f : M \longrightarrow M$ on a compact metric space $M$ and a continuous potential $\phi : M \longrightarrow \mathbb{R}$ we say that $\mu_\phi$ is an equilibrium state associated to $(f, \phi)$ if $\mu_\phi$ is an $f-$invariant probability measure characterized by the following variational principle:

$$P_f(\phi) = h_{\mu_\phi}(f) + \int \phi d\mu_\phi = \sup_{\mu \in \mathcal{M}_f(M)} \left\{ h_\mu(f) + \int \phi d\mu \right\},$$

where $P_f(\phi)$ denotes the topological pressure, $h_\mu(f)$ denotes the metric entropy and the supremum is taken over all $f-$invariant probability measures.

In the uniformly hyperbolic context, which includes uniformly expanding maps, it is well known that equilibrium states always exist and are unique if the potential $\phi$ is Hölder continuous and the dynamics $f$ is transitive. However the scenario beyond the uniformly hyperbolic context is pretty much incomplete, despite several advances obtained by several authors. In particular, we can cite work of Ramos, Viana [26] and Ramos, Siqueira [27]. Under some constraints on the potential and the dynamics, they obtained both existence and uniqueness of equilibrium states and also some statistical properties for those measures.

Theorem A and B of [26] state that for a hyperbolic Hölder continuous potential $\phi$, there exists a conformal measure $\nu$ which happens to be an expanding measure. Also, requiring that the dynamics is transitive, there exists an unique $f-$invariant ergodic equilibrium state $\mu_\phi$ which is absolutely continuous with respect to $\nu$.

In a work in progress by Alves, Ramos, Siqueira, they study statistical stability of equilibrium states in the context of Theorems A and B of [10]. That is, they want to obtain statistical stability for the absolutely continuous measure. The following question takes place:

**Question 6.4.1.** *Is that possible to obtain some kind of continuous variation for the set of conformal measures $\nu$?*

## 6.5 Metric entropy

In [8] Alves, Oliveira and Tahzibi prove that the metric entropy varies continuously when one considers sequences of invariant measures which are absolutely continuous with respect to Lebesgue. Also, in [14], Carvalho and Varandas deal with an analogous problem for diffeomorphisms but using different techniques. We know by Main Theorem that the set of expanding measures varies continuously in compact pieces. One can ask about the continuity of the entropy on the set of expanding measures, at least when we restrict ourselves to measures with bounded parameters.

**Question 6.5.1.** *In statements of Theorems A and B, can we obtain measures $\mu_n \longrightarrow \mu_0$ such that $h_{\mu_n} \longrightarrow h_{\mu_0}$?*

# Appendix A

# Auxiliary results

We state here the so called Pliss Lemma, which is strongly used to ensure the abundance of hyperbolic times in an expanding set.

**Lemma A.0.1.** *Given* $0 < c_1 < c_2 < A$ *let* $\theta = (c_2 - c_1)/(A - c_1)$. *Given real numbers* $a_1, \cdots, a_N$ *satisfying* $a_j \leq A$ *for every* $1 \leq j \leq N$ *and*

$$\sum_{j=1}^{N} a_j \geq c_2 N,$$

*then there are* $l > \theta N$ *and* $l$ *numbers* $1 < n_1 < \cdots < n_l \leq N$ *so that*

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_i - n)$$

*for every* $0 \leq n < n_i$ *and* $i = 1, \cdots, l$.

**Proof:** See [25].

$\square$

The proof of the last Lemma can be found in [25].

**Lemma A.0.2.** *Let* $\{G_j\}_{j\in\mathbb{N}}$ *be a collection of ensembles of* $M$ *such that* $f^j(x) \in G_{n-j}$ $\forall 0 \leq j < n$ $\forall x \in G_n$. *Seja* $B \subset X$ *and let* $x \in B$ *be such that* $\sharp\{j \geq 0;\ x \in G_j \text{ e } f^j(x) \in B\} = \infty$. *Let* $T : \mathcal{O}_f^+(x) \cap B \longrightarrow \mathcal{O}_f^+(x) \cap B$ *be a map given by* $T(y) = f^{g(y)}(y)$, *where* $g : \bigcup_j G_j \longrightarrow \mathbb{N}$ *is a function with* $1 \leq g(y) \leq \min\{j \in \mathbb{N};\ y \in G_j \text{ e } f^j(y) \in B\}$. *Then*

$$\sharp\{1 \leq j \leq n;\ x \in G_j \text{ e } f^j(x) \in B\} \leq \sharp\{j \geq 0;\ \sum_{k=0}^{j} g(T^k(x)) \leq n\}.$$

*Furthermore, if* $\limsup_{n\to\infty} \frac{1}{n} \sharp\{1 \leq j \leq n;\ x \in G_j \text{ e } f^j(x) \in B\} > \Theta > 0$, *then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j(x) \leq \Theta^{-1}.$$

The next result, whose proof can be found in Lemma 2.5 of [20], states that inverse branches vary continuously with the associated map. In fact, even differentiability can be obtained by this method.

**Lemma A.0.3.** *Let $f : M \longrightarrow M$ be a local homeomorphism, where $M$ is a compact, connected Riemannian manifold. Let $B = B_\delta(x) \subset M$ be a ball such that the inverse branches $f_1, \cdots, f_s : B \longrightarrow M$ are well defined as homeomorphisms onto their images. Then the map that assigns to each local homeomorphism $f$ its inverse branches $(f_1, \cdots, f_n)$ is continuous.*

By definition, a periodic point $p$ with period $n$ is a repeller if, and only if, $Df^n(p)$ is well defined and the absolute value of any eigenvalue of $Df^n(p)$ is bigger than 1. We know that

$$\lim_{n\to\infty} \|((Df^n(p))^{-1})^{n_0}\|^{\frac{1}{n_0}} = \min\{\lambda^{-1}; \lambda \text{ é um autovalor de } Df^n(p)\}, \tag{A.1}$$

and so, the periodic point $p$ is a repeller if, and only if, there exists $n_0 \geq 1$ such that $p$ is a periodic point for $\tilde{f} := f^{n_0}$ with period $n$ and such that $\log \|(D\tilde{f}^n(x))^{-1}\|^{-1} > 0$; $\forall x \in \mathcal{O}^+_{\tilde{f}}(p)$ (The "only if" part is immediate. In order to conclude the "if" part, it is enough to take any prime number $n_0 \in \mathbb{N}$ big enough).

One can show that the orbit of a periodic repeller point is an example of an expanding set, as we can see in next result.

**Lemma A.0.4.** *If $p$ is a periodic repeller point with period $n \geq 1$ and $\mathcal{O}^-_{\tilde{f}}(p) \cap \mathcal{C} = \emptyset$ then given any $\lambda_0 > 0$, there exists $\ell \geq 1$ such that $p$ is a periodic point with period $n$ with respect to $\tilde{f} := f^l$ and $\mathcal{O}^-_{\tilde{f}}(p)$ is a $\lambda_0$-expanding set for $\tilde{f}$.*

**Proof:** Proof can be found in [24] Lemma 9.2.

□

In Theorem 2.1.6 we relate a condition involving integrability of return times to the existence of invariant measures. In the next result we utilize a condition involving integrability of the first hyperbolic time (which could be replaced by a hyperbolic return time with some easy adaptations on the proof) to ensure that a measure is in fact $\lambda$–expanding, for some $\lambda > 0$.

**Lemma A.0.5.** *Consider a probability ergodic measure $\mu$ invariant with respect to $f : M \longrightarrow M$ and suppose that the first $(\sigma, \delta)$–hyperbolic time function $h$ is integrable with respect to $\mu$: $\exists K > 0$ such that $\int h d\mu < K$. Then there are $\lambda > 0$ and $0 < \theta < 1$ which depend only on $\sigma$, $\delta$ and $K$ such that $\mu$ is a $\lambda$–expanding measure for $f$ for which almost every point has frequency of hyperbolic times higher than $\theta$.*

**Proof:** Assuming that $\int h d\mu < K$ we obtain by Theorem 1.1 of [37] that there exists a measure $\nu \ll \mu$ that is an ergodic invariant probability with respect to the induced map $F : Y \longrightarrow Y$ given by $F(x) := f^{h(x)}(x)$.

Since $\mu$ is an invariant ergodic probability, we have, by Birkhoff's Theorem that

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{j=0}^{n-1} h \circ F^j(x) = \int h d\mu,$$

for $\mu$–almost every point $x \in M$.

Fix $x \in M$ as a typical point for $\mu$.

**Claim. A.0.6.**

$$\sharp\left\{0 < \ell \le \sum_{j=0}^{n-1} h_s \circ F_s^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\} = n.$$

**Proof:** We argue by induction on $n$. In fact, for $n = 1$, we have that $\sum_{j=0}^{n-1} h \circ F^j(x) = h(x)$. So, $\sharp\left\{\ell \le \sum_{j=0}^{n-1} h \circ F^j(x);\ x \in H_j(\sigma,\delta,f)\right\} = 1$, since $h(x)$ is the first hyperbolic time of $x$. Suppose that the result is valid for $n \in \mathbb{N}$. For $n+1$ we have that

$$\begin{array}{rcl}
\sharp\left\{0 < \ell \le \sum_{j=0}^{n} h \circ F^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\} & = & \\
\sharp\left\{0 < \ell \le \sum_{j=0}^{n-1} h_s \circ F^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\} & + & \\
\sharp\left\{\sum_{j=0}^{n-1} h_s \circ F^j(x) < \ell \le h \circ F^n(x);\ x \in H_\ell(\sigma,\delta,f)\right\} & = & \\
n + \sharp\left\{\sum_{j=0}^{n-1} h \circ F^j(x) < \ell \le h \circ F^n(x);\ x \in H_\ell(\sigma,\delta,f)\right\} & = & n+1,
\end{array}$$

where in the second equality we used the induction hypothesis and in the third equality we used that $\sharp\left\{\sum_{j=0}^{n-1} h \circ F^j(x) < \ell \le h \circ F^n(x);\ x \in H_\ell(\sigma,\delta,f)\right\} = 1$, because if it is higher than 1, we would have that there is a hyperbolic time for $x$ between $\sum_{j=0}^{n-1} h \circ F^j(x)$ and $h \circ F^n(x)$, and so, the first hyperbolic of $F^{n-1}(x)$ would be smaller than $h(F^{n-1}(x))$, what is an absurd. □

Since $\sharp\left\{0 < \ell \le \sum_{j=0}^{n-1} h \circ F^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\} = n$, we know that

$$\frac{\sharp\left\{0 < \ell \le \sum_{j=0}^{n-1} h \circ F^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\}}{\sum_{j=0}^{n-1} h_s \circ F^j(x)} = \frac{n}{\sum_{j=0}^{n-1} h \circ F^j(x)},$$

and by Birkhoff's theorem this last term converges to $\frac{1}{\int h d\nu} \ge \frac{1}{\int h d\mu} > \frac{1}{K}$ for $\mu$–almost every point $x \in M$ (because $\nu \ll \mu$). But the expression

$$\lim_{n\longrightarrow+\infty} \frac{\sharp\left\{0 < \ell \le \sum_{j=0}^{n-1} h \circ F^j(x);\ x \in H_\ell(\sigma,\delta,f)\right\}}{\sum_{j=0}^{n-1} h \circ F^j(x)}$$

indicates precisely the frequency of hyperbolic times of $x$. We conclude that $\mu$–almost every point has frequency of $(\sigma, \delta)$–hyperbolic times higher than $\theta := 1/K$.

Setting $\lambda := -\log\sigma$, we easily conclude that $\limsup_{n\longrightarrow+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df(f^j(x))^{-1}\|^{-1} \geq \log\sigma^{-1} = \lambda > 0$. By Lemma 2.4.5, this condition implies that in fact $\mu$ is a $\lambda$–expanding measure for $f$.

$\square$

# Appendix B

# Continuous variation of the first hyperbolic time map

Denote $h_{\sigma,\delta,f}(x)$ as the first $(\sigma,\delta)-$hyperbolic time for $x$ with respect to $f$. In this section we present auxiliary results that allow us to deal with the frequency of hyperbolic times in terms of the integral of the first hyperbolic time map. We will see that integrability can be extended to dynamics close enough maybe if we have less contraction in hyperbolic times.

**Lemma B.0.1.** *Consider $x \in M$ such that $h_{\sigma,\delta,f}(x) = n \in \mathbb{N}$. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon > 0, \varepsilon \leq \varepsilon_0$, there is a neighborhood $V \ni x$ such that $y \in V \Rightarrow h_{\sigma+\varepsilon,\delta,f}(y) = n$.*

**Proof:** Consider $\varepsilon > 0$. By the continuity of $Df$ we have that the functions $\xi_k := \prod_{j=n-k}^{n-1} \|(Df \circ f^j)^{-1}\|$, $0 \leq k < n$ are continuous on $x$. Since $\xi_k(x) \leq \sigma^k$ there is a neighborhood $V \ni x$ such that $\xi_k(y) \leq (\sigma+\varepsilon)^k$, $\forall 1 \leq k < n$. So $n$ is a $(\sigma+\varepsilon,\delta)-$hyperbolic time for every $y \in V$ (with respect to $f$). In addition, see that we can take $V$ and $\varepsilon$ small enough in such a way that $n$ is the first $(\sigma+\varepsilon,\delta)-$hyperbolic time for every $y \in V$ (with respect to $f$). In fact, if we assume by contradiction that $x$ is accumulated by a sequence of points $y$ with hyperbolic time smaller than $n$ (we can assume, passing to a subsequence if necessary, that $x$ is accumulated by a sequence of points with $(\sigma+\varepsilon,\delta)-$hyperbolic time equal to $s$, for some $1 \leq s < n$) then by continuity of $Df$ we have that $\xi_k(y) \leq (\sigma+\varepsilon)^k$, $\forall 1 \leq k < s$ and so $s$ is a hyperbolic time for $x$ as well. Since $\varepsilon > 0$ was taken arbitrarily, making $\varepsilon \longrightarrow 0$ we conclude that $s$ is also a $(\sigma,\delta)-$hyperbolic time for $x$, what is a contradiction, since $h_{\sigma,\delta,f}(x) = n$. Then, there must exists $\varepsilon_0 > 0$ such that $\varepsilon \leq \varepsilon_0$ implies $h_{\sigma+\varepsilon,\delta,f}(x) = n$ and hence, $h_{\sigma+\varepsilon,\delta,f}(y) = n$ for every $y \in V$. If $\mathcal{C} \neq \varnothing$ we also consider the functions $\tilde{\xi}_k := d_\delta(f^{n-k}, \mathcal{C})$, which are obviously continuous on $x \notin \mathcal{C}$. By utilizing an argument analogous to the one above, the result follows.

$\square$

Denote $H_n^*(\sigma,\delta,f) = \{x \in M,\ h_{\sigma,\delta,f}(x) = n\}$ as the set of points whose first $(\sigma,\delta)-$hyperbolic time with respect to $f$ is $n$. Since $h_{\sigma,\delta,f}$ is integrable with respect to $\mu$, $h_{\sigma+\varepsilon,\delta,f}$ is also integrable with respect to $\mu$ (because $h_{\sigma+\varepsilon,\delta,f}(x) \leq h_{\sigma,\delta,f}(x)$ for $\mu-$almost every $x \in M$). Since the image of $h$ is a subset of the natural numbers, we can write

$$\int h_{\sigma+\varepsilon,\delta,f} d\mu = \sum_{k\geq 1} k \cdot \mu(H_k^*(\sigma+\varepsilon,\delta,f)).$$

Now consider $\mu, x, x_f, x_g$ and $m_0$ as in the proof of Theorem 4.2.1. We will see that (if $m_0$ is large enough) then $\nu_{x_f}$ is close enough to $\mu$ in such a way that we can ensure that $\int h_{\sigma+\varepsilon,\delta,f} d\nu_{x_f} < K$ (remember that, by hypothesis, $\int h_{\sigma,\delta,f} d\mu < K$ and hence $\int h_{\sigma+\varepsilon,\delta,f} d\mu < K$).

We know, by Lemma B.0.1 that for each $j \in \mathbb{N}$, $H_j^*(\sigma+\varepsilon,\delta,f)$ is an open set. By simplicity we will denote here this set by $H_j^*$. Also, since $\mu$ is a probability measure, $\sum_{j\geq 1} \mu(H_j^*) = 1$. Consider $\tau > 0$ and $j_0 \in \mathbb{N}$ big enough such that $\sum_{j\geq j_0} j \cdot \mu(H_j^*) < \tau/2$ (which is possible since $h_{\sigma+\varepsilon,\delta,f}$ is integrable with respect to $\mu$). Thus, by taking $m_0$ large enough, we can ensure that $\mu$ and $\nu_{x_f}$ are close enough in the weak$^*-$topology in such a way that one has

$$|\mu(H_j^*) - \nu_{x_f}(H_j^*)| < \frac{\gamma}{2 \cdot \sum_{i=1}^{j_0-1} i};\ \forall 1 \leq j < j_0.$$

In this way we have that

$$\begin{aligned}
& \left| \int h_{\sigma+\varepsilon,\delta,f} d\mu - \int h_{\sigma+\varepsilon,\delta,f} d\nu_{x_f} \right| \\
= & \left| \int_{\bigcup_{i=1}^{j_0-1} H_i^*} h_{\sigma+\varepsilon,\delta,f} d\mu - \int h_{\sigma+\varepsilon,\delta,f} d\nu_{x_f} + \int_{\bigcup_{i\geq j_0} H_i^*} h_{\sigma+\varepsilon,\delta,f} d\mu \right| \\
\leq & \ \tau/2 + \tau/2 = \tau.
\end{aligned}$$

Taking $\tau = K - \int h_{\sigma+\varepsilon,\delta,f} d\mu$, we have that $\int h_{\sigma+\varepsilon,\delta,f} d\nu_{x_f} < K$.

Fix $x \in M$ and define $\xi_{k,x}(f) := \prod_{j=n-k}^{n-1} \|(Df \circ f^j)^{-1}(x)\|$, $0 \leq k < n$. Consider $\tilde{\varepsilon} > 0$. If we allow $f$ to vary in the $C^1-$topology we obtain that there exists $\gamma_2 > 0$ such that if $d(g,f) \leq \gamma_2$ then $\xi_{k,x}(f) \leq (\sigma+\varepsilon)^k \Rightarrow \xi_{k,x}(g) \leq (\sigma+\tilde{\varepsilon})^k$ for each $0 \leq k < n$. Applying this to $x = x_f$ and since we already know by Lemma B.0.1 that if $\zeta > 0$ is small enough then $d(x_f,x_g) \leq \zeta \Rightarrow \xi_{k,x_g}(g) \leq (\sigma + \tilde{\tilde{\varepsilon}})$. We conclude that there exists $\varepsilon' > 0$ such that $h_{g,\sigma+\varepsilon',\delta}(x_g) = h_{f,\sigma+\varepsilon',\delta}(x_f)$. Then $\int h_{g,\sigma+\varepsilon',\delta}(x_g) d\nu < K$.

**Remark B.0.2.** *Let $f : M \longrightarrow M$ be a non-flat map and $\mu \in \mathcal{M}_{\mathfrak{erp}}(\sigma^{1/2},\ell,\theta,\delta,f)$. If there exists $0 < \lambda < 1$ and $C > 0$ such that for each, $n \geq 1$, $\mu(\Gamma_n) \leq C\lambda_n$ then $\mu(\{h \geq n\}) \leq C\lambda_n$, where $h$ is the first $(\sigma,\delta)-$hyperbolic time map.*

*In fact, Remark 1.3.7 gives us that if we choose, for example, $\sigma = e^{\lambda/4}$ then*

$$\sharp\{1 \leq j \leq n;\ x \in H_j(\sigma,\delta,f)\} \geq \theta' n$$

*whenever* $\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f^j(x)\|^{-1}\geq\lambda$ *and* $\frac{1}{n}\sum_{j=0}^{n-1}-\log\operatorname{dist}_\delta(f^j(x),\mathcal{C})\leq\frac{\lambda}{16\beta}$. *By definition, if* $x\in(\Gamma_n)^c$ *then the expansion time function and the recurrence time function on* $x$ *are smaller than* $n$. *It means that for some* $n_0<n$ *one has*

$$\frac{1}{n}\sum_{j=0}^{n_0-1}\log\|Df(f^j(x)\|^{-1}\geq\lambda$$

*and*

$$\frac{1}{n}\sum_{j=0}^{n_0-1}-\log\operatorname{dist}_\delta(f^j(x),)\leq\frac{\lambda}{\beta}$$

*and so there exists at least one* $(\sigma,\delta)$*–hyperbolic time for* $x$ *smaller than* $n_0<n$. *This fact implies that* $h(x)<n$.

*We have just concluded that* $(\Gamma_n)^c\subset\{h<n\}$. *So* $\{h\geq n\}\subset\Gamma_n$ *and from this we obtain that the estimates made on the tail* $\Gamma_n$ *will be the same for* $\{h\geq n\}$*:*

$$\mu(\Gamma_n)\leq C\sigma^n\Rightarrow\mu(\{h\geq n\})\leq C\sigma^n.$$

*We can see that working with dynamics whose the first hyperbolic time map is integrable isn't a strong restriction, since every expanding measure with exponential decay of the measure of the tail* $\Gamma_n$ *satisfies this hypothesis (in fact, we can see that even if* $\mu(\Gamma_n)$ *has polynomial decay, with order at least* $2$*, then the first hyperbolic time will be integrable with respect to* $\mu$*).*

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