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SCALING LIMITS FOR SLOWED EXCLUSION PROCESS

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Scaling limits for slowed exclusion process

Mariana Tavares de Aguiar

Tese submetida ao corpo docente da pós-graduação e pesquisa do Instituto de Matemática da Universidade Federal da Bahia como parte dos requisitos necessários para a obtenção do grau de doutora em Matemática.

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“Do or do not. There is no try.”

Master Yoda

Resumo

O presente trabalho teve o intuito de estudar os seguintes problemas: Limite Hidrodinâmico para o processo de exclusão simples simétrico (SSEP) com uma membrana lenta e as Flutuações fora do equilíbrio para o SSEP com um elo lento. Mais precisamente, o modelo em estudo do Limite Hidrodinâmico é o SSEP, no toro d-dimensional, que possui uma membrana Λ cuja taxa de passagem é dada por α/N^β, $\alpha > 0$, menor do que a taxa em outros elos. Devido à existência desta membrana lenta, dependendo do regime do parâmetro que regula a lentidão desta membrana, aparecem a nível macroscópico condições de fronteira. Para $\beta \in [0,1)$, a equação hidrodinâmica é dada pela equação de calor no toro contínuo, significando que a membrana lenta não tem efeito no limite. Para $\beta \in (1,\infty]$, a equação hidrodinâmica é dada pela equação de calor com condições de bordo de Neumann, significando que a membrana divide o toro em duas regiões isoladas Λ and $\Lambda^\complement$. E, para o valor crítico $\beta = 1$, a equação hidrodinâmica é dada pela equação de calor com condições de fronteira de Robin, relacionada com a lei de Fick. No caso das Flutuações, o modelo em estudo é o SSEP unidimensional que possui um elo lento. A grande dificuldade no trabalho das Flutuações, foi obter as estimativas precisas de probabilidades de transição de passeios aleatórios de dimensão 1, quando olhamos para a derivada discreta e de dimensão 2 quando olhamos para a função correlação.

Palavras-chave: Sistema de Partículas, limite hidrodinâmico, flutuações, processo de exclusão.

Abstract

The present work aims to study the following problems: The Hydrodynamic Limit for the simple symmetric exclusion processes (SSEP) with a slow membrane and the non-equilibrium fluctuations for the SSEP with a slow bond. more precisely, the model in study of the Hydrodynamic Limit is the SSEP in the d-dimensional torus, bonds crossing the membrane Λ have jump rate α/N^{β}, $\alpha > 0$, lower than the rate in other bonds. Due to the existence of this slow membrane, depending on the regime of the parameter that regulates the slowness of this membrane, boundary conditions appear ate macroscopic level. For $\beta \in [0,1)$, the hydrodynamic equation is given by the usual heat equation on the continuous torus, meaning that the slow membrane has no effect in the limit. For $\beta \in (1,\infty]$, the hydrodynamic equation is the heat equation with Neumann boundary conditions, meaning that the slow membrane divides the torus into two isolated regions Λ and $\Lambda^{\complement}$. And, for the critical value $\beta = 1$, the hydrodynamic equation is the heat equation with certain Robin boundary conditions related to the Fick's Law. In the case of Fluctuations, the model in study is the SSEP one-dimensional with a slow bond. The main difficulty of this work is a precise estimate of transition probabilities of random walks, in 1-d when looking at the discrete derivative and in 2-d when looking at the correlation.

Keywords: Particle Systems, hydrodynamic limit, fluctuations, exclusion process.

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Chapter 1

Introduction

This thesis is focused on the development of two important contributions in the area of scaling limits of interacting particle systems. The first result establishes the hydrodynamic limit for a symmetric simple exclusion process (SSEP) on the d-dimensional discrete torus $\mathbb{T}_N^d$ with a spatial non-homogeneity given by a slow membrane. The slow membrane is defined here as the boundary of a smooth simple connected region Λ on the continuous d-dimensional torus $\mathbb{T}^d$. In this setting, bonds crossing the membrane have jump rate α/N^β and all other bonds have jump rate one, where $\alpha > 0$, $\beta \in [0, \infty]$, and $N \in \mathbb{N}$ is the scaling parameter. In the diffusive scaling we prove that the hydrodynamic limit presents a dynamical phase transition, that is, it depends on the regime of β. For $\beta \in [0, 1)$, the hydrodynamic equation is given by the usual heat equation on the continuous torus, meaning that the slow membrane has no effect in the limit. For $\beta \in (1, \infty]$, the hydrodynamic equation is the heat equation with Neumann boundary conditions, meaning that the slow membrane $\partial\Lambda$ divides $\mathbb{T}^d$ into two isolated regions Λ and $\Lambda^\complement$. And for the critical value $\beta = 1$, the hydrodynamic equation is the heat equation with certain Robin boundary conditions related to the Fick's Law. The second result is the non-equilibrium fluctuations for the one-dimensional symmetric simple exclusion process with a slow bond. This generalizes a result of [8, 10], which dealt with the equilibrium fluctuations. The foundation stone of our proof is a precise estimate on the correlations of the system. To obtain these estimates, we first deduce a spatially discrete PDE for the covariance function and we relate it to the local times of a random walk in a non-homogeneous environment via Duhamel's principle. Projection techniques and coupling arguments reduce the analysis to the problem of studying the local times of the classical random walk. We think that the method developed here can be applied to a variety of models, and we provide a discussion on this matter.

Chapter 2

Hydrodynamic Limit for the SSEP with a slow membrane

2.1 Introduction

A central question of Statistical Mechanics is about how microscopic interactions determine the macroscopic behavior of a given system. Under this guideline, an entire area on scaling limits of interacting random particle systems has been developed, see [18] and references therein.

In the last years, many attention has been given to scaling limits of (spatially) non-homogeneous interacting systems, see for instance [12, 7] among many others. Such an attention is quite natural due to the fact that a non-homogeneity may represent vast physical situations, as impurities, changing of density in the media etc. Among those interacting particles systems, processes of *exclusion type* have special importance: they are, at same time, mathematically tractable and have a physical interaction, leading to precise representation of many phenomena. Being more precise, a random process is called of *exclusion type* if it has the *hard-core interaction*, that is, at most one particle is allowed per site of a given graph. The random evolution of the system (in the symmetric case) can be described as follows: to each edge of the given graph, a Poisson clock is associated, all of them independent. At a ring time of some clock, the occupation values for the vertexes of the corresponding edge are interchanged.

In [12], a quite broad setting for the one-dimensional symmetric exclusion process (SEP) in non-homogeneous medium has been considered, being obtained its hydrodynamic limit, that is, the law of large numbers for the time evolution of the spatial density of particles. The hydrodynamic equation there was given by a PDE related to a Krein-Feller operator. And in [6], the fluctuations for the same model were obtained.

The scenario for the SEP in non-homogeneous medium in dimension $d \geq 2$ up to now is far less understood. In [25], a generalization of [12] to the d-dimensional setting was reached. However, the definition of model there was very specific to permit a reduction to the one-dimensional approach of [12].

In [13], the hydrodynamic limit in the diffusive scaling for the following d-dimensional simple symmetric exclusion process (SSEP) in non-homogeneous medium was proved, where the term *simple* means that only jumps to nearest neighbors are allowed. The underlying graph is the discrete d-dimensional torus, and all bonds of the graph have rate one, except those laying over a $(d-1)$-dimensional closed surface, which have rate given by N^{-1} times a constant depending on the angle between the edge and the normal vector to the surface, where N is the scaling parameter. The hydrodynamic equation obtained was given by a PDE related to a d-dimensional Krein-Feller operator. Despite less broad in certain sense than the setting of [25], the model in [13] cannot be approached by one-dimensional techniques, being truly d-dimensional.

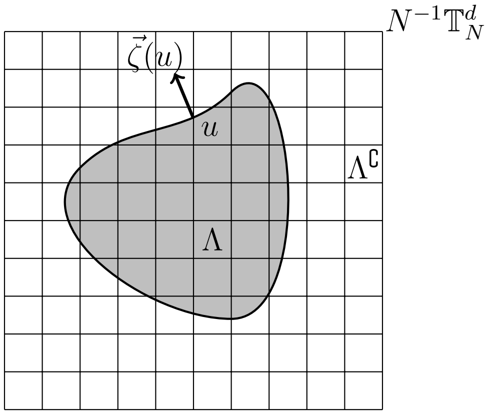


Figure 2.1: The region in gray represents Λ, and the white region represents its complement $\Lambda^{\complement}$. The grid represents $N^{-1}\mathbb{T}_N^d$, the discrete torus embedded on the continuous torus $\mathbb{T}^d$. By $\vec{\zeta}(u)$ we denote the normal exterior unitary vector to Λ at the point $u \in \partial\Lambda$.

In the present paper, we consider a d-dimensional model close to the one in [13] and related to the *slow bond phase transition behavior* of [7, 8, 9]. It is fixed a $(d-1)$-dimensional smooth surface $\partial\Lambda$ in the continuous d-dimensional torus $\mathbb{T}^d$, see Figure 2.1. Edges have rates equal to one, except those intersecting $\partial\Lambda$, which have rate α/N^β, where $\alpha > 0$, $\beta \in [0,\infty]$ and $N \in \mathbb{N}$ is the scaling parameter. Here we prove the hydrodynamic limit, which depends on the range of β, namely, if $\beta \in [0,1)$, $\beta = 1$ or $\beta \in (1,\infty]$.

For $\beta \in [0,1)$, the hydrodynamic equation is given by the usual heat equation: meaning that, in this regime, the slow bonds do not have any effect in the continuum limit. For $\beta \in (1,\infty]$, the hydrodynamic equation is the heat equation with the following Neumann boundary conditions over $\partial\Lambda$:

$$\frac{\partial\rho(t,u^+)}{\partial\vec{\zeta}(u)} = \frac{\partial\rho(t,u^-)}{\partial\vec{\zeta}(u)} = 0, \qquad \forall\, t \geq 0,\, u \in \partial\Lambda,$$

where $\vec{\zeta}$ is the normal unitary vector to $\partial\Lambda$. This means that, in this regime, the slow bonds are so strong that there no flux of mass through $\partial\Lambda$ in the continuum, despite the existence of flux of particles in the discrete for each $N \in \mathbb{N}$. For the critical value $\beta = 1$, the hydrodynamic equation is given by the heat equation with the following Robin boundary conditions:

$$\frac{\partial \rho(t,u^+)}{\partial \vec{\zeta}(u)} = \frac{\partial \rho(t,u^-)}{\partial \vec{\zeta}(u)} = \alpha\Big(\rho(t,u^+) - \rho(t,u^-)\Big)\sum_{j=1}^{d} |\langle \vec{\zeta}(u), e_j\rangle|, \quad t \geq 0,\, u \in \partial\Lambda\,, \quad \textbf{(2.1)}$$

where u^- denotes the limit towards $u \in \partial\Lambda$ through points over Λ while u^+ denotes the limit towards $u \in \partial\Lambda$ through points over $\Lambda^{\complement}$, and $\{e_1 \dots, e_d\}$ is the canonical basis of $\mathbb{R}^d$.

We observe that the Robin boundary condition above is in agreement with the *Fick's Law*: the spatial derivatives are equal due to the conservation of particles, representing the rate at which the mass crosses the boundary. Such a rate is proportional to the difference of concentration on each side of the boundary, being the diffusion coefficient through the boundary at a point $u \in \partial\Lambda$ given by $D(u) = \alpha\sum_{j=1}^{d}|\langle\vec{\zeta}(u), e_j\rangle|$. Since $\vec{\zeta}(u)$ is a unitary vector, the reader can check via Lagrange multipliers that this diffusion coefficient satisfies

$$\alpha \;\leq\; D(u) \;\leq\; \alpha\sqrt{d}$$

in dimension $d \geq 2$. Moreover, in this case $\beta = 1$, the hydrodynamic equation exhibits the phenomena of *non-invariance for isometries*. Let us explain this notion. Consider an isometry $\mathbf{T} : \mathbb{T}^d \to \mathbb{T}^d$, an initial density profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and denote by $(S(t)\rho_0)(u)$ the solution of the usual heat equation with initial condition ρ_0. Then,

$$\big(S(t)(\rho_0 \circ \mathbf{T})\big)(u) \;=\; (S(t)\rho_0)\big(\mathbf{T}(u)\big)\,.$$

In other words, if we isometrically move the initial condition of the usual heat equation, the solution of the PDE under this new initial condition is the equal to the previous solution moved by the same isometry. On the other hand, as we can see in (2.1), the diffusion coefficient $D(u)$ depends on how the surface $\partial\Lambda$ is positioned with respect to the canonical basis. Hence the PDE for $\beta = 1$ *is not invariant for isometries*, differently from the cases $\beta \in [0,1)$ and $\beta \in (1,\infty]$. Note that the diffusion coefficient also says that the underlying graph plays a role in the limit.

Besides the dynamical phase transition itself, this work has the following features. First of all, in contrast with some previous works, the hydrodynamic equations are characterized as classical PDEs, with clear interpretation. In the regime $\beta \in [0,1)$, the proof relies on a sharp replacement lemma which compares occupations of neighbor sites in opposite sides of $\partial\Lambda$. For $\beta = 1$, the proof is based on a precise analysis of the surface integrals and the model drops the *ad hoc* hypothesis adopted in [13]: here the rates for bonds crossing $\partial\Lambda$ are all equal to α/N, with no extra constant depending on the incident angle. Finally, a remark the uniqueness of weak solutions for the cases $\beta = 1$ and $\beta \in (1,\infty]$. Uniqueness of weak solutions are in general a delicate and technical issue, specially for dimension higher than one. In Proposition 2.7.2 we provide a general statement which leads to the uniqueness of weak solutions in both cases $\beta = 1$ and $\beta \in (1,\infty]$. The keystone of the proof is the notion of *Friedrichs extension* for strongly monotone symmetric operators. The uniqueness statement has the feature of being simple, d-dimensional and easily adaptable to many contexts. However, it is strictly limited to the uniqueness of weak solutions of parabolic linear PDEs with linear boundary conditions.

The paper is divided as follows: In Section 2.2 we state definitions and results. In Section 2.3 we draw the strategy of proof for the hydrodynamic limit. In Section 2.4 is reserved to the proof of tightness of the processes. In Section 3.3 we prove the necessary replacement lemmas and energy estimates. In Section 2.6 we characterize limit points as concentrated on weak solutions of the respective PDEs, and in Section 2.7 we assure uniqueness of those weak solutions.

2.2 Definitions and Results

Let $\mathbb{T}^d$ be the continuous d-dimensional torus, which is $[0,1)^d$ with periodic boundary conditions, and let $\mathbb{T}_N^d$ be the discrete torus with N^d points, which can naturally embedded in the continuous torus as $N^{-1}\mathbb{T}_N^d$, see Figure 2.1. We therefore will not distinguish notation for functions defined on $\mathbb{T}^d$ or $N^{-1}\mathbb{T}_N^d$.

By $\eta = (\eta(x))_{x\in\mathbb{T}_N^d}$ we denote configurations in the state space $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$, where $\eta(x) = 0$ means that the site x is empty, and $\eta(x) = 1$ means that the site x is occupied. By a *symmetric simple exclusion process* we mean the Markov Process with configuration space Ω_N and exchange rates $\xi_{x,y}^N > 0$ for $x, y \in \mathbb{T}_N^d$ with $\|x-y\|_1 = 1$. This process can be characterized in terms of the infinitesimal generator $\mathcal{L}_N$ acting on functions $f : \Omega_N \to \mathbb{R}$ as

$$(\mathcal{L}_N f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^{d} \xi_{x,x+e_j}^N \Big[f(\eta^{x,x+e_j}) - f(\eta)\Big]\,,$$

where $\{e_1,\dots,e_d\}$ is the canonical basis of $\mathbb{R}^d$ and $\eta^{x,x+e_j}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+e_j)$, that is,

$$\eta^{x,x+e_j}(y) \;=\; \begin{cases} \eta(x+e_j), & \text{if } y = x\,, \\ \eta(x), & \text{if } y = x+e_j\,, \\ \eta(y), & \text{otherwise.} \end{cases}$$

The Bernoulli product measures $\{\nu_\theta^N : \theta \in [0,1]\}$ are invariant and in fact, reversible, for the symmetric nearest neighbor exclusion process introduced above. Namely, ν_θ^N is a product measure on Ω_N whose marginal at site $x \in \mathbb{T}_N^d$ is given by

$$\nu_\theta^N\{\eta : \eta(x) = 1\} \;=\; \theta\,.$$

Fix now two parameters $\alpha > 0$ and $\beta \in [0,\infty]$ and a simple connected closed region $\Lambda \subset \mathbb{T}^d$ whose boundary $\partial\Lambda$ is a smooth $(d-1)$-dimensional surface. The *symmetric simple exclusion process with slow bonds over* $\partial\Lambda$ (SSEP with slow bonds over $\partial\Lambda$) we define now is the particular simple symmetric exclusion process with exchange rates given by

$$\xi_{x,x+e_j}^N \;=\; \begin{cases} \dfrac{\alpha}{N^\beta}\,, & \text{if } \dfrac{x}{N} \in \Lambda \text{ and } \frac{x+e_j}{N} \in \Lambda^\complement, \text{ or } \dfrac{x}{N} \in \Lambda^\complement \text{ and } \dfrac{x+e_j}{N} \in \Lambda, \\ 1\,, & \text{otherwise,} \end{cases} \tag{2.2}$$

for all $x \in \mathbb{T}_N^d$ and $j = 1, \ldots, d$. That is, the *slow bonds* of the process will be the bonds in $N^{-1}\mathbb{T}_N^d$ for which one of its vertices belongs to Λ and the other one belongs to $\Lambda^{\complement}$. See Figure 2.1 for an illustration.

Note that, when $\beta = \infty$, there are no crossings of particles through the boundary $\partial\Lambda$. From now on, abusing of notation, we will call the generator of the SSEP with slow bonds over $\partial\Lambda$ by $\mathcal{L}_N$, being understood that jump rates will be given by (2.2).

Denote by $\{\eta_t : t \geq 0\}$ the Markov process with state space Ω_N and generator $N^2\mathcal{L}_N$, where the N^2 factor is the so-called *diffusive scaling*. This Markov process depends on N, but it will not be indexed on it to not overload notation. Let $D(\mathbb{R}_+, \Omega_N)$ be the *Skorohod space* of càdlàg trajectories taking values in Ω_N. For a measure μ_N on Ω_N, denote by $\mathbb{P}_{\mu_N}^N$ the probability measure on $D(\mathbb{R}_+, \Omega_N)$ induced by the initial state μ_N and the Markov process $\{\eta_t : t \geq 0\}$. Expectation with respect to $\mathbb{P}_{\mu_N}^N$ will be denoted by $\mathbb{E}_{\mu_N}^N$.

In the sequel, we present the partial differential equations governing the time evolution of the density profile for the different regimes of β, defining the notion of weak solution for each one of those equations. Denote by ρ_t a function $\rho(t,\cdot)$ and denote by $C^n(\mathbb{T}^d)$ the set of continuous functions from $\mathbb{T}^d$ to $\mathbb{R}$ with continuous derivatives of order up to n. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm in $L^2(\mathbb{T}^d)$, that is,

$$\langle f, g \rangle \;=\; \int_{\mathbb{T}^d} f(u)\, g(u)\, du \quad \text{and} \quad \|f\| = \sqrt{\langle f, f \rangle}\,, \quad \forall\, f, g \in L^2(\mathbb{T}^d)\,. \tag{2.3}$$

Fix once and for all a measurable density profile $\rho_0 : \mathbb{T}^d \to [0,1]$. Note that ρ_0 is bounded.

Definition 1. *A bounded function $\rho : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is said to be a weak solution of the heat equation*

$$\begin{cases} \partial_t \rho(t,u) \;=\; \Delta\rho(t,u), & t \geq 0,\, u \in \mathbb{T}^d, \\ \rho(0,u) \;=\; \rho_0(u), & u \in \mathbb{T}^d. \end{cases} \tag{2.4}$$

if, for all functions $H \in C^2(\mathbb{T}^d)$ and all $t \in [0,T]$, the function $\rho(t,\cdot)$ satisfies the integral equation

$$\langle \rho_t, H \rangle \;-\; \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \Delta H \rangle\, ds \;=\; 0\,.$$

We recall next the definition of Sobolev Space from [5]. Let U be an open set of $\mathbb{R}^d$ or $\mathbb{T}^d$. The Sobolev Space $\mathcal{H}^1(U)$ consists of all locally summable functions $\kappa : U \to \mathbb{R}$ such that there exist functions $\partial_{u_j}\kappa \in L^2(U)$, $j = 1, \ldots, d$, satisfying

$$\int_{\mathbb{T}^d} \partial_{u_j} H(u) \kappa(u)\, du \;=\; -\int_{\mathbb{T}^d} H(u) \partial_{u_j}\kappa(u)\, du$$

for all $H \in C^\infty(U)$ with compact support. Furthermore, for $\kappa \in \mathcal{H}^1(U)$, we define the norm $\|\kappa\|_{\mathcal{H}^1(U)} = \Big(\sum_{j=1}^d \int_U \big|\partial_{u_j}\kappa\big|^2 du \Big)^{1/2}$. Finally, we define the space

$L^2([0,T],\mathcal{H}^1(U))$, which consists of all measurable functions $\tau:[0,T]\to\mathcal{H}^1(U)$ such that

$$\|\tau\|_{L^2([0,T],\mathcal{H}^1(U))} \;:=\; \Big(\int_0^T \|\tau_t\|^2_{\mathcal{H}^1(U)}\,dt\Big)^{1/2} \;<\; \infty\,.$$

Note that $U=\mathbb{T}^d\backslash\partial\Lambda$ is an open subset of $\mathbb{T}^d$.

The following notation will be used several times along the text. Given a function $f:\mathbb{T}^d\backslash\partial\Lambda\to\mathbb{R}$ and $u\in\partial\Lambda$, we denote

$$f(u^+) \;:=\; \lim_{\substack{v\to u\\ v\in\Lambda^{\complement}}} f(v) \quad \text{and } f(u^-) \;:=\; \lim_{\substack{v\to u\\ v\in\Lambda}} f(v)\,, \tag{2.5}$$

that is, $f(u^+)$ is the limit of $f(v)$ as v approaches $u\in\partial\Lambda$ through *the complement of* Λ, while $f(u^-)$ is the limit of $f(v)$ as v approaches $u\in\partial\Lambda$ through Λ. Let $\mathbf{1}_A$ be the indicator function of a set A, that is, $\mathbf{1}_A(a)=1$ if $a\in A$ and zero otherwise. Denote by $\vec{\zeta}(u)$ the normal unitary exterior vector to the region Λ at the point $u\in\partial\Lambda$ and by $\partial/\partial\vec{\zeta}$ the directional derivative with respect to $\vec{\zeta}(u)$.

Below, by $\langle\vec{u},\vec{v}\rangle$ we denote the canonical inner product of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^d$, which shall not be misunderstood with the inner product in $L^2(\mathbb{T}^d)$ as defined in (2.3). By dS we indicate a surface integral.

Definition 2. *A bounded function $\rho:[0,T]\times\mathbb{T}^d\to\mathbb{R}$ is said to be a weak solution of the following heat equation with Robin boundary conditions*

$$\begin{cases} \partial_t\rho(t,u) \;=\; \Delta\rho(t,u), & t\geq 0,\; u\in\mathbb{T}^d,\\ \dfrac{\partial\rho(t,u^+)}{\partial\vec{\zeta}(u)} = \dfrac{\partial\rho(t,u^-)}{\partial\vec{\zeta}(u)} = \alpha\Big(\rho(t,u^+)-\rho(t,u^-)\Big)\displaystyle\sum_{j=1}^d|\langle\vec{\zeta}(u),e_j\rangle|, & t\geq 0,\; u\in\partial\Lambda,\\ \rho(0,u) \;=\; \rho_0(u), & u\in\mathbb{T}^d\,. \end{cases} \tag{2.6}$$

if $\rho\in L^2([0,T],\mathcal{H}^1(\mathbb{T}^d\backslash\partial\Lambda))$ and, for all functions $H=h_1\mathbf{1}_\Lambda+h_2\mathbf{1}_{\Lambda^\complement}$ with $h_1,h_2\in C^2(\mathbb{T}^d)$ and for all $t\in[0,T]$, the following the integral equation holds:

$$\begin{aligned}&\langle\rho_t,H\rangle-\langle\rho_0,H\rangle-\int_0^t\langle\rho_s,\Delta H\rangle\,ds-\int_0^t\int_{\partial\Lambda}\rho_s(u^+)\sum_{j=1}^d\partial_{u_j}H(u^+)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\ &+\int_0^t\int_{\partial\Lambda}\rho_s(u^-)\sum_{j=1}^d\partial_{u_j}H(u^-)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\ &+\int_0^t\int_{\partial\Lambda}\alpha\,(\rho_s(u^-)-\rho_s(u^+))(H(u^+)-H(u^-))\Big(\sum_{j=1}^d|\langle\vec{\zeta}(u),e_j\rangle|\Big)\,dS(u)ds \;=\; 0\,.\end{aligned}$$

The reader should note that the function H is (possibly) discontinuous at the boundary $\partial\Lambda$. Note also that the expression $\sum_{j=1}^d\partial_{u_j}H(u^\pm)\langle\vec{\zeta}(u),e_j\rangle$ appearing in the integral equation above is nothing but $\partial H(u^\pm)/\partial\vec{\zeta}$ due to linearity of the directional derivative.

Definition 3. *A bounded function $\rho : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is said to be a weak solution of the heat equation with Neumann boundary conditions*

$$\begin{cases} \partial_t \rho(t,u) \;=\; \Delta \rho(t,u), & t \geq 0,\, u \in \mathbb{T}^d, \\ \dfrac{\partial \rho(t,u^+)}{\partial \vec{\zeta}(u)} = \dfrac{\partial \rho(t,u^-)}{\partial \vec{\zeta}(u)} = 0, & t \geq 0,\, u \in \partial\Lambda, \\ \rho(0,u) \;=\; \rho_0(u), & u \in \mathbb{T}^d\,, \end{cases} \tag{2.7}$$

if $\rho \in L^2([0,T], \mathcal{H}^1(\mathbb{T}^d \backslash \partial\Lambda))$ and, for all functions $H = h_1 \mathbf{1}_\Lambda + h_2 \mathbf{1}_{\Lambda^\complement}$ with $h_1, h_2 \in C^2(\mathbb{T}^d)$ and for all $t \in [0,T]$, the following integral equation holds:

$$\langle \rho_t, H \rangle \;-\; \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \Delta H \rangle \, ds - \int_0^t \int_{\partial\Lambda} \rho_s(u^+) \sum_{j=1}^d \partial_{u_j} H(u^+) \langle \vec{\zeta}(u), e_j \rangle \, dS(u) ds$$

$$+ \int_0^t \int_{\partial\Lambda} \rho_s(u^-) \sum_{j=1}^d \partial_{u_j} H(u^-) \langle \vec{\zeta}(u), e_j \rangle \, dS(u) ds \;=\; 0\,.$$

Since in Definitions 2 and 3 we impose $\rho \in L^2([0,T], \mathcal{H}^1(\mathbb{T}^d \backslash \partial\Lambda))$, the integrals above are well-defined on the boundary due to the notion of trace in Sobolev spaces, see [5] on the subject. We clarify that the notion of weak solutions above have been defined in the standard way of Analysis: the reader can check that a strong solution of (2.4), (2.6) or (2.7) is indeed a weak solution of the respective PDE.

Fix a measurable density profile $\rho_0 : \mathbb{T}^d \to [0,1]$. For each $N \in \mathbb{N}$, let μ_N be a probability measure on Ω_N. A sequence of probability measures $\{\mu_N : N \geq 1\}$ is said to be *associated to a profile* $\rho_0 : \mathbb{T}^d \to [0,1]$ if, for every $\delta > 0$ and every continuous function $H : \mathbb{T}^d \to \mathbb{R}$ the following limit holds:

$$\lim_{N\to\infty} \mu_N \left\{ \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N) \eta(x) - \int H(u) \rho_0(u) du \right| > \delta \right\} \;=\; 0\,. \tag{2.8}$$

Below, we establish the main result of this paper, the hydrodynamic limit for the exclusion process with slow bonds, which depends on the regime of β.

Theorem 2.2.1. *Fix $\beta \in [0,\infty]$. Consider the exclusion process with slow bonds over $\partial\Lambda$ with rate $\alpha N^{-\beta}$ at each one of these slow bonds. Fix a Borel measurable initial profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and consider a sequence of probability measures $\{\mu_N\}_{N\in\mathbb{N}}$ on Ω_N associated to ρ_0 in the sense of* (2.8)*. Then, for each $t \in [0,T]$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N} \left[\eta : \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N)\, \eta_t(x) - \int_{\mathbb{T}^d} H(u)\, \rho(t,u) du \right| > \delta \right] \;=\; 0\,,$$

for every $\delta > 0$ and every function $H \in C(\mathbb{T}^d)$ where:

- *If $\beta \in [0,1)$, then ρ is the unique weak solution of* (2.4).
- *If $\beta = 1$, then ρ is the unique weak solution of* (2.6).
- *If $\beta \in (1,\infty]$, then ρ is the unique weak solution of* (2.7).

The assumption that Λ is simple and connected may be dropped, being imposed only for the sake of clarity. Otherwise, notation would be highly overloaded.

2.3 Scaling Limit and Proof's Structure

Let $\mathcal{M}$ be the space of positive Radon measures on $\mathbb{T}^d$ with total mass bounded by one, endowed with the weak topology. Let $\pi_t^N \in \mathcal{M}$ the empirical measure at time t associated to η_t, it is a measure on $\mathbb{T}^d$ obtained rescaling space by N:

$$\pi_t^N(du) \;=\; \pi_t^N(\eta_t, du) \;:=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} \eta_t(x)\delta_{x/N}(du)\,,$$

where δ_u denotes the Dirac measure concentrated on $u \in \mathbb{T}^d$. For a measurable function $H : \mathbb{T}^d \to \mathbb{R}$ which is π-integrable, denote by $\langle \pi_t^N, H\rangle$ the integral of H with respect to π_t^N:

$$\langle \pi_t^N, H\rangle \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\left(\tfrac{x}{N}\right)\eta_t(x)\,.$$

Note that this notation $\langle\cdot,\cdot\rangle$ is also used as the inner product of $L^2(\mathbb{T}^d)$. Fix once and for all a time horizon $T > 0$. Let $D([0,T],\mathcal{M})$ be the space of $\mathcal{M}$-valued *càdlàg* trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *Skorohod* topology. Then, the $\mathcal{M}$-valued process $\{\pi_t^N : t \geq 0\}$ is a random element of $D([0,T],\mathcal{M})$ determined by $\{\eta_t : t \geq 0\}$. For each probability measure μ_N on Ω_N, denote by $\mathbb{Q}_{\mu_N}^{\beta,N}$ the distribution of $\{\pi_t^N : t \geq 0\}$ on the path space $D([0,T],\mathcal{M})$, when η_0^N has distribution μ_N.

Fix a continuous Borel measurable profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on Ω_N associated to ρ_0. Let $\mathbb{Q}^\beta$ be the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where:

- if $\beta \in [0,1)$, then ρ is the unique weak solution of (2.4),
- if $\beta = 1$, then ρ is the unique weak solution of (2.6),
- if $\beta \in (1,\infty]$, then ρ is the unique weak solution of (2.7).

Proposition 2.3.1. *For any $\beta \in [0,\infty]$, the sequence of probability measures $\mathbb{Q}_{\mu_N}^{\beta,N}$ converges weakly to $\mathbb{Q}^\beta$ as N goes to infinity.*

The proof of this result is divided into three parts. In the next section, we show that tightness of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$. In Section 3.3, we prove a suitable *Replacement Lemma* for each regime of β, which will be crucial in the task of characterizing limit points. In Section 2.6 we characterize the limit points of the sequence for each regime of the parameter β. Finally, the uniqueness of weak solutions is presented in Section 2.7 and this implies the uniqueness of limit points of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$.

Finally, we note that Theorem 2.2.1 is a consequence of Proposition 2.3.1. Actually, since $\mathbb{Q}_{\mu_N}^{\beta,N}$ weakly converges to $\mathbb{Q}^\beta$ for all continuous functions $H : \mathbb{T}^d \to \mathbb{R}$, it follows that the path $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ converges in distribution to $\{\langle \pi_t, H\rangle :$

$0 \le t \le T\}$. Since $\{\langle \pi_t, H\rangle : 0 \le t \le T\}$ is a deterministic path, convergence in distribution is equivalent to convergence in probability. Therefore,

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N}\Big\{ \Big| \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} H(x/N)\,\eta_t(x) - \int_{\mathbb{T}^d} H(u)\rho(t,u)du\Big| > \delta \Big\}$$
$$= \lim_{N\to\infty} \mathbb{Q}^{\beta,N}_{\mu_N}\big\{ |\langle \pi^N_t, H\rangle - \langle \pi_t, H\rangle| > \delta \big\} \;=\; 0\,,$$

for all $\delta > 0$ and $0 \le t \le T$. This gives the strategy of proof for the hydrodynamic limit. Next, we make some general observations.

Since particles in the exclusion process evolve independently as a nearest neighbor random walk, except for exclusion rule, the exclusion process with slow bonds over $\partial\Lambda$ is related to the random walk on $N^{-1}\mathbb{T}^d_N$ that describes the evolution of the system with a single particle. To be used throughout the paper we introduce the generator of the random walk described above, which is

$$\mathbb{L}_N H\big(\tfrac{x}{N}\big) = \sum_{j=1}^{d} \Big\{ \xi^N_{x,x+e_j}\Big[H\big(\tfrac{x+e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big] + \xi^N_{x,x-e_j}\Big[H\big(\tfrac{x-e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big]\Big\} \tag{2.9}$$

for every $H : N^{-1}\mathbb{T}^d_N \to \mathbb{R}$ and every $x \in \mathbb{T}^d_N$. Above, it is understood that $\xi_{x\pm e_j,x} = \xi_{x,x\pm e_j}$. By Dynkin's formula (see A.1.5.1 in [18]),

$$M^N_t(H) \;=\; \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t N^2 \mathcal{L}_N \langle \pi^N_s, H\rangle ds$$

is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta^N_s \,:\, s \le t)$. By some elementary calculations,

$$N^2\mathcal{L}_N\langle \pi^N_s, H\rangle \;=\; \frac{1}{N^{d-2}}\sum_{x\in\mathbb{T}^d_N} \eta_s(x)\mathbb{L}_N H\Big(\frac{x}{N}\Big) \;=\; \langle \pi^N_s, N^2\mathbb{L}_N H\rangle\,,$$

hence the martingale can be rewritten as

$$M^N_t(H) \;=\; \langle \pi^N_t, H\rangle - \langle \pi^N_0, H\rangle - \int_0^t \langle \pi^N_s, N^2\mathbb{L}_N H\rangle ds\,. \tag{2.10}$$

Note that this observation stands for any jump rates. The particular form of jump rates for the SSEP with slow bonds over $\partial\Lambda$ will play a role when characterizing limit points and proving replacement lemmas.

2.4 Tightness

This section deals with the issue of tightness for the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \ge 1\}$ of probability measures on $D([0,T],\mathcal{M})$.

Proposition 2.4.1. *For any fixed $\beta \in [0,\infty]$, the sequence of measures $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \ge 1\}$ is tight in the Skorohod topology of $D([0,T],\mathcal{M})$.*

Proof. In order to prove tightness of $\{\pi_t^N : 0 \leq t \leq T\}$, it is enough to show tightness of the real-valued process $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ for $H \in C(\mathbb{T}^d)$. In fact, (cf. Proposition 1.7, chapter 4 of [18]) it is enough to show tightness of $\{\langle \pi_t^N, H\rangle : 0 \leq t \leq T\}$ in $D([0,T],\mathbb{R})$ for a dense set of functions in $C(\mathbb{T}^d)$ with respect to the uniform topology.

For that purpose, fix $H \in C^2(\mathbb{T}^d)$. Since the sum of tight processes is tight, in order to prove tightness of $\{\langle \pi_t^N, H\rangle : N \geq 1\}$, it is enough to assure tightness of each term in (2.10). The quadratic variation of $M_t^N(H)$ is given by

$$\langle M^N(H)\rangle_t = \int_0^t \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \frac{\xi^N_{x,x+e_j}}{N^{2d-2}} \Big[(\eta_s(x) - \eta_s(x+e_j))(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}))\Big]^2 ds, \qquad (2.11)$$

implying that

$$\langle M^N(H)\rangle_t \;\leq\; \frac{\alpha t}{N^d} \sum_{j=1}^d \|\partial_{u_j} H\|_\infty^2\,, \qquad (2.12)$$

where $\|H\|_\infty := \sup_{u\in\mathbb{T}^d} |H(u)|$, hence M_t^N converges to zero as $N\to\infty$ in $L^2(\mathbb{P}^\beta_{\mu_N})$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu_N}\Big[\sup_{0\leq t\leq T} |M_t^N(H)| > \delta\Big] \;=\; 0\,, \qquad (2.13)$$

which implies tightness of the sequence of martingales $\{M_t^N(H) : N \geq 1\}$. Next, we will prove tightness for the integral term in (2.10). Let Γ_N be the set of vertices in $\mathbb{T}_N^d$ having some incident edge with exchange rate not equal to one, that is,

$$\Gamma_N = \Big\{x \in \mathbb{T}_N^d : \text{ for some } j = 1,\dots,d,\quad \xi^N_{x,x+e_j} = \frac{\alpha}{N^\beta} \text{ or } \xi^N_{x,x-e_j} = \frac{\alpha}{N^\beta}\Big\}. \qquad (2.14)$$

The term $\langle \pi_s^N, N^2\mathbb{L}_N H\rangle$ appearing inside the time integral in (2.10) can be then written as

$$\frac{1}{N^d}\sum_{j=1}^d \sum_{x\notin\Gamma_N} \eta_s(x) N^2 \Big[H(\tfrac{x+e_j}{N}) + H(\tfrac{x-e_j}{N}) - 2H(\tfrac{x}{N})\Big]$$
$$+ \frac{1}{N^{d-1}}\sum_{j=1}^d \sum_{x\in\Gamma_N} \eta_s(x) \Big[\xi^N_{x,x+e_j} N\big(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\big) + \xi^N_{x,x-e_j} N\big(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})\big)\Big]$$

since $\xi_{x,x+e_j} = \xi_{x+e_j,x} = 1$ for every $x \notin \Gamma_N$. By a Taylor expansion on $H \in C^2(\mathbb{T}^d)$, the absolute value of the summand in the first double sum above is bounded by $\|\Delta H\|_\infty$. Since there are $\mathcal{O}(N^{d-1})$ elements in Γ_N, and $\xi_{x,x+e_j} \leq \alpha$, the absolute value of summand in second double sum above is bounded by $\sum_{j=1}^d \alpha\|\partial_{u_j} H\|_\infty$. Therefore, there exists $C > 0$, depending only on H, such that $|N^2\mathbb{L}_N\langle \pi_s^N, H\rangle| \leq C$, which yields

$$\Big|\int_s^t N^2\mathbb{L}_N\langle \pi_s^N, H\rangle dr\Big| \;\leq\; C|t-s|\,.$$

By [18, Proposition 4.1.6], last inequality implies tightness of the integral term, concluding the proof of the proposition. □

2.5 Replacement Lemma and Energy Estimates

This section gives a fundamental result that allow us to replace a mean occupation of a site by the mean density of particles in a small macroscopic box around this site. We start by introducing some tools to be used in the sequel.

Denote by $H_N(\mu_N|\nu_\theta)$ the relative entropy of μ_N with respect to the invariant state ν_θ. For a precise definition and properties of the entropy, we refer the reader to [18]. Assuming $0 < \theta < 1$, the formula in [18, Theorem A1.8.3] assures the existence a finite constant $\kappa_0 = \kappa_0(\theta)$ such that

$$H_N(\mu_N|\nu_\theta) \;\leq\; \kappa_0 N^d \tag{2.15}$$

for any probability measure μ_N on $\{0,1\}^{\mathbb{T}_N^d}$. Denote by $\mathfrak{D}_N$ the Dirichlet form of the process, which is the functional acting on functions $f:\{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$\mathfrak{D}_N(f) \;:=\; \langle f, -\mathcal{L}_N f\rangle_{\nu_\theta} = \sum_{j=1}^{d} \sum_{x\in\mathbb{T}_N^d} \frac{\xi^N_{x,x+e_j}}{2} \int \big(f(\eta^{x,x+e_j}) - f(\eta)\big)^2\, \nu_\theta(d\eta)\,. \tag{2.16}$$

In the sequence, we will make use of the functional $\mathfrak{D}_N(\sqrt{f})$, where f is a probability density with respect to ν_θ.

2.5.1 Replacement Lemma for $\beta \in [0,1)$

Below, we define the local density of particles, which corresponds a to the mean occupation in a box around a given site. Abusing of notation, we denote by $\varepsilon N - 1$ the integer part of $\varepsilon N - 1$. For $\beta \in [0,1)$, we define the local mean by

$$\eta^{\varepsilon N}(x) \;=\; \frac{1}{(\varepsilon N)^d} \sum_{j_1,j_2,\dots,j_d=0}^{\varepsilon N-1} \eta\left(x + j_1 e_1 + \ldots + j_d e_d\right)\,. \tag{2.17}$$

Note that the sum on the right hand side of above may contain sites in and out of Λ in the sense that $x/N \in \Lambda$ or $x/N \in \Lambda^{\complement}$. By $\mathcal{O}(f(N))$ we will mean a function bounded in modulus by a constant times $f(N)$.

Lemma 2.5.1. *Fix $\beta \in [0,1)$. Let f be a density with respect to the invariant measure ν_θ, $\lambda_N : \mathbb{T}_N^d \to \mathbb{R}$ a function such that $\|\lambda_N\|_\infty \leq M < \infty$ and $\gamma > 0$. Then,*

$$\begin{aligned}
&\int \gamma N \sum_{x\in\Gamma_N} \lambda_N(x)\big\{\eta(x) - \eta^{\varepsilon N}(x)\big\} f(\eta)\nu_\theta(d\eta)\\
&\leq\; \frac{\gamma^2 M^2 \mathcal{O}(N^d)}{2}\Big(\frac{N^{\beta-1}}{\alpha} + d\varepsilon\Big) + N^2 \mathfrak{D}_N(\sqrt{f})\,.
\end{aligned}$$

Proof. By the definition (2.17) of local mean $\eta^{\varepsilon N}(x)$,

$$\int \lambda_N(x)\Big\{\eta(x) - \eta^{\varepsilon N}(x)\Big\} f(\eta)\nu_\theta(d\eta) \;=$$

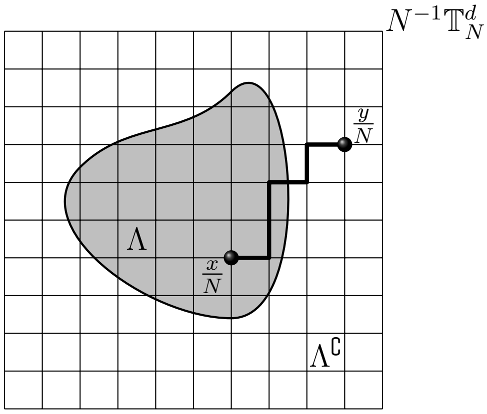


Figure 2.2: Illustration (in dimension 2) of a polygonal path joining the sites x and $y = x + j_1 e_1 + j_2 e_2$, with $j_1 = j_2 = 3$. Note the embedding in the continuous torus $\mathbb{T}^d$.

$$= \int \lambda_N(x) \frac{1}{\varepsilon^d N^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \Big\{ \eta(x) - \eta(x + j_1 e_1 + \ldots + j_d e_d) \Big\} f(\eta) \nu_\theta(d\eta) . \qquad (2.18)$$

The next step is to write $\eta(x) - \eta(x + j_1 e_1 + \cdots + j_d e_d)$ as a telescopic sum:

$$\eta(x) - \eta(x + j_1 e_1 + \ldots + j_d e_d) \; = \; \sum_{\ell=1}^{j_1+\cdots+j_d} \eta(a_{\ell-1}) - \eta(a_\ell) ,$$

where $a_0 = x$, $a_{j_1+\cdots+j_\ell} = x + j_1 e_1 + \cdots + j_d e_d$, and $\|a_{\ell-1} - a_\ell\|_1 = 1$ for any $\ell = 1, \ldots, j_1 + \cdots + j_d$. Note that the path $a_0, a_1, \ldots, a_{j_1+\cdots+j_\ell}$ depends on the initial point x and the final point $x + j_1 e_1 + \cdots + j_d e_d$. See Figure 2.2 for an illustration and keep in mind that the length of this path is bounded by $d\varepsilon N$. Inserting the previous equality into (2.18), we get

$$\int \lambda_N(x) \frac{1}{(\varepsilon N)^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \Big\{ \sum_{\ell=1}^{j_1+\cdots+j_d} \eta(a_{\ell-1}) - \eta(a_\ell) \Big\} f(\eta)\, \nu_\theta(d\eta) .$$

Rewriting the expression above as twice the half and performing the transformation $\eta \mapsto \eta^{a_{\ell-1},a_\ell}$ for which the probability measure ν_θ is invariant, expression above becomes:

$$\frac{1}{2(\varepsilon N)^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \sum_{\ell=1}^{j_1+\cdots+j_d} \int \lambda_N(x) \left(\eta(a_{\ell-1}) - \eta(a_\ell) \right) \left(f\left(\eta^{a_\ell,a_{\ell-1}}\right) - f\left(\eta\right) \right) \, d\nu_\theta .$$

Since $ab = \sqrt{c} a \frac{b}{\sqrt{c}} \leq \frac{1}{2} c a^2 + \frac{1}{2} \frac{b^2}{c}$, which holds for any $c > 0$, the previous expression is smaller or equal than

$$\frac{1}{2(\varepsilon N)^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \sum_{\ell=1}^{j_1+\cdots+j_d} \left[\frac{\xi^N_{a_{\ell-1},a_\ell}}{2A} \int \left(\sqrt{f\left(\eta^{a_\ell,a_{\ell-1}}\right)} - \sqrt{f\left(\eta\right)} \right)^2 d\nu_\theta \right.$$
$$\left. + \frac{A}{2\xi^N_{a_{\ell-1},a_\ell}} \int \lambda_N^2(x) \left(\eta(a_\ell) - \eta(a_{\ell-1}) \right)^2 \left(\sqrt{f\left(\eta^{a_\ell,a_{\ell-1}}\right)} + \sqrt{f\left(\eta\right)} \right)^2 d\nu_\theta \right] .$$

Summing over $x \in \Gamma_N$, we can bound the last expression by

$$\frac{1}{2(\varepsilon N)^d} \sum_{x\in\Gamma_N} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \sum_{\ell=1}^{j_1+\cdots+j_d} \Bigg[\frac{\xi^N_{a_{\ell-1},a_\ell}}{2A} \int \Big(\sqrt{f\left(\eta^{a_\ell,a_{\ell-1}}\right)} - \sqrt{f\left(\eta\right)} \Big)^2 d\nu_\theta$$
$$+ \sum_{x\in\Gamma_N} \frac{A}{2\xi^N_{a_{\ell-1},a_\ell}} \int \lambda_N^2(x) \left(\eta(a_\ell) - \eta(a_{\ell-1})\right)^2 \Big(\sqrt{f\left(\eta^{a_\ell,a_{\ell-1}}\right)} + \sqrt{f\left(\eta\right)} \Big)^2 d\nu_\theta \Bigg].$$

Recalling (2.16), we can bound the first parcel in the sum above by

$$\frac{1}{2(\varepsilon N)^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \frac{1}{A} \mathfrak{D}_N(\sqrt{f}) \;=\; \frac{1}{2A} \mathfrak{D}_N(\sqrt{f})\,.$$

Since f is a density and $|\lambda_N(x)| \leq M$, the second parcel is bounded by

$$\begin{aligned}
&\frac{1}{2(\varepsilon N)^d} \sum_{x\in\Gamma_N} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} \sum_{\ell=1}^{j_1+\cdots+j_d} \frac{A}{2} \cdot \frac{4M^2}{\xi^N_{a_{\ell-1},a_\ell}} \\
&\leq\; \frac{1}{(\varepsilon N)^d} \sum_{j_1,\dots,j_d=0}^{\varepsilon N-1} AM^2 \mathcal{O}(N^{d-1}) \Big(\frac{N^\beta}{\alpha} + d\varepsilon N \Big) \\
&=\; AM^2 \mathcal{O}(N^{d-1}) \Big(\frac{N^\beta}{\alpha} + d\varepsilon N \Big)\,.
\end{aligned}$$

Up to here we have achieved that

$$\begin{aligned}
&\int \sum_{x\in\Gamma_N} \lambda_N(x) \big\{ \eta(x) - \eta^{\varepsilon N}(x) \big\} f(\eta) \nu_\theta(d\eta) \\
&\leq\; AM^2 \mathcal{O}(N^{d-1}) \Big(\frac{N^\beta}{\alpha} + d\varepsilon N \Big) + \frac{1}{2A} \mathfrak{D}_N(\sqrt{f})\,.
\end{aligned}$$

We point out that the quantity of sites on Γ_N is of order $\mathcal{O}(N^{d-1})$, which is a consequence of the fact that $\partial\Lambda$ is a smooth surface of dimension $d-1$. Then, multiplying the inequality above by γN gives us

$$\begin{aligned}
&\int \gamma N \sum_{x\in\Gamma_N} \lambda_N(x) \big\{ \eta(x) - \eta^{\varepsilon N}(x) \big\} f(\eta) \nu_\theta(d\eta) \\
&\leq\; A\gamma \mathcal{O}(N^d) M^2 \Big[\frac{N^\beta}{\alpha} + d\varepsilon N \Big] + \frac{\gamma N}{2A} \mathfrak{D}_N(\sqrt{f})\,.
\end{aligned}$$

Now choosing $A = \gamma N^{-1}/2$ the proof ends. □

Recall the definition of Γ_N in (2.14).

Lemma 2.5.2 (Replacement lemma)**.** *Fix $\beta \in [0,1)$. Let $\lambda_N : \mathbb{T}^d_N \to \mathbb{R}$ be a sequence of functions such that $\|\lambda_N\|_\infty \leq M < \infty$. Then,*

$$\varlimsup_{\varepsilon\to 0} \varlimsup_{N\to\infty} \mathbb{E}^\beta_{\mu_N} \Big[\Big| \int_0^t \frac{1}{N^{d-1}} \sum_{x\in\Gamma_N} \lambda_N(x) \{ \eta_s^{\varepsilon N}(x) - \eta_s(x) \}\, ds \Big| \Big] \;=\; 0\,.$$

Proof. Using the variational formula for entropy, for any $\gamma \in \mathbb{R}$ (which will be chosen large *a posteriori*),

$$
\begin{aligned}
&\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}\lambda_N(x)\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}ds\Big|\Big]\\
&=\frac{1}{\gamma N^d}\mathbb{E}^{\beta}_{\mu_N}\Big[\gamma N\Big|\int_0^t \sum_{x\in\Gamma_N}\lambda_N(x)\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}ds\Big|\Big]\\
&\leq \frac{H_N(\mu_N|\nu_\theta)}{\gamma N^d}+\frac{1}{\gamma N^d}\log\mathbb{E}_{\nu_\theta}\Big[\exp\Big(\gamma N\Big|\int_0^t \sum_{x\in\Gamma_N}\lambda_N(x)\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}ds\Big|\Big)\Big]. \qquad (2.19)
\end{aligned}
$$

By the estimate (2.15) on the entropy, the first parcel of above is negligible as $N\to\infty$ since we will choose γ arbitrarily large. Therefore, we can focus on the second parcel. Using that $e^{|x|}\leq e^x+e^{-x}$ and

$$
\varlimsup_{N\to\infty}\frac{1}{N^d}\log(a_N+b_N) \;=\; \max\Big\{\varlimsup_{N\to\infty}\frac{1}{N^d}\log a_N,\, \varlimsup_{N\to\infty}\frac{1}{N^d}\log b_N\Big\} \qquad (2.20)
$$

for any sequences $a_N, b_N>0$, one can see that the second parcel on the right hand side of (2.19) is less than or equal to the sum of

$$
\varlimsup_{N\to\infty}\frac{1}{\gamma N^d}\log\Big\{\mathbb{E}_{\nu_\theta}\Big[\exp\Big(\gamma N\int_0^t\sum_{x\in\Gamma_N}\lambda_N(x)\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}ds\Big)\Big]\Big\} \qquad (2.21)
$$

and

$$
\varlimsup_{N\to\infty}\frac{1}{\gamma N^d}\log\Big\{\mathbb{E}_{\nu_\theta}\Big[\exp\Big(-\gamma N\int_0^t\sum_{x\in\Gamma_N}\lambda_N(x)\{\eta_s(x)-\eta_s^{\varepsilon N}(x)\}ds\Big)\Big]\Big\}. \qquad (2.22)
$$

We handle only (2.21), being (2.22) analogous. By Feynman-Kac's formula, see [18, Appendix 1, Lemma 7.2], expression (2.21) is bounded by

$$
\varlimsup_{N\to\infty}\frac{1}{\gamma N^d}\log\Big\{\exp\Big(\int_0^t\Phi_N\,ds\Big)\Big\} \;=\; \varlimsup_{N\to\infty}\frac{t\,\Phi_N^1}{\gamma N^d},
$$

where

$$
\Phi_N^1=\sup_{f\text{ density}}\Big\{\int\gamma N\sum_{x\in\Gamma_N}\lambda_N(x)\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\theta(d\eta)-N^2\mathfrak{D}_N(\sqrt{f})\Big\}.
$$

Applying Lemma 2.5.1 finishes the proof. □

2.5.2 Replacement Lemma for $\beta\in[1,\infty]$

Here, some additional notation is required. The idea is actually very simple: the local mean shall be over a region avoiding slow bonds. Let $B_N[x,\ell]\subset\mathbb{T}_N^d$ be the discrete box centered on $x\in\mathbb{T}_N^d$ which edge has size 2ℓ, that is, $B_N[x,\ell]=\{y\in\mathbb{T}_N^d:$

$\|y - x\|_\infty \le \ell\}$, where we have written $\|\cdot\|_\infty$ for the supremum norm on $\mathbb{T}_N^d$, that is, $\|(x_1, \ldots, x_d)\|_\infty = \max\big\{|x_1| \wedge |N - x_1|, \ldots, |x_d| \wedge |N - x_d|\big\}$.

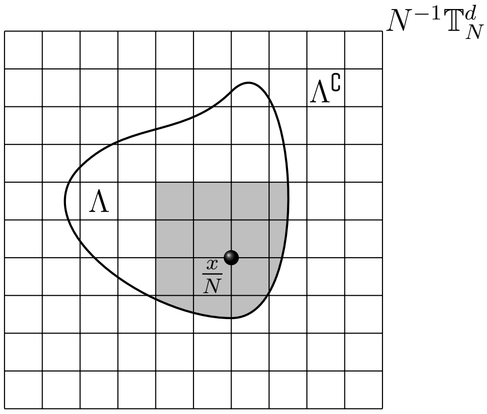


Figure 2.3: Illustration in dimension two of $C_N[x, 2]$. The sites in $C_N[x, 2]$ are those laying in the gray region.

Let $\Lambda_N = \{x \in \mathbb{T}_N^d : \frac{x}{N} \in \Lambda\}$ the set of sites in $\frac{1}{N}\mathbb{T}_N^d$ belonging to Λ. We define now the region $C_N[x, \ell] \subset \mathbb{T}_N^d$ by

$$C_N[x, \ell] \;:=\; \begin{cases} B_N[x, \ell] \cap \Lambda_N & \text{if } \frac{x}{N} \in \Lambda\,, \\ B_N[x, \ell] \cap \Lambda_N^{\complement} & \text{if } \frac{x}{N} \in \Lambda^{\complement}\,, \end{cases} \tag{2.23}$$

see Figure 2.3 for an illustration. For $\beta \in [1, \infty]$, we define the local density as the average over $C_N[x, \ell]$, that is,

$$\eta^{\varepsilon N}(x) \;:=\; \frac{1}{\# C_N[x, \varepsilon N]} \sum_{y \in C_N[x, \varepsilon N]} \eta(y)\,. \tag{2.24}$$

Lemma 2.5.3. *Fix $\beta \in [1, \infty]$. Let f be a density with respect to the invariant measure ν_θ, let $\lambda_N : \mathbb{T}_N^d \to \mathbb{R}$ a function such that $\|\lambda_N\|_\infty \le M < \infty$ and $\gamma > 0$. Then, the following inequalities hold:*

$$\int \gamma N \sum_{x \in \Gamma_N} \lambda_N(x) \big\{\eta(x) - \eta^{\varepsilon N}(x)\big\} f(\eta) \nu_\theta(d\eta) \;\le\; \tfrac{1}{2}\gamma^2 M^2 \mathcal{O}(N^d) d\varepsilon + N^2 \mathfrak{D}_N(\sqrt{f}) \tag{2.25}$$

and

$$\int \gamma \sum_{x \in \mathbb{T}_N^d} \lambda_N(x) \{\eta(x) - \eta^{\varepsilon N}(x)\} f(\eta) \nu_\theta(d\eta) \;\le\; \tfrac{1}{2}\gamma^2 M^2 \mathcal{O}(N^{d-1}) d\varepsilon + N^2 \mathfrak{D}_N(\sqrt{f})\,. \tag{2.26}$$

Proof. Let us prove the inequality (2.26). As commented in the beginning of this subsection, the local average $\eta^{\varepsilon N}$ is taken over $C_N[x, \varepsilon N]$. Thus, we can write

$$\int \lambda_N(x) \{\eta(x) - \eta^{\varepsilon N}(x)\} f(\eta) \nu_\theta(d\eta)$$

$$= \int \lambda_N(x)\Big\{\frac{1}{\#C_N[x,\varepsilon N]}\sum_{y\in C_N[x,\varepsilon N]}\big(\eta(x)-\eta(y)\big)\Big\}f(\eta)\nu_\theta(d\eta)\,. \qquad \textbf{(2.27)}$$

For each $y \in C[x, \varepsilon N]$, let $\gamma(x, y)$ be a polygonal path of minimal length connecting x to y which does not crosses $\partial\Lambda$. That is, $\gamma(x, y)$ is a sequence of sites $(a_0, \dots, a_M)$ such that $x = a_0$, $y = a_M$, $\|a_i - a_{i+1}\|_1 = 1$ and $\xi_{a, a_{i+1}} = 1$ for $i = 0, \dots, M-1$, and $\gamma(x, y)$ has minimal length, that is, $M = M(x, y) = \|x - y\|_1 + 1$. Now we repeat the steps in the proof of Lemma 2.5.1, observing that in this case the sum will be over $\mathbb{T}^d_N$, obtaining that (2.27) is bounded from above by

$$\frac{1}{2\#C_N[x,\varepsilon N]}\sum_{x\in\mathbb{T}^d_N}\sum_{y\in C_N[x,\varepsilon N]}\sum_{\ell=1}^{M(x,y)-1}\Bigg[\frac{1}{2A}\int\Big(\sqrt{f(\eta^{a_\ell,a_{\ell-1}})}-\sqrt{f(\eta)}\Big)^2 d\nu_\theta$$
$$+\frac{A}{2}\int\Big(\lambda_N(x)\Big)^2(\eta(a_\ell)-\eta(a_{\ell-1}))^2\Big(\sqrt{f(\eta^{a_\ell,a_{\ell-1}})}+\sqrt{f(\eta)}\Big)^2 d\nu_\theta\Bigg]\,.$$

We can bound the first parcel in the sum above by $\frac{1}{2A}\mathfrak{D}_N(\sqrt{f})$ and the second parcel by

$$\frac{1}{2\#C_N[x,\varepsilon N]}\sum_{x\in\mathbb{T}^d_N}\sum_{y\in C_N[x,\varepsilon N]}\sum_{\ell=1}^{M(x,y)-1}\frac{4AM^2}{2}$$
$$\leq \frac{1}{\#C_N[x,\varepsilon N]}\sum_{y\in C_N[x,\varepsilon N]} AM^2\mathcal{O}(N^d)d\varepsilon N \;=\; AM^2\mathcal{O}(N^d)d\varepsilon N\,.$$

We hence have

$$\int\sum_{x\in\mathbb{T}^d_N}\lambda_N(x)\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\theta(d\eta) \;\leq\; AM^2\mathcal{O}(N^d)d\varepsilon N + \frac{1}{2A}\mathfrak{D}_N(\sqrt{f})\,.$$

Then, multiplying the inequality above by γ gives us

$$\int\gamma\sum_{x\in\mathbb{T}^d_N}\lambda_N(x)\{\eta(x)-\eta^{\varepsilon N}(x)\}f(\eta)\nu_\theta(d\eta) \;\leq\; A\gamma\mathcal{O}(N^d)M^2d\varepsilon N + \frac{\gamma}{2A}\mathfrak{D}_N(\sqrt{f})\,.$$

Now choosing $A = \gamma N^{-2}/2$ the proof of (2.25) ends. The proof of inequality (2.25) similar to the proof of Lemma 2.5.1, under the additional feature that rates of bonds over a path connecting two sites will be always equal to one, which facilitates the argument. □

Lemma 2.5.4 (Replacement lemma). *Fix $\beta \in [1, \infty]$. Let $\lambda_N : \mathbb{T}^d_N \to \mathbb{R}$ be a sequence of functions such that $\|\lambda_N\|_\infty \leq c < \infty$. Then,*

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}\lambda_N(x)\{\eta^{\varepsilon N}_s(x)-\eta_s(x)\}\,ds\Big|\Big] \;=\; 0$$

and

$$\varlimsup_{\varepsilon\to 0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\Big|\int_0^t\frac{1}{N^{d}}\sum_{x\in\mathbb{T}^d_N}\lambda_N(x)\{\eta^{\varepsilon N}_s(x)-\eta_s(x)\}\,ds\Big|\Big] \;=\; 0\,.$$

Proof. The proof is similar to the one of Lemma 2.5.2, being sufficient to show that expressions

$$\Phi_N^2 := \sup_{f \text{ density}} \Big\{ \int \gamma N \sum_{x \in \Gamma_N} \lambda_N(x)\{\eta^{\varepsilon N}(x) - \eta(x)\} f(\eta) d\nu_\theta - N^2 \mathfrak{D}_N(\sqrt{f}) \Big\},$$

$$\Phi_N^3 := \sup_{f \text{ density}} \Big\{ \int \gamma \sum_{x \in \mathbb{T}_N^d} \lambda_N(x)\{\eta^{\varepsilon N}(x) - \eta(x)\} f(\eta) d\nu_\theta - N^2 \mathfrak{D}_N(\sqrt{f}) \Big\}$$

satisfy

$$\lim_{N\to\infty} \frac{t\Phi_N^2}{\gamma N^d} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \frac{t\Phi_N^3}{\gamma N^d} = 0\,,$$

which is a consequence of Lemma 2.5.3, finishing the proof. □

2.5.3 Energy Estimates

In this subsection, consider $\beta \in [1,\infty]$. Our goal here is to prove that any limit point $\mathbb{Q}_*^\beta$ of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N > 1\}$ is concentrated on trajectories $\rho(t,u)du$ with *finite energy*, meaning that $\rho(t,u)$ belongs to a suitable Sobolev space.

This result plays a both role in the uniqueness of weak solutions of (2.7) and in the characterization of limit points. The fact that $\mathbb{Q}_*^\beta$ is concentrated in trajectories with density with respect to the Lebesgue measure of the form $\rho(t,u)du$, with $0 \leq \rho \leq 1$, is a consequence of maximum of one particle per site, see [18]. The issue here is to prove that the density $\rho(t,u)$ belongs to the Sobolev space $L^2\big([0,T];\mathcal{H}^1(\mathbb{T}^d\backslash\partial\Lambda)\big)$, see Section 2.2 for its definition.

Assume without loss of generality that the entire sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ weakly converges to $\mathbb{Q}_*^\beta$. Let $B[u,\varepsilon] := \{r \in \mathbb{T}^d : \|r-u\|_\infty < \varepsilon\}$ and

$$C[u,\varepsilon] := \begin{cases} B[u,\varepsilon] \cap \Lambda & \text{if } u \in \Lambda\,, \\ B[u,\varepsilon] \cap \Lambda^{\complement} & \text{if } u \in \Lambda^{\complement}\,, \end{cases}$$

where we have written $\|\cdot\|_\infty$ for the supremum norm on the continuous torus $\mathbb{T}^d = [0,1)^d$, that is, $\|(u_1,\dots,u_d)\|_\infty = \max\big\{|u_1| \wedge |1-u_1|, \dots, |u_d| \wedge |1-u_d|\big\}$. See Figure 2.4 for an illustration.

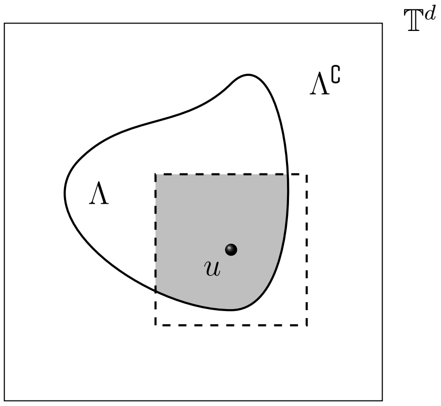


Figure 2.4: Illustration in dimension two of $C[u,\varepsilon]$, which is represented by the region in gray, while $B[u,\varepsilon]$ is represented by the square delimited by the dashed line. Note that $C[u,\varepsilon]$ is the continuous counterpart of $C_N[x,\ell]$ defined in (2.23).

We define an approximation of the identity ι_ε in the continuous torus $\mathbb{T}^d$ by

$$\iota_\varepsilon(u,v) \;:=\; \frac{1}{|C[u,\varepsilon]|}\mathbf{1}_{C[u,\varepsilon]}(v)\,, \tag{2.28}$$

where $|C[u,\varepsilon]|$ above denotes the Lebesgue measure of the set $C[u,\varepsilon]$. Recall that the convolution of a measure π with ι_ε is defined by

$$(\pi * \iota_\varepsilon)(u) \;=\; \int_{\mathbb{T}^d} \iota_\varepsilon(u,v)\pi(dv) \quad \text{for any } u\in\mathbb{T}^d\,. \tag{2.29}$$

Given a function ρ, the convolution $\rho * \iota_\varepsilon$ shall be understood as the convolution of the measure $\rho(v)dv$ with ι_ε. An important remark now is the equality

$$(\pi_t^N * \iota_\varepsilon)\big(\tfrac{x}{N}\big) \;=\; \eta_t^{\varepsilon N}(x) + \mathcal{O}\big((\varepsilon N)^{1-d}\big)\,, \tag{2.30}$$

where $\eta_t^{\varepsilon N}$ has been defined in (2.24), being the small error above due to the fact that sites on the boundary of $C_N[x,\ell]$ may or may not belong to $C[u,\varepsilon]$ when taking $u = x/N$ and $\ell = \varepsilon N$. Given a function $H:\mathbb{T}^d\to\mathbb{R}$, let

$$V_N(\varepsilon,j,H,\eta) := \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} H\big(\tfrac{x}{N}\big)\frac{\{\eta(x)-\eta(x+\varepsilon N e_j)\}}{\varepsilon} - \frac{2}{N^d}\sum_{x\in\mathbb{T}_N^d}\Big(H\big(\tfrac{x}{N}\big)\Big)^2. \tag{2.31}$$

Lemma 2.5.5. *Consider $H_1,\dots,H_k$ functions in $C^{0,1}([0,T]\times\mathbb{T}^d)$ with compact support contained in $[0,T]\times(\mathbb{T}^d\backslash\partial\Lambda)$. Hence, for every $\varepsilon>0$ and $j=1,\dots,d$,*

$$\varlimsup_{\delta\to 0}\varlimsup_{N\to\infty}\mathbb{E}^\beta_{\mu_N}\Big[\max_{1\le i\le k}\Big\{\int_0^T V_N(\varepsilon,j,H_i(s,\cdot),\eta_s^{\delta N})\,ds\Big\}\Big] \;\le\; \kappa_0\,, \tag{2.32}$$

where κ_0 has been defined in (2.15).

Proof. Provided by Lemma 2.5.4, it is enough to prove that

$$\varlimsup_{N\to\infty}\;\mathbb{E}^\beta_{\mu_N}\Big[\max_{1\le i\le k}\Big\{\int_0^t V_N(\varepsilon,j,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big] \;\le\; \kappa_0\,.$$

By the entropy inequality, for each fixed N, the expectation above is smaller than

$$\frac{H(\mu^N|\nu_\theta)}{N^d} + \frac{1}{N^d}\log \mathbb{E}_{\nu_\theta}\Big[\exp\Big\{\max_{1\le i\le k} N^d\Big\{\int_0^T V_N(\varepsilon,j,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big\}\Big]\,.$$

Using (2.15), we bound the first parcel above by κ_0. Since $\exp\big\{\max_{1\le i\le k} a_j\big\} \le \sum_{1\le i\le k}\exp\{a_j\}$ and by (2.20), we conclude that the limsup as $N\uparrow\infty$ of the second parcel above is less than or equal to

$$\begin{aligned}&\varlimsup_{N\to\infty}\frac{1}{N^d}\log \mathbb{E}_{\nu_\theta}\Big[\sum_{1\le i\le k}\exp\Big\{N^d\int_0^T V_N(\varepsilon,j,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\\ &=\max_{1\le i\le k}\varlimsup_{N\to\infty}\frac{1}{N^d}\log \mathbb{E}_{\nu_\theta}\Big[\exp\Big\{N^d\int_0^T V_N(\varepsilon,j,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big]\,.\end{aligned}$$

Thus, in order to conclude the proof, it is enough to show that the limsup above is non positive for each $i=1,\dots,k$. By the Feynman-Kac formula (see [18, p. 332, Lemma 7.2]) for each fixed N and $d\ge 2$,

$$\frac{1}{N^d}\log \mathbb{E}_{\nu_\theta}\Big[\exp\Big\{N^d\int_0^T V_N(\varepsilon,j,H_i(s,\cdot),\eta_s)\,ds\Big\}\Big] \tag{2.33}$$

$$\le \int_0^T \sup_f\Big\{\int V_N(\varepsilon,j,H_i(s,\cdot),\eta)f(\eta)d\nu_\theta - N^{2-d}\mathfrak{D}_N(\sqrt{f})\Big\}\,ds\,, \tag{2.34}$$

where the supremum above is taken over all probability densities f with respect to ν_θ. By assumption, each of the functions $\{H_i : i=1,\dots,k\}$ vanishes in a neighborhood of $\partial\Lambda$. Thus, we make following observation about the first sum in the RHS of (2.31): for small ε, non-zero summands are such that x/N and $(x+\varepsilon N e_j)N$ lay both in Λ or both in $\Lambda^{\complement}$. Henceforth, in such a case, it is possible to find a path no slow bonds connecting x and $x+\varepsilon N e_j$. Keeping this in mind, we can repeat the arguments in the proof of Lemma 2.5.3 to deduce that

$$\begin{aligned}&\int \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} H\big(\tfrac{x}{N}\big)\frac{\{\eta(x)-\eta(x+\varepsilon N e_j)\}}{\varepsilon} f(\eta)d\nu_\theta\\ &\le N^{2-d}\mathfrak{D}_N(\sqrt{f}) + \frac{2}{N^d}\sum_{x\in\mathbb{T}^d_N}\Big(H\big(\tfrac{x}{N}\big)\Big)^2\,.\end{aligned}$$

Plugging this inequality into (2.34) implies that (2.33) has a nonpositive limsup, showing (2.5.3) and therefore finishing the proof. □

Lemma 2.5.6.

$$\mathbb{E}_{\mathbb{Q}^\beta_*}\Big[\sup_H\Big\{\int_0^T\int_{\mathbb{T}^d}(\partial_{u_j}H)(s,u)\rho(s,u)duds - 2\int_0^T\int_{\mathbb{T}^d}(H(s,u))^2\,duds\Big\}\Big] \;\le\; \kappa_0\,,$$

where the supremum is carried over all functions $H\in C^{0,1}([0,T]\times\mathbb{T}^d)$ with compact support contained in $[0,T]\times(\mathbb{T}^d\backslash\partial\Lambda)$.

Proof. Consider a sequence $\{H_i \,:\, i \geq 1\}$ dense in the subset of $C^2([0,t] \times \mathbb{T}^d)$ of functions with support contained in $[0,T] \times (\mathbb{T}^d \backslash \partial\Lambda)$, being the density with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$. Recall we are assuming that $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ converges to $\mathbb{Q}^\beta_*$. Then, by (2.32) and the Portmanteau Theorem,

$$\overline{\lim_{\delta\to 0}}\, \mathbb{E}_{\mathbb{Q}^\beta_*}\Big[\max_{1\leq i\leq k} \Big\{ \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} H_i(s,u))\{\rho^\delta_s(u) - \rho^\delta_s(u+\varepsilon e_j)\}\, duds - 2\int_0^T \int_{\mathbb{T}^d} (H_i(s,u))^2\, duds \Big\}\Big] \;\leq\; \kappa_0,$$

where $\rho^\delta_s(u) = (\rho_s * \iota_\delta)(u)$ as defined in (2.29). Letting $\delta \downarrow 0$, the Lebesgue Differentiation Theorem assures that $\rho^\delta_s(u)$ converges almost surely to ρ_s. Then, performing a change of variables and letting $\varepsilon \downarrow 0$, we obtain that

$$\mathbb{E}_{\mathbb{Q}^\beta_*}\Big[\max_{1\leq i\leq k} \Big\{ \int_0^T \int_{\mathbb{T}^d} (\partial_{u_j} H_i(s,u))\rho_s(u)\, duds - 2\int_0^T \int_{\mathbb{T}^d} (H_i(s,u))^2\, duds \Big\}\Big] \leq \kappa_0.$$

Since the maximum increases to the supremum, we conclude the lemma by applying the Monotone Convergence Theorem to $\{H_i \,:\, i \geq 1\}$, which is a dense sequence in the subset of functions $C^2([0,T] \times \mathbb{T}^d)$ with compact support contained in $[0,T] \times (T^d \backslash \partial\Lambda)$. □

Proposition 2.5.7. *The measure $\mathbb{Q}^\beta_*$ is concentrated on paths $\pi(t,u) = \rho(t,u)du$ such that $\rho \in L^2\big([0,T]; \mathcal{H}^1(\mathbb{T}^d \backslash \partial\Lambda)\big)$.*

Proof. Denote by $\ell : C^2([0,T] \times \mathbb{T}^d) \to \mathbb{R}$ the linear functional defined by

$$\ell(H) \;=\; \int_0^T \int_{\mathbb{T}^d} (\partial_{u_j} H)(s,u)\rho(s,u)\, du\, ds\,.$$

Since the set of functions $H \in C^2([0,T] \times \mathbb{T}^d)$ with support contained in $[0,T] \times (\mathbb{T}^d \backslash \partial\Lambda)$ is dense in $L^2([0,T] \times \mathbb{T}^d)$ and since by Lemma 2.5.6 ℓ is a $\mathbb{Q}^\beta_*$-a.s. bounded functional in $C^2([0,T] \times \mathbb{T}^d)$, we can extend it to a $\mathbb{Q}^\beta_*$-a.s. bounded functional in $L^2([0,T] \times \mathbb{T}^d)$, which is a Hilbert space. Then, by the Riesz Representation Theorem, there exists a function $G \in L^2([0,T] \times \mathbb{T}^d)$ such that

$$\ell(H) \;=\; -\int_0^T \int_{\mathbb{T}^d} H(s,u)G(s,u)\, du\, ds\,,$$

concluding the proof. □

2.6 Characterization of limit points

Before going into the details of each regime $\beta \in [0,1)$, $\beta = 1$ or $\beta \in (1,\infty]$, we make some useful considerations for all cases.

We will prove in this section that all limit points of the sequence $\{\mathbb{Q}^{\beta,N}_{\mu_N} : N \geq 1\}$ are concentrated on trajectories of measures $\pi(t,du) = \rho(t,u)\, du$, whose density

$\rho(t,u)$ with respect to the Lebesgue measure is the weak solution of the hydrodynamic equation (2.4), (2.6) or (2.7) for each corresponding value of β. Provided by tightness, let $\mathbb{Q}_*^\beta$ be a limit point of the sequence $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ and assume, without loss of generality, that $\{\mathbb{Q}_{\mu_N}^{\beta,N} : N \geq 1\}$ converges to $\mathbb{Q}_*^\beta$.

Since there is at most one particle per site, it is easy to show that $\mathbb{Q}_*^\beta$ is concentrated on trajectories $\pi(t,du)$ which are absolutely continuous with respect to the Lebesgue measure $\pi(t,du) = \rho(t,u)\,du$ and whose density $\rho(t,\cdot)$, is nonnegative and bounded by one. Recall the martingale $M_t^N(H)$ in (2.10).

Lemma 2.6.1. *If*

a) $\beta \in [0,1)$ *and* $H \in C^2(\mathbb{T}^d)$*, or*

b) $\beta \in [1,\infty]$ *and* $H \in C^2(\mathbb{T}^d \backslash \partial\Lambda)$*,*

then, for all $\delta > 0$*,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}^N \Big[\sup_{0\leq t\leq T} |M_t^N(H)| > \delta \Big] \;=\; 0\,. \tag{2.35}$$

Proof. Item a) has been already proved in (2.13). For item b), recalling (2.11) note that

$$\langle M^N(H)\rangle_t \;\leq\; \frac{T}{N^{2d-2}} \sum_{j=1}^d \sum_{x\in\mathbb{T}_N^d} \xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big]^2. \tag{2.36}$$

Since $H \in C^2(\mathbb{T}^d \backslash \partial\Lambda)$, H is differentiable with bounded derivative except over $\partial\Lambda$. Therefore, if the edge $x,\, x+e_j$ is not a slow bond, then

$$\xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N}) \Big]^2 \;\leq\; \frac{1}{N^2} \|\partial_{u_j} H\|_\infty^2\,. \tag{2.37}$$

On the other hand, if the edge $x,\, x+e_j$ is a slow bond, then

$$\xi_{x,x+e_j}^N \Big[H(\tfrac{x+e_j}{N}) - H_t(\tfrac{x}{N}) \Big]^2 \;\leq\; \frac{4\alpha \|H\|_\infty^2}{N^\beta}\,. \tag{2.38}$$

Since the number of slow bonds is of order $\mathcal{O}(N^{d-1})$, plugging (2.37) and (2.38) into (2.36) gives us $\langle M^N(H_t)\rangle_t \leq \mathcal{O}(1/N^d)$. ' Then, Doob's inequality concludes the proof. □

2.6.1 Characterization of limit points for $\beta \in [0,1)$.

Proposition 2.6.2. *Let* $H \in C^2(\mathbb{T}^d)$*. Then, for any* $\delta > 0$*,*

$$\mathbb{Q}_*^\beta \Big[\pi_\cdot : \sup_{0\leq t\leq T} \Big| \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \Delta H\rangle\, ds \Big| > \delta \Big] \;=\; 0\,.$$

Proof. Since $\mathbb{Q}_{\mu_N}^{\beta,N}$ converges weakly to Q_*^β, by Portmanteau's Theorem (see [2, Theorem 2.1]),

$$\mathbb{Q}_*^\beta \Big[\pi_\cdot : \sup_{0\leq t\leq T} \Big| \langle \pi_t, H\rangle \;-\; \langle \pi_0, H\rangle \;-\; \int_0^t \langle \pi_s, \Delta H\rangle\, ds \Big| > \delta \Big]$$

$$\leq \varlimsup_{N\to\infty} \mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi_\cdot : \sup_{0\leq t\leq T}\Big|\langle \pi_t, H\rangle \,-\, \langle \pi_0, H\rangle \,-\, \int_0^t \langle \pi_s, \Delta H\rangle\, ds\Big| > \delta\Big] \tag{2.39}$$

since the supremum above is a continuous function in the Skorohod metric, see Proposition 2.8.1. Recall that $\mathbb{Q}^{\beta,N}_{\mu_N}$ is the probability measure induced by $\mathbb{P}^{\beta}_{\mu_N}$ via the empirical measure. With this in mind and then adding and subtracting $\langle \pi^N_s, N^2\mathbb{L}_N H\rangle$, expression (2.39) can be bounded from above by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\pi_\cdot : \sup_{0\leq t\leq T}\Big|\langle \pi^N_t, H\rangle \,-\, \langle \pi^N_0, H\rangle \,-\, \int_0^t \langle \pi^N_s, N^2\mathbb{L}_N H\rangle\, ds\Big| > \delta/2\Big]$$
$$+ \varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\pi_\cdot : \sup_{0\leq t\leq T}\Big|\int_0^t \langle \pi^N_s, \Delta H - N^2\mathbb{L}_N H\rangle\, ds\Big| > \delta/2\Big]\,.$$

By Lemma 2.6.1, the first term above is null. Since there is at most one particle per site, the second term in last expression is bounded by

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\frac{T}{N^d}\sum_{x\notin\Gamma_N}\Big|\Delta H\Big(\frac{x}{N}\Big) - N^2\mathbb{L}_N\Big(\frac{x}{N}\Big)\Big| > \delta/4\Big]$$
$$+ \varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \frac{1}{N^d}\sum_{x\in\Gamma_N}\Big\{\Delta H\Big(\frac{x}{N}\Big) - N^2\mathbb{L}_N\Big(\frac{x}{N}\Big)\Big\}\eta_s(x)\, ds\Big| > \delta/4\Big]\,.$$

Outside Γ_N, the operator $N^2\mathbb{L}_N$ coincides with the discrete Laplacian. Since $H \in C^2(\mathbb{T}^d)$, the first probability above vanishes for N sufficiently large. Recall that the number of elements in Γ_N is of order N^{d-1}. Applying the triangular inequality, the second expression in the previous sum becomes bounded by the sum of

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\mathcal{O}(N^{-1})T\|\Delta H\|_\infty > \delta/8\Big] \tag{2.40}$$

and

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_N} N\mathbb{L}_N\Big(\frac{x}{N}\Big)\eta_s(x)\, ds\Big| > \delta/8\Big]\,. \tag{2.41}$$

For large N, the probability in (2.40) vanishes. We deal now with (2.41). Let $x \in \Gamma_N$. By definition of Γ_N, some adjacent bond to x is a *slow bond*. Thus, the opposite vertex to x with respect to this bond is also in Γ_N, see Figure 2.5.

Recall the definition of $\mathbb{L}_N$ in (2.9). Whenever $\{x, x-e_j\}$ neither $\{x, x+e_j\}$ are slow bonds, the expression

$$\xi^N_{x,x+e_j}\Big[H\big(\tfrac{x+e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big] + \xi^N_{x,x-e_j}\Big[H\big(\tfrac{x-e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big]$$

is of order $\mathcal{O}(N^{-2})$ due to assumption $H \in C^2(\mathbb{T}^d)$. Therefore, in (2.41) we can disregard terms of this kind, reducing the proof that (2.41) is null to prove that

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e \text{ is a slow bond}}} \mathbf{A}(e)\, ds\Big| > \delta/16\Big] \;=\; 0\,, \tag{2.42}$$

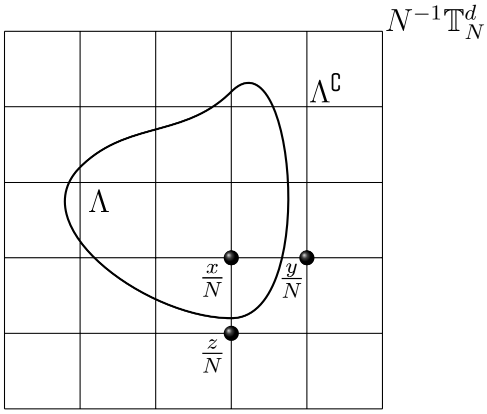


Figure 2.5: Illustration of sites $x, y, z \in \Gamma_N$. We note that two adjacent edges to x are slow bonds, and two adjacent edges are not. Besides, any opposite vertex to x will be of the form $x \pm e_j$.

where

$$\begin{aligned}\mathbf{A}(e) \;=\; & \left[\alpha N^{1-\beta}\Big(H\big(\tfrac{x+e_j}{N}\big)-H\big(\tfrac{x}{N}\big)\Big)+\frac{H\big(\frac{x-e_j}{N}\big)-H\big(\frac{x}{N}\big)}{1/N}\right]\eta_s(x)\\ & + \left[\frac{H\big(\frac{x+2e_j}{N}\big)-H\big(\frac{x+e_j}{N}\big)}{1/N}+\alpha N^{1-\beta}\Big(H\big(\tfrac{x}{N}\big)-H\big(\tfrac{x+e_j}{N}\big)\Big)\right]\eta_s(x+e_j)\,.\end{aligned}$$

Since H is smooth, the terms inside parenthesis involving $N^{1-\beta}$ are of order $\mathcal{O}(N^{-\beta})$ and hence negligible. On the other hand, the remaining terms are close to plus or minus the derivative of H at x/N. We have thus reduced the proof of (2.42) to the proof of

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e\text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s(x+e_j)-\eta_s(x)\big)\,ds\Big| > \delta/32\Big] \;=\; 0\,. \tag{2.43}$$

Let $t_0 = 0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$ with mesh bounded by an arbitrary $\tilde{\varepsilon} > 0$. Via the triangular inequality, if we prove that

$$\sum_{k=0}^{n}\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t_k} \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e\text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s(x+e_j)-\eta_s(x)\big)\,ds\Big| \;>\; \delta\Big]$$

vanishes, then we will conclude that (2.43) vanishes as well. Therefore, it is enough now to show that, for any $\delta > 0$ and any $t \in [0,T]$,

$$\varlimsup_{N\to\infty} \mathbb{P}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e\text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s(x+e_j)-\eta_s(x)\big)\,ds\Big| \;>\; \delta\Big] \;=\; 0\,.$$

Markov's inequality then allows us to bound the expression above by

$$\varlimsup_{N\to\infty} \delta^{-1}\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^{t} \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e\text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s(x+e_j)-\eta_s(x)\big)\,ds\Big|\Big]\,. \tag{2.44}$$

Adding and subtracting $\eta_s^{\varepsilon N}(x)$ and $\eta_s^{\varepsilon N}(x+e_j)$, we bound (2.44) from above by

$$\varlimsup_{N\to\infty} \delta^{-1}\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e \text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s(x+e_j)-\eta_s^{\varepsilon N}(x+e_j)\big)\,ds\Big|\Big]$$
$$+\varlimsup_{N\to\infty} \delta^{-1}\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e \text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s^{\varepsilon N}(x+e_j)-\eta_s^{\varepsilon N}(x)\big)\,ds\Big|\Big]$$
$$+\varlimsup_{N\to\infty} \delta^{-1}\mathbb{E}^{\beta}_{\mu_N}\Big[\Big|\int_0^t \frac{1}{N^{d-1}}\sum_{\substack{e=\{x,x+e_j\}\\ e \text{ is a slow bond}}} \partial_{u_j}H\big(\tfrac{x}{N}\big)\big(\eta_s^{\varepsilon N}(x)-\eta_s(x)\big)\,ds\Big|\Big].$$

Since $|\{\eta_s^{\varepsilon N}(x+e_j)-\eta_s^{\varepsilon N}(x)\}|\leq\frac{2(\varepsilon N)^{d-1}}{(\varepsilon N)^d}=\frac{2}{\varepsilon N}$, $|\Gamma_N|$ is of order N^{d-1} and $\|\partial_{u_j}H\|_\infty<\infty$, the second term above vanishes. For the remaining terms, we apply Lemma 2.5.2, finishing the proof. □

2.6.2 Characterization of limit points for $\beta = 1$.

This subsection is devoted to the proof of the next proposition. Keep in mind that Proposition 2.5.7 allows us to write $\pi(t,u)=\rho(t,u)du$ when considering the measure $\mathbb{Q}^{\beta}_*$.

Proposition 2.6.3. *Let $H\in C^2(\mathbb{T}^d\backslash\partial\Lambda)$. For all $\delta>0$,*

$$\begin{aligned}\mathbb{Q}^{\beta}_*\Big[\pi_\cdot : \sup_{0\leq t\leq T}&\Big|\langle\rho_t,H\rangle-\langle\rho_0,H\rangle-\int_0^t\langle\rho_s,\Delta H\rangle\,ds\\
&-\int_0^t\int_{\partial\Lambda}\rho_s(u^+)\sum_{j=1}^d\partial_{u_j}H(u^+)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\
&+\int_0^t\int_{\partial\Lambda}\rho_s(u^-)\sum_{j=1}^d\partial_{u_j}H(u^-)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\
&+\int_0^t\int_{\partial\Lambda}\alpha(\rho_s(u^-)-\rho_s(u^+))(H(u^+)-H(u^-))\sum_{j=1}^d|\langle\vec{\zeta}(u),e_j\rangle|\,dS(u)ds\Big|>\delta\Big]=0.\end{aligned}\tag{2.45}$$

Let us gather some ingredients for the proof of above. The first one is a suitable expression for $N\mathbb{L}_N$ over Γ_N. Define

$$\begin{aligned}\Gamma_{N,-}&=\Gamma_N\cap\big\{x\in\mathbb{T}^d_N : \tfrac{x}{N}\in\Lambda\big\}\quad\text{and}\\
\Gamma_{N,+}&=\Gamma_N\cap\big\{x\in\mathbb{T}^d_N : \tfrac{x}{N}\in\Lambda^\complement\big\}\end{aligned}\tag{2.46}$$

Such a notation has been chosen to agree with (2.5). Let us focus on $\Gamma_{N,-}$, being the analysis for $\Gamma_{N,+}$ completely analogous. It is convenient to consider the decomposition $\Gamma_{N,-}=\bigcup_{j=1}^d\Gamma^j_{N,-}$, where

$$\Gamma^j_{N,-}=\Gamma^{j,\text{left}}_{N,-}\cup\Gamma^{j,\text{right}}_{N,-},\qquad\text{with}$$

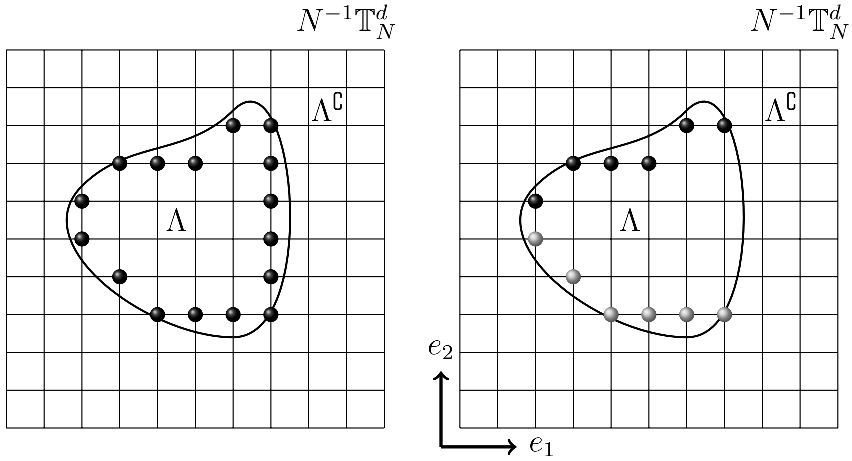


Figure 2.6: In the left, an illustration of the set $\Gamma_{N,-}$, whose elements are represented by black balls. In the right, an illustration of the sets $\Gamma_{N,-}^{j,\text{left}}$ and $\Gamma_{N,-}^{j,\text{right}}$ for $j=2$, whose elements are represented by gray and black balls, respectively.

$$\Gamma_{N,-}^{j,\text{left}} = \Big\{x \in \Gamma_{N,-} \,:\, \frac{x-e_j}{N} \in \Lambda^{\complement}\Big\} \text{ and } \Gamma_{N,-}^{j,\text{right}} = \Big\{x \in \Gamma_{N,-} \,:\, \frac{x+e_j}{N} \in \Lambda^{\complement}\Big\}\,,$$

see Figure 2.6 for an illustration. Note that $\Gamma_{N,-}^{j,\text{right}}$ and $\Gamma_{N,-}^{j,\text{left}}$ are not necessarily disjoint for a fixed j. Nevertheless, due to the smoothness of $\partial\Lambda$, the number of elements in the intersection of these two sets is of order $\mathcal{O}(N^{d-2})$, hence negligible to our purposes. We will henceforth assume that $\Gamma_{N,-}^{j,\text{right}}$ and $\Gamma_{N,-}^{j,\text{left}}$ are disjoint sets for all $j=1,\dots,d$.

Remark 2.6.4. At first sight, the reader may imagine that $\Gamma_{N,-}$ is equal to $\Gamma_{N,-}^{j,\text{left}} \cup \Gamma_{N,-}^{j,\text{right}}$ for any j, or at least very to close to. This is false, as illustrated by Figure 2.6. Moreover, for $i \neq j$ and large N, the sets $\Gamma_{N,-}^{j}$ and $\Gamma_{N,-}^{i}$ in general are not disjoint with a no negligible intersection.

Define now

$$N\mathbb{L}_N^j H(\tfrac{x}{N}) = N\xi_{x,x+e_j}^N\big(H(\tfrac{x+e_j}{N}) - H(\tfrac{x}{N})\big) + N\xi_{x,x-e_j}^N\big(H(\tfrac{x-e_j}{N}) - H(\tfrac{x}{N})\big)\,.$$

Then, by By Fubini's Lemma,

$$\begin{aligned}\sum_{x\in\Gamma_{N,-}} N\mathbb{L}_N H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x) &= \sum_{x\in\Gamma_{N,-}}\sum_{j=1}^{d} N\mathbb{L}_N^j H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\\ &= \sum_{j=1}^{d}\Big\{\sum_{x\in\Gamma_{N,-}^{j,\text{right}}} N\mathbb{L}_N^j H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x) + \sum_{x\in\Gamma_{N,-}^{j,\text{left}}} N\mathbb{L}_N^j H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\Big\}\,. \qquad (2.47)\end{aligned}$$

If $x \in \Gamma_{N,-}^{j,\text{right}}$, then $\xi_{x,x+e_j}^N = \alpha/N$ and $\xi_{x,x-e_j}^N = 1$, see Figure 2.5. In this case,

$$N\mathbb{L}_N^j H\big(\tfrac{x}{N}\big) = \alpha\Big(H\big(\tfrac{x+e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big) - \partial_{u_j} H\big(\tfrac{x}{N}\big) + \mathcal{O}(N^{-1})\,.$$

On the other hand, if $x \in \Gamma_{N,-}^{j,\text{left}}$, then $\xi_{x,x-e_j}^N = \alpha/N$ and $\xi_{x,x+e_j}^N = 1$. In this case,

$$N\mathbb{L}_N^j H\big(\tfrac{x}{N}\big) \;=\; \partial_{u_j} H\big(\tfrac{x}{N}\big) + \alpha\Big(H\big(\tfrac{x-e_j}{N}\big) - H\big(\tfrac{x}{N}\big)\Big) + \mathcal{O}(N^{-1})\,.$$

Now, let $\mathbf{u} : \mathbb{T}^d \to \partial\Lambda$ be a function such that

$$\|\mathbf{u}(u) - u\| \;=\; \min_{v\in\partial\Lambda} \|v - u\|\,, \tag{2.48}$$

and $\mathbf{u}$ is continuous in a neighborhood of $\partial\Lambda$. That is, $\mathbf{u}$ maps $u \in \mathbb{T}^d$ to some of its closest points over $\partial\Lambda$ and $\mathbf{u}$ is continuous on the set $(\partial\Lambda)^\varepsilon = \{u \in \mathbb{T}^d : \text{dist}(u, \partial\Lambda) < \varepsilon\}$ for some small $\varepsilon > 0$. There are more than one function fulfilling (2.48), but any choice among them will be satisfactory for our purposes, once this function is continuous near $\partial\Lambda$. With this mind we can rewrite (2.47), achieving the formula

$$\begin{aligned}
&\frac{1}{N^{d-1}} \sum_{x\in\Gamma_{N,-}} N\mathbb{L}_N H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\\
&= \frac{1}{N^{d-1}} \sum_{j=1}^d \Bigg\{ \sum_{x\in\Gamma_{N,-}^{j,\text{right}}} \Big[\alpha\big(H(\mathbf{u}^+) - H(\mathbf{u}^-)\big) - \partial_{u_j} H(\mathbf{u}^-)\Big]\eta_s^{\varepsilon N}(x)\\
&\qquad + \sum_{x\in\Gamma_{N,-}^{j,\text{left}}} \Big[\partial_{u_j} H(\mathbf{u}^-) + \alpha\big(H(\mathbf{u}^+) - H(\mathbf{u}^-)\big)\Big]\eta_s^{\varepsilon N}(x)\Bigg\}.
\end{aligned} \tag{2.49}$$

plus a negligible error, where by $H(\mathbf{u}^-)$ and $H(\mathbf{u}^+)$ are the sided limits of H at $\mathbf{u}$. The dependence of $\mathbf{u}$ on x/N will be dropped to not overload notation. Defining

$$\Gamma_{N,+}^j \;=\; \Gamma_{N,+}^{j,\text{left}} \cup \Gamma_{N,+}^{j,\text{right}}\,, \qquad \text{with}$$

$$\Gamma_{N,+}^{j,\text{left}} = \Big\{x \in \Gamma_{N,+} \;:\; \frac{x+e_j}{N} \in \Lambda\Big\} \;\text{ and }\; \Gamma_{N,+}^{j,\text{right}} = \Big\{x \in \Gamma_{N,+} \;:\; \frac{x-e_j}{N} \in \Lambda\Big\}\,,$$

we similarly have

$$\begin{aligned}
&\frac{1}{N^{d-1}} \sum_{x\in\Gamma_{N,+}} N\mathbb{L}_N H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\\
&= \frac{1}{N^{d-1}} \sum_{j=1}^d \Bigg\{ \sum_{x\in\Gamma_{N,+}^{j,\text{right}}} \Big[\partial_{u_j} H(\mathbf{u}^+) + \alpha\big(H(\mathbf{u}^-) - H(\mathbf{u}^+)\big)\Big]\eta_s^{\varepsilon N}(x)\\
&\qquad + \sum_{x\in\Gamma_{N,+}^{j,\text{left}}} \Big[\alpha\big(H(\mathbf{u}^-) - H(\mathbf{u}^+)\big) - \partial_{u_j} H(\mathbf{u}^+)\Big]\eta_s^{\varepsilon N}(x)\Bigg\}.
\end{aligned} \tag{2.50}$$

The second ingredient is about convergence of sums over Γ_N towards integrals over $\partial\Lambda$. Let us review some standard facts about integrals over surfaces. Consider a smooth compact manifold $\mathcal{M} \subset \mathbb{R}^d$ of dimension $(d-1)$. Assume that $\mathcal{M}$ is the graph of a function $f : R \subset \mathbb{R}^{d-1} \to \mathbb{R}$, that is, $\mathcal{M} = \{(x, f(x)) : x \in R\}$. Then, given a smooth function $g : \mathcal{M} \to \mathbb{R}$, the surface integral of g over $\mathcal{M}$ will be given by

$$\begin{aligned}\int_{\mathcal{M}} g(u)\, dS(u) \;&=\; \int_R g(x, f(x)) \frac{dx}{|\cos(\gamma(x, f(x)))|} \\ = \;&\int_R g\big(x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1})\big) \frac{dx_1 \cdots dx_{d-1}}{|\langle \vec{\zeta}(x_1, \dots, x_{d-1}), e_d\rangle|}\,,\end{aligned} \tag{2.51}$$

where $\gamma(x, f(x))$ is defined as the angle between the normal exterior vector $\vec{\zeta}(u) = \vec{\zeta}(x_1, \dots, x_{d-1})$ and e_d, the d-th element of the canonical basis of $\mathbb{R}^d$. Of course, a manifold in general is only locally a graph of a function as above. Nevertheless, the notion of partition of unity allows to use this local property to evaluate a surface integral. Recall the definition of u given in (2.48).

Lemma 2.6.5. *Let $g : \Lambda \backslash (\partial\Lambda) \subset \mathbb{T}^d \to \mathbb{R}$ be a function which is continuous near $\partial\Lambda$ with an extension to Λ which is also continuous near $\partial\Lambda$. Then,*

$$\int_{\partial\Lambda} g(u^-)|\langle \vec{\zeta}(u), e_j\rangle|\, dS(u) \;=\; \lim_{N\to\infty} \frac{1}{N^{d-1}} \sum_{x \in \Gamma^j_{N,-}} g\big(\tfrac{x}{N}\big) \quad \textit{and} \tag{2.52}$$

$$\int_{\partial\Lambda} g(u^-)\langle \vec{\zeta}(u), e_j\rangle\, dS(u) \;=\; \lim_{N\to\infty} \frac{1}{N^{d-1}} \Bigg[\sum_{x \in \Gamma^{j,\text{right}}_{N,-}} g\big(\tfrac{x}{N}\big) - \sum_{x \in \Gamma^{j,\text{left}}_{N,-}} g\big(\tfrac{x}{N}\big) \Bigg]. \tag{2.53}$$

Analogously, if $g : \Lambda^{\complement} \subset \mathbb{T}^d \to \mathbb{R}$ is a function which is continuous near $\partial\Lambda$ with an extension to the closure of $\Lambda^{\complement}$ which is also continuous near $\partial\Lambda$, then

$$\int_{\partial\Lambda} g(u^+)|\langle \vec{\zeta}(u), e_j\rangle|\, dS(u) \;=\; \lim_{N\to\infty} \frac{1}{N^{d-1}} \sum_{x \in \Gamma^j_{N,+}} g\big(\tfrac{x}{N}\big) \quad \textit{and} \tag{2.54}$$

$$\int_{\partial\Lambda} g(u^+)\langle \vec{\zeta}(u), e_j\rangle\, dS(u) \;=\; \lim_{N\to\infty} \frac{1}{N^{d-1}} \Bigg[\sum_{x \in \Gamma^{j,\text{right}}_{N,+}} g\big(\tfrac{x}{N}\big) - \sum_{x \in \Gamma^{j,\text{left}}_{N,+}} g\big(\tfrac{x}{N}\big) \Bigg]. \tag{2.55}$$

Proof. In view of the previous discussion, we claim that

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \sum_{x \in \Gamma^j_{N,-}} \frac{h\big(\frac{x}{N}\big)}{|\langle \zeta\big(\mathbf{u}(\frac{x}{N})\big), e_j\rangle|} \;=\; \int_{\partial\Lambda} h(u^-)\, dS(u)\,. \tag{2.56}$$

for any continuous function $h : \Lambda \to \mathbb{R}$ such that $h(u) = 0$ on the set $\{u \in \partial\Lambda : \langle \vec{\zeta}(u), e_j\rangle = 0\}$. This is due to the fact that the sum in the left hand side of (2.52) is equal to a Riemann sum for the integral on the right hand side of (2.51) modulus a small error. To see this, it is enough to note that if $x \in \Gamma_{N,-}$, then x/N is at a distance less or equal than $1/N$ to $\partial\Lambda$, and recall that Λ is compact, thus any continuous function over Λ is uniformly continuous.

Consider now the function $h : \Lambda \to \mathbb{R}$ given by

$$h(u) \;:=\; g(u)\, |\langle \vec{\zeta}\big(\mathbf{u}(u)\big), e_j\rangle|\,.$$

Since $\mathbf{u}(u) = u$ for $u \in \partial\Lambda$, we have that $h(u) = 0$ on the set $\{u \in \partial\Lambda : \langle \vec{\zeta}(u), e_j\rangle = 0\}$. Then, considering this particular function h in (2.56) leads to (2.52). The limit (2.53) can be derived from (2.52) noticing that, for N sufficiently large,

- if $x \in \Gamma_{N,-}^{j,\text{right}}$, then $\langle \vec{\zeta}\big(\mathbf{u}(x/N)\big), e_j\rangle > 0$ and
- if $x \in \Gamma_{N,-}^{j,\text{left}}$, then $\langle \vec{\zeta}\big(\mathbf{u}(x/N)\big), e_j\rangle < 0$,

see Figure 2.5 for support. The proofs for (2.54) and (2.55) are analogous. □

Proof of Proposition 2.6.3. The fact that boundary integrals are not well-defined in the whole Skorohod space $\mathcal{D}([0,T],\mathcal{M})$ forbids us to directly apply Portmanteau's Theorem. To circumvent this technical obstacle, fix $\varepsilon > 0$ which will be taken small later. Adding and subtracting the convolution of $\rho(t,u)$ with the approximation of identity ι_ε defined in (2.28), we bound the probability in (2.45) by the sum of

$$\begin{aligned}
\mathbb{Q}_*^\beta\Big[\pi_\cdot : \sup_{0\le t\le T}\Big|&\langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \Delta H\rangle\, ds \\
&- \int_0^t \int_{\partial\Lambda} (\rho_s * \iota_\varepsilon)(u^+) \sum_{j=1}^d \partial_{u_j} H(u^+) \langle \vec{\zeta}(u), e_j\rangle\, dS(u) ds \\
&+ \int_0^t \int_{\partial\Lambda} (\rho_s * \iota_\varepsilon)(u^-) \sum_{j=1}^d \partial_{u_j} H(u^-) \langle \vec{\zeta}(u), e_j\rangle\, dS(u) ds \\
&+ \int_0^t \int_{\partial\Lambda} \alpha((\rho_s * \iota_\varepsilon)(u^-) - (\rho_s * \iota_\varepsilon)(u^+)) \\
&\qquad \times (H(u^+) - H(u^-)) \sum_{j=1}^d |\langle \vec{\zeta}(u), e_j\rangle|\, dS(u) ds\Big| > \delta/2\Big]
\end{aligned} \tag{2.57}$$

and

$$\begin{aligned}
\mathbb{Q}_*^\beta\Big[\pi_\cdot : \sup_{0\le t\le T}\Big| &\int_0^t \int_{\partial\Lambda} \Big((\rho_s * \iota_\varepsilon)(u^+) - \rho_s(u^+)\Big) \sum_{j=1}^d H(u^+) \langle \vec{\zeta}(u), e_j\rangle\, dS(u) ds \\
&- \int_0^t \int_{\partial\Lambda} \Big((\rho_s * \iota_\varepsilon)(u^-) - \rho_s(u^-)\Big) \sum_{j=1}^d \partial_{u_j} H(u^-) \langle \vec{\zeta}(u), e_j\rangle\, dS(u) ds \\
&- \int_0^t \int_{\partial\Lambda} \alpha\Big((\rho_s * \iota_\varepsilon)(u^-) - \rho_s(u^-)\Big)(H(u^+) - H(u^-)) \sum_{j=1}^d |\langle \vec{\zeta}(u), e_j\rangle|\, dS(u) ds \\
&+ \int_0^t \int_{\partial\Lambda} \alpha\Big((\rho_s * \iota_\varepsilon)(u^+) - \rho_s(u^+)\Big) \\
&\qquad \times (H(u^+) - H(u^-)) \sum_{j=1}^d |\langle \vec{\zeta}(u), e_j\rangle|\, dS(u) ds\Big| > \delta/2\Big].
\end{aligned} \tag{2.58}$$

where ι_ε and the convolution $\rho_s * \iota_\varepsilon$ were defined in (2.29). Adapting results of [1, Chapter III] to our context, the reader can check that functions in the Sobolev space $L^2\big([0,T];\mathcal{H}^1(\mathbb{T}^d\backslash\partial\Lambda)\big)$ are continuous in $\mathbb{T}^d\backslash\partial\Lambda$. Thus, Lemma 2.5.7 gives us that (2.58) vanishes as $\varepsilon\to 0$. It remains to deal with (2.57). By Portmanteau's Theorem, (2.57) is bounded from above by

$$
\begin{aligned}
&\varlimsup_{N\to\infty}\mathbb{Q}^{\beta,N}_{\mu_N}\Big[\pi_\cdot:\sup_{0\le t\le T}\Big|\langle\pi_t,H\rangle-\langle\pi_0,H\rangle-\int_0^t\langle\pi_s,\Delta H\rangle\,ds\\
&-\int_0^t\int_{\partial\Lambda}(\pi_s*\iota_\varepsilon)(u^+)\sum_{j=1}^d\partial_{u_j}H(u^+)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\
&+\int_0^t\int_{\partial\Lambda}(\pi_s*\iota_\varepsilon)(u^-)\sum_{j=1}^d\partial_{u_j}H(u^-)\langle\vec{\zeta}(u),e_j\rangle\,dS(u)ds\\
&+\int_0^t\int_{\partial\Lambda}\alpha((\pi_s*\iota_\varepsilon)(u^-)-(\pi_s*\iota_\varepsilon)(u^+))\\
&\qquad\qquad\times(H(u^+)-H(u^-))\sum_{j=1}^d|\langle\vec{\zeta}(u),e_j\rangle|\,dS(u)ds\Big|>\delta/2\Big],
\end{aligned}
$$

since the supremum above is a continuous function in the Skorohod metric. Now, recalling that $\mathbb{Q}^{\beta,N}_{\mu_N}$ is the probability induced by $\mathbb{P}^{\beta}_{\mu_N}$ via the empirical measure, adding and subtracting the terms $\langle\pi^N_s,N^2\mathbb{L}_NH\rangle$ and $\frac{1}{N^{d-1}}\sum_{x\in\Gamma_N}N\mathbb{L}_NH(\frac{x}{N})\eta^{\varepsilon N}_s(x)$, applying (2.30) and the Lemma 2.6.5, we can bound the previous expression by the sum of

$$
\varlimsup_{N\to\infty}\mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\langle\pi^N_t,H\rangle-\langle\pi^N_0,H\rangle-\int_0^t\langle\pi^N_s,N^2\mathbb{L}_NH\rangle\,ds\Big|>\delta/8\Big]\,, \tag{2.59}
$$

$$
\varlimsup_{N\to\infty}\mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t\sum_{x\notin\Gamma_N}\Big(N^2\mathbb{L}_NH\big(\tfrac{x}{N}\big)-\Delta H\big(\tfrac{x}{N}\big)\Big)\eta_s(x)\,ds\Big|>\delta/8\Big]\,, \tag{2.60}
$$

$$
\varlimsup_{N\to\infty}\mathbb{P}^{\beta}_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\frac{1}{N^{d-1}}\int_0^t\sum_{x\in\Gamma_N}N\mathbb{L}_NH\big(\tfrac{x}{N}\big)(\eta_s(x)-\eta^{\varepsilon N}_s(x))\,ds\Big|>\delta/8\Big] \tag{2.61}
$$

and

$$\varlimsup_{N\to\infty} \mathbb{P}^\beta_{\mu_N}\Big[\sup_{0\le t\le T}\Big|\int_0^t \sum_{x\in\Gamma_N} N\mathbb{L}_N H\Big(\frac{x}{N}\Big)\eta_s^{\varepsilon N}(x)\,ds$$
$$+\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,-}^{j,\text{right}}}\eta_s^{\varepsilon N}(x)\partial_{u_j}H(\mathbf{u}^-)\,ds$$
$$-\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,-}^{j,\text{left}}}\eta_s^{\varepsilon N}(x)\partial_{u_j}H(\mathbf{u}^-)\,ds$$
$$-\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,+}^{j,\text{right}}}\eta_s^{\varepsilon N}(x)\partial_{u_j}H(\mathbf{u}^+)\,ds \tag{2.62}$$
$$+\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,+}^{j,\text{left}}}\eta_s^{\varepsilon N}(x)\partial_{u_j}H(\mathbf{u}^+)\,ds$$
$$+\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,-}^{j}}\alpha\,\eta_s^{\varepsilon N}(x)(H(\mathbf{u}^+)-H(\mathbf{u}^-))\,ds$$
$$-\sum_{j=1}^d\int_0^t \frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,+}^{j}}\alpha\,\eta_s^{\varepsilon N}(x)(H(\mathbf{u}^+)-H(\mathbf{u}^-))\,ds+\mathbf{err}(N)\Big|>\delta/8\Big],$$

where err(N) is a error that goes in modulus to zero as $N\to\infty$. Proposition 2.6.1 tells us that (2.59) is null. The approximation of the continuous Laplacian by the discrete Laplacian assures that (2.60) is null. Since $N\mathbb{L}_N H$ is a sequence of uniformly bounded functions, Lemma 2.5.4 allows we conclude that (2.61) vanishes as $\varepsilon\searrow 0$. Finally, provided by formulas (2.49) and (2.50) and recalling the decomposition $\Gamma_N=\Gamma_{N,+}\cup\Gamma_{N,-}$, we can see that, except for the error term, all terms inside the supremum in (2.62) cancel. This concludes the proof. □

2.6.3 Characterization of limit points for $\beta\in(1,\infty]$.

Proposition 2.6.6. *Let* $H\in C^2(\mathbb{T}^d\backslash\partial\Lambda)$. *For all* $\delta>0$,

$$\mathbb{Q}^\beta_*\Big[\pi_\cdot:\sup_{0\le t\le T}\Big|\langle\rho_t,H\rangle-\langle\rho_0,H\rangle-\int_0^t\langle\rho_s,\Delta H\rangle\,ds$$
$$-\int_0^t\int_{\partial\Lambda}\rho_s(u^+)\sum_{j=1}^d\partial_{u_j}H(u^+)\langle\vec{\zeta},e_j\rangle\,dS(u)ds \tag{2.63}$$
$$+\int_0^t\int_{\partial\Lambda}\rho_s(u^-)\sum_{j=1}^d\partial_{u_j}H(u^-)\langle\vec{\zeta},e_j\rangle\,dS(u)ds\Big|>\delta\Big]\;=\;0.$$

Proof. The proof of this proposition is similar, in fact, simpler than the one of

Proposition 2.6.3. In this case,

$$
\begin{aligned}
&\frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,-}} N\mathbb{L}_N H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\\
&= \frac{1}{N^{d-1}}\sum_{j=1}^{d}\Bigg\{\sum_{x\in\Gamma_{N,-}^{j,\text{right}}}\Big[\alpha N^{1-\beta}\big(H(\mathbf{u}^+)-H(\mathbf{u}^-)\big)-\partial_{u_j}H(\mathbf{u}^-)\Big]\eta_s^{\varepsilon N}(x)\\
&\qquad+\sum_{x\in\Gamma_{N,-}^{j,\text{left}}}\Big[\partial_{u_j}H(\mathbf{u}^-)+\alpha N^{1-\beta}\big(H(\mathbf{u}^+)-H(\mathbf{u}^-)\big)\Big]\eta_s^{\varepsilon N}(x)\Bigg\}.
\end{aligned}
\tag{2.64}
$$

and

$$
\begin{aligned}
&\frac{1}{N^{d-1}}\sum_{x\in\Gamma_{N,+}} N\mathbb{L}_N H\big(\tfrac{x}{N}\big)\eta_s^{\varepsilon N}(x)\\
&= \frac{1}{N^{d-1}}\sum_{j=1}^{d}\Bigg\{\sum_{x\in\Gamma_{N,+}^{j,\text{right}}}\Big[\partial_{u_j}H(\mathbf{u}^+)+\alpha N^{1-\beta}\big(H(\mathbf{u}^-)-H(\mathbf{u}^+)\big)\Big]\eta_s^{\varepsilon N}(x)\\
&\qquad+\sum_{x\in\Gamma_{N,+}^{j,\text{left}}}\Big[\alpha N^{1-\beta}\big(H(\mathbf{u}^-)-H(\mathbf{u}^+)\big)-\partial_{u_j}H(\mathbf{u}^+)\Big]\eta_s^{\varepsilon N}(x)\Bigg\}.
\end{aligned}
\tag{2.65}
$$

Since $\beta \in (1,\infty]$, we conclude that all terms above involving α disappear in the limit as $N \to \infty$. Noting that there are no surface integrals in (2.63) involving α, it is a simple game to repeat the steps in the proof of Proposition 2.6.3 to finally conclude (2.63). □

2.7 Uniqueness of weak solutions

The hydrodynamic equation (2.4) is the classical heat equation, which does not need any consideration about uniqueness of weak solutions. Thus, we only need to guarantee that weak solutions of (2.6) and (2.7) are unique.

Let us trace the strategy for the proof of uniqueness, which works for both (2.6) and (2.7). Considering in each case $\beta = 1$ or $\beta \in (1,\infty]$ a suitable set of test functions, we can annul all surface integrals. Being more precise, consider the following definitions:

Definition 4. *Let $\mathfrak{D}^{\text{Rob}} \subset L^2(\mathbb{T}^d)$ be the set of functions $H : \mathbb{T}^d \to \mathbb{R}$ such that $H(u) = h_1(u)\mathbf{1}_\Lambda(u) + h_2(u)\mathbf{1}_\Lambda^{\complement}(u)$, where*

(i) $h_i \in C^2(\mathbb{T}^d)$ *for* $i = \{1,2\}$.

(ii) $\langle\nabla h_1(u),\vec{\zeta}(u)\rangle = \langle\nabla h_2(u),\vec{\zeta}(u)\rangle = \big(h_2(u)-h_1(u)\big)\sum_{j=1}^{d}|\langle\vec{\zeta}(u),e_j\rangle|$, $\forall u \in \partial\Lambda$.

Define the operator $\mathfrak{L}^{\text{Rob}} : \mathfrak{D}^{\text{Rob}} \to L^2(\mathbb{T}^d)$ *by*

$$\mathfrak{L}^{\text{Rob}} H(u) \;=\; \begin{cases} \Delta h_1(u), & \textit{if } \; u \in \Lambda\,, \\ \Delta h_2(u), & \textit{if } \; u \in \Lambda^{\complement}\,. \end{cases}$$

Definition 5. *Let* $\mathfrak{D}^{\text{Neu}} \subset L^2(\mathbb{T}^d)$ *be the set of functions* $H : \mathbb{T}^d \to \mathbb{R}$ *such that* $H(u) = h_1(u)\mathbf{1}_\Lambda(u) + h_2(u)\mathbf{1}^{\complement}_\Lambda(u)$*, where:*

(i) $h_i \in C^2(\mathbb{T}^d)$ *for* $i = \{1, 2\}$.

(ii) $\langle \nabla h_1(u), \vec{\zeta}(u)\rangle = \langle \nabla h_2(u), \vec{\zeta}(u)\rangle = 0$ *,* $\forall u \in \partial\Lambda$.

Define the operator $\mathfrak{L}^{\text{Neu}} : \mathfrak{D}^{\text{Neu}} \to L^2(\mathbb{T}^d)$ *by*

$$\mathfrak{L}^{\text{Neu}} H(u) \;=\; \begin{cases} \Delta h_1(u) & \textit{if } \; u \in \Lambda\,, \\ \Delta h_2(u) & \textit{if } \; u \in \Lambda^{\complement}\,. \end{cases}$$

It is straightforward to check that, if ρ is a weak solution of (2.6), then

$$\langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \mathfrak{L}^{\text{Rob}} H\rangle\, ds \;=\; 0\,, \quad \forall H \in \mathfrak{D}^{\text{Rob}}\,, \forall t \in [0, T]\,, \tag{2.66}$$

while, if ρ is a weak solution of (2.7), then

$$\langle \rho_t, H\rangle - \langle \rho_0, H\rangle - \int_0^t \langle \rho_s, \mathfrak{L}^{\text{Neu}} H\rangle\, ds \;=\; 0\,, \quad \forall H \in \mathfrak{D}^{\text{Neu}}\,, \forall t \in [0, T]\,. \tag{2.67}$$

In both cases, if an orthonormal basis of $L^2(\mathbb{T}^d)$ composed of eigenfunctions for the corresponding operator (associated to nonpositive eigenvalues) is available, this would easily lead to the proof of uniqueness, as we shall see later. However, this is not the case. So, to overcome this situation we extend the corresponding operator via a *Friedrichs extension* (see [26] on the subject) to achieve the desired orthonormal basis.

Let us briefly explain the notion of Friedrichs extension. Let X be a Hilbert space and denote by $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ its inner product and norm, respectively. Consider a linear, strongly monotone and symmetric operator $\mathcal{A} : \mathfrak{D} \subset X \to X$, where by *strongly monotone* we mean that there exists $c > 0$ such that

$$\langle \mathcal{A}H, H\rangle \;\geq\; c\|H\|^2\,, \quad \forall\, H \in \mathfrak{D}\,.$$

Denote by $\langle \cdot, \cdot\rangle_{\mathcal{E}(\mathcal{A})}$ the so-called *energetic* inner product on $\mathfrak{D}$ associated to $\mathcal{A}$, which is defined by

$$\langle F, G\rangle_{\mathcal{E}(\mathcal{A})} \;:=\; \langle F,\, \mathcal{A}G\rangle\,.$$

Let $\mathcal{H}_{\text{Fried}}$ be the set of all functions F in X for which there exists a sequence $\{F_n : n \geq 1\}$ in $\mathfrak{D}$ such that F_n converges to F in X and F_n is Cauchy for the inner product $\langle \cdot, \cdot\rangle_{\mathcal{E}(\mathcal{A})}$. A sequence $\{F_n : n \geq 1\}$ with these properties will be called an *admissible sequence* for F. For F, G in $\mathcal{H}_{\text{Fried}}$, let

$$\langle F, G\rangle_{\text{Fried}} \;:=\; \lim_{n\to\infty} \langle F_n, G_n\rangle_{\mathcal{E}(\mathcal{A})}\,, \tag{2.68}$$

where $\{F_n : n \geq 1\}$, $\{G_n : n \geq 1\}$ are admissible sequences for F and G, respectively. By [26, Proposition 5.3.3], the limit exists and does not depend on the admissible sequence chosen and, moreover, the space $\mathcal{H}_{\text{Fried}}$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{\text{Fried}}$ is a real Hilbert space, usually called the *energetic space* associated to $\mathcal{A}$.

The Friedrichs extension $\mathcal{A}_{\text{Fried}} : \mathfrak{D}_{\text{Fried}} \to X$ of the operator $\mathcal{A}$ is then defined as follows. Let $\mathfrak{D}_{\text{Fried}}$ be the set of vectors in $F \in \mathcal{H}_{\text{Fried}}$ for which there exists a vector $f \in X$ such that

$$\langle F, G \rangle_{\text{Fried}} \;=\; \langle f, G \rangle \,, \quad \forall G \in \mathcal{H}_{\text{Fried}} \,.$$

and let $\mathcal{A}_{\text{Fried}} F = f$. See the excellent book [26] for why this operator $\mathcal{A}_{\text{Fried}} : \mathfrak{D}_{\text{Fried}} \to X$ is indeed an extension of $\mathcal{A} : \mathfrak{D} \to X$ and more details on the construction. The main result about Friedrichs extensions and eigenfunctions we cite here is the next one.

Theorem 2.7.1 ([26], Theorem 5.5C)**.** *Let $\mathcal{A} : \mathfrak{D} \subseteq X \to X$ be a linear, symmetric and strongly monotone operator and let $\mathcal{A}_{\text{Fried}} : \mathfrak{D}_{\text{Fried}} \subseteq X \to X$ be its Friedrichs extension. Assume additionally that the embedding $\mathcal{H}_{\text{Fried}} \hookrightarrow X$ is compact. Then,*

- *(a) The eigenvalues of $-\mathcal{A}_{\text{Fried}}$ form a countable set $0 < c \leq \mu_1 \leq \mu_2 \leq \cdots$ with $\lim_{n\to\infty} \mu_n = \infty$, and all these eigenvalues have finite multiplicity.*
- *(b) There exists a complete orthonormal basis of X composed of eigenvectors of $\mathcal{A}_{\text{Fried}}$.*

Denote by $\mathbb{I}$ the identity operator. If $\mathfrak{L} : \mathfrak{D} \subseteq X \to X$ is a symmetric nonpositive operator, then $\mathbb{I} - \mathfrak{L} : \mathfrak{D} \to X$ is symmetric and strongly monotone with $c = 1$. In fact,

$$\langle (\mathbb{I} - \mathfrak{L})H, H \rangle \;=\; \|H\|^2 + \langle -\mathfrak{L}H, H \rangle \;\geq\; \|H\|^2 \,, \quad \forall\, H \in \mathfrak{D} \,.$$

Therefore, under the hypothesis that $\mathfrak{L} : \mathfrak{D} \subseteq X \to X$ is a symmetric and nonpositive linear operator, we may consider the Friedrichs extension of $(\mathbb{I} - \mathfrak{L})$.

Proposition 2.7.2. *Let $\mathfrak{L} : \mathfrak{D} \subseteq X \to X$ be a symmetric nonpositive operator. Denote by $(\mathbb{I} - \mathfrak{L})_{\text{Fried}} : \mathfrak{D}_{\text{Fried}} \to X$ the Friedrichs extension of $(\mathbb{I} - \mathfrak{L}) : \mathfrak{D} \to X$ and by $\mathcal{H}_{\text{Fried}}$ the corresponding energetic space. Assume that the embedding $\mathcal{H}_{\text{Fried}} \hookrightarrow X$ is compact. Then, there exists at most one measurable function $\rho : [0,T] \to X$ such that*

$$\sup_{t\in[0,T]} \|\rho_t\| \;<\; \infty \tag{2.69}$$

and

$$\langle \rho_t, H \rangle - \langle \rho_0, H \rangle - \int_0^t \langle \rho_s, \mathfrak{L}H \rangle \, ds \;=\; 0 \,, \quad \forall H \in \mathfrak{D} \,, \forall t \in [0,T] \,.$$

where ρ_0 is a fixed element of X.

Proof. Consider ρ^1, ρ^2 two solutions of above and write $\rho = \rho^1 - \rho^2$. By linearity,

$$\langle \rho_t, H \rangle - \int_0^t \langle \rho_s, \mathfrak{L}H \rangle \, ds \;=\; 0 \,, \quad \forall H \in \mathfrak{D} \,, \forall t \in [0,T] \,.$$

which is the same as

$$\langle \rho_t, H\rangle + \int_0^t \langle \rho_s, (\mathbb{I}-\mathfrak{L})H\rangle\, ds - \int_0^t \langle \rho_s, H\rangle\, ds \;=\; 0\,, \quad \forall H\in\mathfrak{D}\,, \forall t\in[0,T]\,.$$

Since $\mathfrak{D}_{\textbf{Fried}} \subseteq \mathcal{H}_{\textbf{Fried}}$, the last equation can be extended to

$$\langle \rho_t, H\rangle + \int_0^t \langle \rho_s, (\mathbb{I}-\mathfrak{L})_{\textbf{Fried}}H\rangle\, ds - \int_0^t \langle \rho_s, H\rangle\, ds \;=\; 0\,, \quad \forall H\in\mathfrak{D}_{\textbf{Fried}}\,, \forall t\in[0,T]\,. \tag{2.70}$$

By Theorem 2.7.1, the Friedrichs extension $(\mathbb{I}-\mathfrak{L})_{\textbf{Fried}} : \mathfrak{D}_{\textbf{Fried}} \to X$ has eigenvalues $1\leq \lambda_1 \leq \lambda_2 \leq \cdots$, all of them having finite multiplicity with $\lim_{n\to\infty}\lambda_n = \infty$, and there exists a complete orthonormal basis $\{\Psi_j\}_{i\in\mathbb{N}}$ of $L^2(\mathbb{T}^d)$ composed of eigenfunctions. Denote

$$\mathfrak{L}_{\textbf{Fried}} \;:=\; \mathbb{I} - (\mathbb{I}-\mathfrak{L})_{\textbf{Fried}}\,.$$

Thus, $\{\Psi_j\}_{j\in\mathbb{N}}$ is also a set of eigenfunctions for the operator $\mathfrak{L}_{\textbf{Fried}}$ whose eigenvalues are given by $\mu_j = 1-\lambda_j \leq 0$. Define

$$R(t) \;=\; \sum_{j=1}^{\infty} \frac{1}{j^2(1-\mu_j)}\langle \rho_t, \Psi_j\rangle^2 \quad \text{for } t\in[0,T]\,.$$

Since ρ satisfy (2.70), we have that

$$\frac{d}{dt}\langle \rho_t, \Psi_j\rangle^2 \;=\; 2\langle \rho_t, \Psi_j\rangle\langle \rho_t, \mathfrak{L}_{\textbf{Fried}}\Psi_j\rangle \;=\; 2\mu_j\langle \rho_t, \Psi_j\rangle^2\,. \tag{2.71}$$

By (2.69) and the Cauchy-Schwarz inequality, we have that

$$\sum_{j=1}^{\infty} \frac{2|\mu_j|}{j^2(1-\mu_j)}\langle \rho_t, \Psi_j\rangle^2 \;\leq\; \sum_{j=1}^{\infty} \frac{2|\mu_j|}{j^2(1-\mu_j)}\Big(\sup_{t\in[0,T]}\|\rho_t\|^2\Big) \;<\; \infty\,,$$

which together with (2.71) implies that

$$\frac{d}{dt}R(t) \;=\; \sum_{j=1}^{\infty} \frac{2\mu_j}{j^2(1-\mu_j)}\langle \rho_t, \Psi_j\rangle^2 \;\leq\; 0\,.$$

Since $R(t)\geq 0$, $R(0)=0$, and $dR/dt\leq 0$, we conclude that $R(t)=0$ for all $t\in[0,T]$ and hence $\langle \rho_t, \Psi_j\rangle^2 = 0$ for any $t\in[0,T]$. Due to $\{\Psi_j\}_{j\in\mathbb{N}}$ be a complete orthonormal basis of X, we deduce that $\rho\equiv 0$, finishing the proof. □

In view of (2.66) and (2.67), considering X as the Hilbert space $L^2(\mathbb{T}^d)$ and applying the last proposition, to achieve the uniqueness of weak solutions of (2.6) and (2.7) it is enough to assure that

1. The operators $\mathbb{I}-\mathfrak{L}^{\text{Rob}} : \mathfrak{D}^{\text{Rob}} \subseteq L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ and $\mathbb{I}-\mathfrak{L}^{\text{Neu}} : \mathfrak{D}^{\text{Neu}} \subseteq L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ are symmetric nonpositive linear operators.

2. Denoting by $\mathcal{H}^{\text{Rob}}_{\text{Fried}}$ and $\mathcal{H}^{\text{Neu}}_{\text{Fried}}$ their respective energetic spaces, the embeddings $\mathcal{H}^{\text{Rob}}_{\text{Fried}} \hookrightarrow L^2(\mathbb{T}^d)$ and $\mathcal{H}^{\text{Neu}}_{\text{Fried}} \hookrightarrow L^2(\mathbb{T}^d)$ are compact.

This is precisely what we are going to do in the next four propositions. Denote by $\overleftarrow{\zeta}(u) = -\vec{\zeta}(u)$ the normal exterior vector to the region $\Lambda^{\complement}$ at $u \in \partial\Lambda$. Recall that $\langle\cdot,\cdot\rangle$ is used for both the inner products in $L^2(\mathbb{T}^d)$ and in $\mathbb{R}^d$.

Proposition 2.7.3. *The operator $-\mathfrak{L}^{\text{Rob}} : \mathfrak{D}^{\text{Rob}} \to L^2(\mathbb{T}^d)$ is symmetric and nonnegative.*

Proof. Let $H, G \in \mathfrak{D}^{\text{Rob}}$. We can write $H = h_1\mathbf{1}_\Lambda + h_2\mathbf{1}_\Lambda^{\complement}$ and $G = g_1\mathbf{1}_\Lambda + g_2\mathbf{1}_\Lambda^{\complement}$, where $h_1, h_2, g_1, g_2 \in C^2(\mathbb{T}^d)$. By the third Green identity (see Appendix 2.8, Theorem 2.8.2),

$$\int_{\mathbb{T}^d} \big(h\Delta g - g\Delta h\big)\, du \;=\; \int_{\partial\Lambda} \Big(h\langle\nabla g, \vec{\zeta}\rangle - g\langle\nabla h, \vec{\zeta}\rangle\Big)\, dS\,,$$

where dS is an infinitesimal volume element of $\partial\Lambda$. Thus,

$$\begin{aligned}
\langle H, -\mathfrak{L}^{\text{Rob}} G\rangle =& \langle h_1\mathbf{1}_\Lambda + h_2\mathbf{1}_{\Lambda^{\complement}}, -\Delta g_1\mathbf{1}_\Lambda - \Delta g_2\mathbf{1}_{\Lambda^{\complement}}\rangle \\
=& -\int_\Lambda h_1\Delta g_1\, du - \int_{\Lambda^{\complement}} h_2\Delta g_2\, du \\
=& -\int_\Lambda g_1\Delta h_1\, du - \int_{\partial\Lambda} \Big(h_1\langle\nabla g_1, \vec{\zeta}\rangle - g_1\langle\nabla h_1, \vec{\zeta}\rangle\Big)\, dS \\
& -\int_{\Lambda^{\complement}} g_2\Delta h_2\, du - \int_{\partial\Lambda^{\complement}} \Big(h_2\langle\nabla g_2, \overleftarrow{\zeta}\rangle - g_2\langle\nabla h_2, \overleftarrow{\zeta}\rangle\Big)\, dS \\
=& -\int_\Lambda g_1\Delta h_1\, du - \int_{\partial\Lambda} \Big(h_1\langle\nabla g_1, \vec{\zeta}\rangle - g_1\langle\nabla h_1, \vec{\zeta}\rangle\Big)\, dS \\
& -\int_{\Lambda^{\complement}} g_2\Delta h_2\, du - \int_{\partial\Lambda^{\complement}} \Big(g_2\langle\nabla h_2, \vec{\zeta}\rangle - h_2\langle\nabla g_2, \vec{\zeta}\rangle\Big)\, dS\,.
\end{aligned}$$

Using the boundary condition in the item (ii) of Definition 4 and $\partial\Lambda^{\complement} = \partial\Lambda$, we conclude that the last expression above is equal to

$$\begin{aligned}
& -\int_\Lambda g_1\Delta h_1\, du - \int_{\Lambda^{\complement}} g_2\Delta h_2\, du \\
& -\int_{\partial\Lambda}\Big((h_1-h_2)\sum_{j=1}^d |\langle\vec{\zeta}, e_j\rangle|(g_2-g_1) - (g_1-g_2)\sum_{j=1}^d |\langle\vec{\zeta}, e_j\rangle|(h_2-h_1)\Big)dS \\
=& -\int_\Lambda g_1\Delta h_1\, du - \int_{\Lambda^{\complement}} g_2\Delta h_2\, du\,.
\end{aligned}$$

Then, $\langle H, -\mathfrak{L}^{\text{Rob}} G\rangle = -\int_\Lambda g_1\Delta h_1\, du - \int_{\Lambda^{\complement}} g_2\Delta g_2\, du = \langle -\mathfrak{L}^{\text{Rob}} H, G\rangle$. For the nonnegativeness, note that

$$\begin{aligned}
\langle H, -\mathfrak{L}^{\text{Rob}} H\rangle \;=\; & -\int_\Lambda h_1\Delta h_1\, du - \int_{\Lambda^{\complement}} h_2\Delta h_2\, du \\
=& \int_\Lambda |\nabla h_1|^2\, du + \int_{\Lambda^{\complement}} |\nabla h_2|^2\, du - \int_{\partial\Lambda} \Big(\langle\nabla h_1, \vec{\zeta}\rangle h_1 + \langle\nabla h_2, \vec{\zeta}\rangle h_2\Big)\, dS
\end{aligned}$$

where the second equality above holds by the second Green identity, see Appendix, Theorem 2.8.2, and $\partial(\Lambda^{\complement}) = \partial\Lambda$. Since $\int_\Lambda |\nabla h_i|^2\, du \geq 0$, for $i = 1, 2$, it is enough to check that $-\int_{\partial\Lambda}\Big(\langle\nabla h_1,\vec{\zeta}\rangle h_1+\langle\nabla h_2,\vec{\zeta}\rangle h_2\Big)\,dS\;\geq\;0$. In fact,

$$
\begin{aligned}
&-\int_{\partial\Lambda}\Big(\langle\nabla h_1,\vec{\zeta}\rangle h_1+\langle\nabla h_2,\vec{\zeta}\,\rangle h_2\Big)\,dS\;=\;-\int_{\partial\Lambda}\Big(\langle\nabla h_1,\vec{\zeta}\rangle h_1-\langle\nabla h_2,\vec{\zeta}\rangle h_2\Big)\,dS\\
&=\int_{\partial\Lambda}\sum_{j=1}^{d}|\langle\vec{\zeta},e_j\rangle|\Big((h_2-h_1)h_2-(h_2-h_1)h_1\Big)dS\\
&=2\int_{\partial\Lambda}\sum_{j=1}^{d}|\langle\vec{\zeta},e_j\rangle|(h_2-h_1)^2\,dS\;\geq\;0\,,
\end{aligned}
$$

where the second equality holds by item (ii) of Definition 4. □

Proposition 2.7.4. *The embedding $\mathcal{H}^{\mathrm{Rob}}_{\mathrm{Fried}}\hookrightarrow L^2(\mathbb{T}^d)$ is compact.*

Proof. Let $\{H_n\}$ be a bounded sequence in $\mathcal{H}^{\mathrm{Rob}}_{\mathrm{Fried}}$. Fix $\{F_n\}$ a sequence in $\mathfrak{D}^{\mathrm{Rob}}$ such that $\|F_n-H_n\|\to 0$ when $n\to\infty$ and $\{F_n\}$ is also bounded in $\mathcal{H}^{\mathrm{Rob}}_{\mathrm{Fried}}$. Thus, to show the compact embedding we need prove that $\{H_n\}$ have a convergent subsequence in $L^2(\mathbb{T}^d)$. To get a convergent subsequence of $\{H_n\}$, it is sufficient to find a convergent subsequence of $\{F_n\}$ in $L^2(\mathbb{T}^d)$. Write $F_n=f_n\mathbf{1}_\Lambda+\tilde{f}_n\mathbf{1}_{\Lambda^\complement}$, with $f_n,\tilde{f}_n\in C^2(\mathbb{T}^d)$. Then,

$$
\begin{aligned}
&\langle F_n,F_n\rangle_{\mathcal{E}(\mathbb{I}-\mathfrak{L}^{\mathrm{Rob}})}=\langle F_n,F_n\rangle+\langle F_n,-\mathfrak{L}^{\mathrm{Rob}}F_n\rangle\\
&=\langle f_n\mathbf{1}_\Lambda+\tilde{f}_n\mathbf{1}_{\Lambda^\complement},f_n\mathbf{1}_\Lambda+\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\rangle+\langle f_n\mathbf{1}_\Lambda+\tilde{f}_n\mathbf{1}_{\Lambda^\complement},-\Delta f_n\mathbf{1}_\Lambda-\Delta\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\rangle\,.
\end{aligned}
$$

Expanding the right hand side of above and using Green identity (see Appendix 2.8, Theorem 2.8.2), we get that

$$
\begin{aligned}
&\int_\Lambda f_n^2\,du+\int_{\Lambda^\complement}\tilde{f}_n^{\,2}\,du-\int_\Lambda f_n\Delta f_n\,du-\int_{\Lambda^\complement}\tilde{f}_n\Delta\tilde{f}_n\,du\\
&=\|f_n\mathbf{1}_\Lambda\|^2+\|\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\|^2+\|\nabla f_n\mathbf{1}_\Lambda\|^2+\|\nabla\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\|^2\\
&+2\int_{\partial\Lambda}\sum_{j=1}^{d}|\langle\vec{\zeta},e_j\rangle|(f_n-\tilde{f}_n)^2dS\,.
\end{aligned}
$$

Under the hypotheses of boundedness of the sequence $\{F_n\}$ in the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{E}(\mathbb{I}-\mathfrak{L}^{\mathrm{Rob}})}$, the sequences $\{\|f_n\mathbf{1}_\Lambda\|^2\}$, $\{\|\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\|^2\}$, $\{\|\nabla f_n\mathbf{1}_\Lambda\|^2\}$ and $\{\|\nabla\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\|^2\}$ are bounded. By the Rellich-Kondrachov compactness theorem (see [5, Theorem 5.7.1]), $\{f_n\mathbf{1}_\Lambda\}$, $\{\tilde{f}_n\mathbf{1}_{\Lambda^\complement}\}$ have a common convergent subsequence in $L^2(\mathbb{T}^d)$. This implies that $\{F_n\}$ has a convergent subsequence. □

Proposition 2.7.5. *The operator $-\mathfrak{L}^{\mathrm{Neu}}:\mathfrak{D}^{\mathrm{Neu}}\to L^2(\mathbb{T}^d)$ is symmetric and nonnegative.*

Proof. Let $H,G\in\mathfrak{D}^{\mathrm{Neu}}$. We can write $H=h_1\mathbf{1}_\Lambda+h_2\mathbf{1}_\Lambda^\complement$ and $G=g_1\mathbf{1}_\Lambda+g_2\mathbf{1}_\Lambda^\complement$, where $h_1,h_2,g_1,g_2\in C^2(\mathbb{T}^d)$. By the third Green identity, see Appendix 2.8, Theorem 2.8.2, we have that

$$
\int_{\mathbb{T}^d}h\Delta g\,du-g\Delta h\,du\;=\;\int_{\partial\Lambda}h\langle\nabla g,\vec{\zeta}\,\rangle-g\langle\nabla h,\vec{\zeta}\,\rangle\,dS\;=\;0\,,
$$

where dS is the infinitesimal volume element of $\partial\Lambda$. Thus,

$$\begin{aligned}\langle H, -\mathfrak{L}^{\text{Neu}} G\rangle &= \langle h_1 \mathbf{1}_\Lambda + h_2 \mathbf{1}_{\Lambda^\complement}, -\Delta g_1 \mathbf{1}_\Lambda - \Delta g_2 \mathbf{1}_{\Lambda^\complement}\rangle \\ &= -\int_\Lambda h_1 \Delta g_1 du - \int_{\Lambda^\complement} h_2 \Delta g_2 du = -\int_\Lambda g_1 \Delta h_1 du - \int_{\Lambda^\complement} g_2 \Delta g_2 du = \langle -\mathfrak{L}^{\text{Neu}} H, G\rangle.\end{aligned}$$

For nonnegativeness,

$$\langle H, -\mathcal{L}_\Lambda H\rangle = -\int_\Lambda h_1 \Delta h_1 \, du - \int_{\Lambda^\complement} h_2 \Delta h_2 \, du = \int_\Lambda |\nabla h_1|^2 \, du + \int_{\Lambda^\complement} |\nabla h_2|^2 \, du \geq 0,$$

where the second equality above holds due to the second Green identity, see Appendix 2.8, Theorem 2.8.2. □

Lemma 2.7.6. *The embedding $\mathcal{H}^{\text{Neu}}_{\text{Fried}} \hookrightarrow L^2(\mathbb{T}^d)$ is compact.*

Proof. Let $\{H_n\}$ be a bounded sequence in $\mathcal{H}^{\text{Neu}}$. Fix a sequence $\{F_n\}$ of functions in $\mathfrak{D}^{\text{Neu}}$ such that $\|F_n - H_n\| \to 0$ when $n \to \infty$ and $\{F_n\}$ is also bounded in $\mathcal{H}^{\text{Neu}}_{\text{Fried}}$. Thus, to show the compact embedding we need to prove that $\{H_n\}$ has a convergent subsequence in $L^2(\mathbb{T}^d)$. To get a convergent subsequence of $\{H_n\}$, it is sufficient to find a convergent subsequence of $\{F_n\}$ in $L^2(\mathbb{T}^d)$. Write $F_n = f_n \mathbf{1}_\Lambda + \tilde{f}_n \mathbf{1}_{\Lambda^\complement}$, with $f_n \in C^2(\mathbb{T}^d)$. Then,

$$\begin{aligned}\langle F_n, F_n\rangle_{\mathcal{E}(\mathbb{I}-\mathfrak{L}^{\text{Neu}})} &= \langle F_n, F_n\rangle + \langle F_n, -\mathfrak{L}^{\text{Neu}} F_n\rangle \\ &= \langle f_n \mathbf{1}_\Lambda + \tilde{f}_n \mathbf{1}_{\Lambda^\complement}, f_n \mathbf{1}_\Lambda + \tilde{f}_n \mathbf{1}_{\Lambda^\complement}\rangle + \langle f_n \mathbf{1}_\Lambda + \tilde{f}_n \mathbf{1}_{\Lambda^\complement}, -\Delta f_n \mathbf{1}_\Lambda - \Delta \tilde{f}_n \mathbf{1}_{\Lambda^\complement}\rangle.\end{aligned}$$

Expanding the right hand side and using Green identity, see Appendix 2.8, Theorem 2.8.2, we get that

$$\begin{aligned}&\int_\Lambda f_n^2 \, du + \int_{\Lambda^\complement} \tilde{f}_n^{\,2} \, du - \int_\Lambda f_n \Delta f_n \, du - \int_{\Lambda^\complement} \tilde{f}_n^{\,2} \Delta \tilde{f}_n \, du \\ &= \|f_n \mathbf{1}_\Lambda\|^2 + \|\tilde{f}_n \mathbf{1}_{\Lambda^\complement}\|^2 + \|\nabla f_n \mathbf{1}_\Lambda\|^2 + \|\nabla \tilde{f}_n \mathbf{1}_{\Lambda^\complement}\|^2 .\end{aligned}$$

Under the hypotheses of boundedness of the sequence $\{F_n\}$ in the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{E}(\mathbb{I}-\mathfrak{L}^{\text{Neu}})}$, the sequences $\{\|f_n \mathbf{1}_\Lambda\|^2\}$, $\{\|\tilde{f}_n \mathbf{1}_{\Lambda^\complement}\|^2\}$, $\{\|\nabla f_n \mathbf{1}_\Lambda\|^2\}$ and $\{\|\nabla \tilde{f}_n \mathbf{1}_{\Lambda^\complement}\|^2\}$ are bounded. By the Rellich-Kondrachov Compactness Theorem, $\{f_n \mathbf{1}_\Lambda\}$, $\{\tilde{f}_n \mathbf{1}_{\Lambda^\complement}\}$ have a common convergent subsequence in $L^2(\mathbb{T}^d)$. This implies that $\{F_n\}$ has a convergent subsequence. □

2.8 Auxiliary results

Proposition 2.8.1 ([7]). *Let G_1, G_2, G_3 are continuous functions defined on the torus d-dimensional $\mathbb{T}^d$. Then, the application from $D([0,T],\mathcal{M})$ to $\mathbb{R}$ that associates to a trajectory $\{\pi_t : 0 \leq t \leq T\}$ the number*

$$\sup_{0\leq t\leq T} \left| \langle \pi_t, G_1\rangle - \langle \pi_0, G_2\rangle - \int_0^t \langle \pi_s, G_3\rangle \, ds \right|$$

is continuous in the Skorohod metric of $D([0,T],\mathcal{M})$.

Theorem 2.8.2 (Green's formulas, see for instance Appendix C of [5])**.** *Let $u, v \in C^2(\bar{U})$, where U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Denote by $\cdot$ the inner product in $\mathbb{R}^n$, and by ν the normal exterior unitary vector to U at ∂U. Then,*

(i) $\int_U \Delta u dx \ = \ \int_{\partial U} \frac{\partial u}{\partial \nu}\, dS$,

(ii) $\int_U \nabla v \cdot \nabla u\, dx \ = \ -\int_U u\Delta v\, dx + \int_{\partial U} \frac{\partial u}{\partial \nu} u\, dS$,

(iii) $\int_U u\Delta v - v\Delta u\, dx \ = \ \int_{\partial U} u\frac{\partial u}{\partial \nu} - v\frac{\partial u}{\partial \nu}\, dS$.

Chapter 3

Non-equilibrium Fluctuations for the SSEP with a slow bond

3.1 Introduction

One of the most challenging problems in the field of interacting particle systems is the derivation of the non-equilibrium fluctuations around the hydrodynamic limit and up to now there is not a satisfactory and robust theory that one can apply successfully. The main difficulty that one faces is to understand the precise asymptotic behavior of the long range correlations of the system. To be more precise, when letting the interacting system start from a general measure (typically a non invariant measure for which the hydrodynamic limit can be obtained), the correlations between any two sites are not null, but decay to zero as the scaling parameter n grows.

In many situations a uniform bound on the correlation function of order $O(1/n)$ is sufficient to obtain the non-equilibrium fluctuations of the system (see [4, 22] for instance). For the model that we are going to describe in the sequel, the uniform bound on the correlation function happens to be of order $O(\log n/n)$, demanding new efforts both on the derivation of such a bound and on the application of such a bound on the proof of the non-equilibrium fluctuations.

To be more specific, here we study the symmetric simple exclusion process (SSEP) evolving on $\mathbb{Z}$ when a slow bond is added to it. The dynamics of this model is defined as follows. On $\mathbb{Z}$, particles at the vertexes of the bond $\{x, x+1\}$ exchange positions at rate 1, except at the particular bond $\{0, 1\}$, where the rate of exchange is given by α/n, with $\alpha \in (0, +\infty)$. Since the rate at the bond $\{0, 1\}$ is slower with respect to the rates at other bonds, the bond $\{0, 1\}$ coined the name *slow bond*. Particles move on the one-dimensional lattice according to those rates of exchange and they are not created nor annihilated, being the spatial disposition of particles the object of interest.

The investigation on the behaviour of this process was initiated in [7] where the hydrodynamic limit was derived (see also [12, 9]). By this we mean that the density of particles of the system converges to a function $\rho_t(\cdot)$ which is a weak solution to a partial differential equation, called the *hydrodynamic equation*. For the choice of the rates given above, the corresponding hydrodynamic equation is the one-dimensional heat equation with a boundary condition of Robin type:

$$\begin{cases} \partial_t \rho(t,u) \;=\; \partial_{uu}^2 \rho(t,u)\,, \quad \text{for } u \neq 0, \\ \partial_u \rho(t,0^+) \;=\; \partial_u \rho(t,0^-) \;=\; \alpha\big[\rho(t,0^+) - \rho(t,0^-)\big], \\ \rho(0,u) \;=\; \rho_0(u), \end{cases} \tag{3.1}$$

where 0^+ and 0^- denote the side limits at zero from the right and from the left, respectively.

In fact, in [7] a more general choice for the rates was considered, and three different hydrodynamical behaviours were obtained. There, the slow bond was taken as the bond $\{-1,0\}$ instead of $\{0,1\}$, and the rate of exchange at that bond was given by $\frac{\alpha}{n^\beta}$, with $\beta \geq 0$ and α as given above. The choice of the slow bond as $\{0,1\}$ or $\{-1,0\}$ is essentially a matter of notation, having no special relevance. On the other hand, depending on the range of β, the boundary conditions of the hydrodynamic equation can be of Neumann type (when $\beta > 1$), which corresponds to (3.1) with $\alpha = 0$; or there is an absence of boundary conditions (when $\beta \in [0,1)$). The model we approach here corresponds to the choice $\beta = 1$ in [7].

The effect of the slow bond at a microscopic level is obvious: it narrows down the passage of particles across it. At a macroscopic level, its presence leads to boundary conditions in the partial differential equation. By looking at the hydrodynamic equation (3.1), we see that the boundary conditions characterize the current of the system through the macroscopic position $u = 0$. The boundary conditions state that the current is proportional to the difference of concentration of the intervals $(0,+\infty)$ and $(-\infty,0)$ near the boundary, which is in agreement with *Fick's Law*.

The *equilibrium fluctuations* for this model were presented in [8] and three different Ornstein-Uhlenbeck processes were obtained, which again had the corresponding boundary conditions as seen at the hydrodynamical level. We extend here the results of [8] by allowing the system to start from any measure and not necessarily from the stationary measure, namely the Bernoulli product measure, as required in [8]. The choice of rates as described above is restricted to $\beta = 1$ so that we are in the Robin's regime.

As the main theorem, we prove the non-equilibrium fluctuations and show that they are given by an *Ornstein-Uhlenbeck process with Robin boundary conditions*. By an Ornstein-Uhlenbeck process with Robin boundary conditions it should be understood, in the same spirit as in [8], that these boundary conditions are encoded in the space of test functions, see (3.7) below. Microscopically, the role of the boundary conditions at the level of the test functions is to force some additive functionals that appear in the Dynkin martingale to vanish as n grows. If we do not impose the boundary conditions of (3.7) on the test functions, then we would need some extra arguments to control those additive functionals. This is left to a future work.

The proof's structure is the standard one in the theory of stochastic processes: tightness for the sequence of density fields together with uniqueness of limit points. Let us discuss next the features of the work, besides the non-equilibrium result itself. And at same time we give the outline of the paper.

The biggest difficulty we face in our proof is undoubtedly the fact that the slow bond decreases the speed at which correlations vanish. In the usual SSEP, where all bonds have rate one, correlations are of order $O(1/n)$. In our case however, correlations are of order $O(\log n/n)$, therefore bigger than in the usual SSEP. For sites on the same side of the slow bond this fact is intuitive: correlations should actually increase since it is more difficult for particles to cross the slow bond. Curiously, our proof shows that the same happens for sites at different sides of the slow bond, that is, correlations are of order $O(\log n/n)$ on the entire line. An intuition of why this happens is given in Remark 3.4.3, and a discussion of why the bound $O(\log n/n)$ is sharp is made in Subsection 3.4.3.

In Section 3.2 we define the symmetric simple exclusion process in the presence of a slow bond at $\{0, 1\}$, we introduce notations and we state the main results of the article. At the end of this section, three related open problems are presented.

In Section 3.4 we establish connections between the two-point correlation function and the discrete derivative with the expected occupation time of a site of two-dimensional and one-dimensional random walks, respectively, in an inhomogeneous medium. This is one of the features: the way itself to estimate correlations via local times of random walks, which we believe may be applied to different contexts. The idea behind that is actually simple. We express both the discrete derivative and the correlation function as solutions to some discrete equations, then we use Duhamel's Principle to write each one of these solutions in terms of transition probabilities of random walks, in 1-d when looking at the discrete derivative and in 2-d when looking at the correlation function. Then, the local times of these random walks show up naturally from these arguments and we need to establish optimal bounds for them.

Since the necessary estimates for local times of random walks were not yet available in the literature, we derive them in Section 3.3 by means of projection of Markov chains (also known as *lumping*) and couplings. The statements of those estimates may look artificial at first glance, but they naturally appear when one looks for estimates on the discrete derivative of the occupation average at a site and for the two-point correlation function, as aforementioned.

An additional feature is about uniqueness of the Ornstein-Uhlenbeck process with Robin boundary conditions in the non-equilibrium setting, where the variance is governed by the PDE (3.1). Suitably adapting the proofs of [14, 18], we give a slightly more general version of uniqueness, which permits to consider more general starting measures than the usual slowly varying product measure. The generalization here consists on supposing that the density field associated to the initial measure does not necessarily converge to a *Gaussian field*, but only to *some field*. Moreover, this proof of uniqueness has a pedagogical importance, since the original proof of uniqueness for the Ornstein-Uhlenbeck process in the non-equilibrium setting, to the best of our knowledge, is not available in the literature.

Finally, in Section 3.5 we present the proof of the density fluctuations, which relies on the estimates of the discrete derivative of expected occupation number at a site, and on the two-point correlation function. A small but important detail is the fact that the estimate on the discrete derivative is sufficient for our purposes. In previous works ([4, 22]), the proof of non-equilibrium fluctuations was based on the convergence of the spatially discretized heat equation towards the continuum heat equation. Such an approximation is quite good, of order $O(n^{-2})$, and quite hard to adapt to the non-homogeneous medium set up without some uniform ellipticity assumption as in [17]. On the other hand, the discrete derivative estimate for the spatially discretized PDE is much easier to reach, as seen here. This idea on making use of the discrete derivative first appeared in [11], but its utility becomes more evident now.

3.2 Statement of results

3.2.1 The model

We fix a parameter $\alpha > 0$, and we consider the symmetric simple exclusion process $\{\eta_t : t \geq 0\}$ with a slow bound as defined in [7]. More precisely, $\{\eta_t : t \geq 0\}$ is the Markov process with state space $\Omega \stackrel{\text{def}}{=} \{0,1\}^{\mathbb{Z}}$, and infinitesimal generator $\mathcal{L}_n$ acting on local functions $f : \Omega \to \mathbb{R}$ via

$$(\mathcal{L}_n f)(\eta) \;=\; \sum_{x\in\mathbb{Z}} \xi^n_{x,x+1}\Big(f(\eta^{x,x+1}) - f(\eta)\Big) \tag{3.2}$$

where

$$\xi^n_{x,x+1} \;\stackrel{\text{def}}{=}\; \begin{cases} 1\,, & \text{if } x \neq 0\,,\\ \frac{\alpha}{n}\,, & \text{if } x = 0\,. \end{cases} \tag{3.3}$$

Here, for any $x \in \mathbb{Z}$, the configuration $\eta^{x,x+1}$ is obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, i.e.,

$$(\eta^{x,x+1})(y) \;=\; \begin{cases} \eta(x+1)\,, & \text{if } y = x\,,\\ \eta(x)\,, & \text{if } y = x+1\,,\\ \eta(y)\,, & \text{otherwise,} \end{cases}$$

see Figure 3.1 for an illustration of the jump rates. Given $\eta \in \{0,1\}^{\mathbb{Z}}$, we then say that the site $x \in \mathbb{Z}$ is vacant if $\eta(x) = 0$ and occupied if $\eta(x) = 1$.

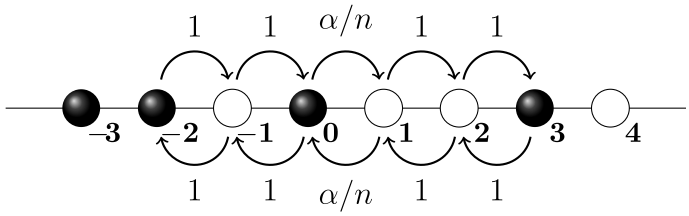


Figure 3.1: Jump rates. The bond $\{0,1\}$ has a particular jump rate associated to it, which is given by α/n.

3.2.2 Hydrodynamic limit

Fix a measurable density profile $\rho_0 : \mathbb{R} \to [0,1]$. For each $n \in \mathbb{N}$, let μ_n be a probability measure on Ω. We say that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is *associated* with the profile $\rho_0(\cdot)$ if, for any $\delta > 0$ and any continuous function of compact support $f : \mathbb{R} \to \mathbb{R}$, the following holds:

$$\lim_{n\to\infty} \mu_n \Big[\eta : \Big| \frac{1}{n} \sum_{x\in\mathbb{Z}} f(\tfrac{x}{n})\, \eta(x) - \int f(u)\, \rho_0(u)\, du \Big| > \delta \Big] \;=\; 0\,. \tag{3.4}$$

Fix $T > 0$, and let $\mathcal{D}([0,T],\Omega)$ be the space of trajectories which are right continuous, with left limits and taking values in Ω. Denote by $\mathbb{P}_{\mu_n}$ the probability measure on $\mathcal{D}([0,T],\Omega)$ induced by the SSEP with a slow bond accelerated by n^2, i.e., the Markov process with generator $n^2\mathcal{L}_n$, and initial measure μ_n. With a slight abuse of notation, we also use the notation $\{\eta_t : t \in [0,T]\}$ for the accelerated process. Denote by $\mathbb{E}_{\mu_n}$ the expectation with respect to $\mathbb{P}_{\mu_n}$. In [7, 9] the *hydrodynamical behaviour* was studied. We note that the process there was studied in *finite volume*, i.e., the model was considered on the discrete torus embedded into the continuous one-dimensional torus. However, since the extension to infinite volume is just a topological issue, the statement below can be obtained via an adaptation of the original approach:

Theorem 3.2.1 ([7, 9])**.** *Suppose that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is associated to the profile $\rho_0(\cdot)$ in the sense of* (3.4)*. Then, for each $t \in [0,T]$, for any $\delta > 0$ and any continuous function $f : \mathbb{R} \to \mathbb{R}$ with compact support,*

$$\lim_{n\to+\infty} \mathbb{P}_{\mu_n} \Big[\eta_\cdot : \Big| \frac{1}{n} \sum_{x\in\mathbb{Z}} f(\tfrac{x}{n})\, \eta_t(x) - \int_{\mathbb{R}} f(u)\, \rho(t,u)\, du \Big| > \delta \Big] \;=\; 0\,,$$

where $\rho(t,\cdot)$ is the unique weak solution of the heat equation with Robin boundary conditions given by

$$\begin{cases} \partial_t \rho(t,u) = \;\partial^2_{uu}\rho(t,u), & t \geq 0,\, u \in \mathbb{R}\backslash\{0\}, \\ \partial_u \rho(t,0^+) = \partial_u\rho(t,0^-) = \alpha\big[\rho(t,0^+) - \rho(t,0^-)\big], & t \geq 0, \\ \rho(0,u) = \;\rho_0(u), & u \in \mathbb{R}. \end{cases} \tag{3.5}$$

Here, $\rho(t,0^+)$ and $\rho(t,0^-)$ denote the limit from the right and from the left at zero, respectively. The notation $0^\pm$ will be used throughout the article.

3.2.3 Space of test functions and semigroup

In this section we introduce a space of test functions, that is suitable for our purposes, and which, basically, coincides with the one in [10]. Here, functions are continuous from the left at zero, while in [10] functions are continuous from the right. This subtle difference is due to choice of slow bond's position, which is $\{0,1\}$ here and $\{-1,0\}$ in [10].

Definition 6. *We denote by $\mathcal{S}_\alpha(\mathbb{R})$ the space of functions $f : \mathbb{R} \to \mathbb{R}$ such that:*

(i) f *is smooth on* $\mathbb{R}\backslash\{0\}$*, i.e.* $f \in C^\infty(\mathbb{R}\backslash\{0\})$*,*

(ii) f *is continuous from the left at 0,*

(iii) for all non-negative integers k, ℓ*, the function* f *satisfies*

$$\|f\|_{k,\ell} \;:=\; \sup_{u\neq 0}\Big|(1+|u|^\ell)\frac{d^k f}{du^k}(u)\Big| \;<\; \infty\,. \tag{3.6}$$

(iv) for any integer $k \geq 0$*,*

$$\frac{d^{2k+1}f}{du^{2k+1}}(0^+) \;=\; \frac{d^{2k+1}f}{du^{2k+1}}(0^-) \;=\; \alpha\left(\frac{d^{2k}f}{du^{2k}}(0^+) - \frac{d^{2k}f}{du^{2k}}(0^-)\right). \tag{3.7}$$

Moreover, $\mathcal{S}'_\alpha(\mathbb{R})$ *denotes the topological dual of* $\mathcal{S}_\alpha(\mathbb{R})$*.*

In plain words, $\mathcal{S}_\alpha(\mathbb{R})$ essentially consists of the space of functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ that are not necessarily smooth at the origin. It is a consequence of (3.6) that $\frac{d^k f}{du^k}(0^+)$ and $\frac{d^k f}{du^k}(0^-)$ exist for all integers $k \geq 0$. As in [8], one may show that $S_\alpha(\mathbb{R})$ is a Fréchet space (this fact was only used when showing tightness, see [21]). We recall below the explicit formula for the semigroup that corresponds to the PDE (3.5).

Proposition 3.2.2 ([8]). *Denote by* g_{even} *and* g_{odd} *the even and odd parts of a function* $g:\mathbb{R}\to\mathbb{R}$*, respectively. That is, for* $u\in\mathbb{R}$*,*

$$g_{\text{even}}(u) = \frac{g(u)+g(-u)}{2} \quad \textit{and} \quad g_{\text{odd}}(u) = \frac{g(u)-g(-u)}{2}\,.$$

The solution of (3.5) *with initial condition* $g \in \mathcal{S}_\alpha(\mathbb{R})$ *is given by*

$$\begin{aligned} T_t^\alpha g(u) = \frac{1}{\sqrt{4\pi t}}\Bigg\{ &\int_{\mathbb{R}} e^{-\frac{(u-y)^2}{4t}} g_{\text{even}}(y)\,dy \\ &+ e^{2\alpha u}\int_u^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[(\tfrac{z-y+4\alpha t}{2t})e^{-\frac{(z-y)^2}{4t}} + (\tfrac{z+y-4\alpha t}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]\, g_{\text{odd}}(y)\,dy\,dz\Bigg\}, \end{aligned}$$

for $u > 0$*, and*

$$\begin{aligned} T_t^\alpha g(u) = \frac{1}{\sqrt{4\pi t}}\Bigg\{ &\int_{\mathbb{R}} e^{-\frac{(u-y)^2}{4t}} g_{\text{even}}(y)\,dy \\ &- e^{-2\alpha u}\int_{-u}^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[(\tfrac{z-y+4\alpha t}{2t})e^{-\frac{(z-y)^2}{4t}} + (\tfrac{z+y-4\alpha t}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]\, g_{\text{odd}}(y)\,dy\,dz\Bigg\}, \end{aligned}$$

for $u < 0$*.*

The next proposition connects T_t^α with the space of test functions $\mathcal{S}_\alpha(\mathbb{R})$.

Proposition 3.2.3 ([10]). *The operator T_t^α defines a semigroup $T_t^\alpha : \mathcal{S}_\alpha(\mathbb{R}) \to \mathcal{S}_\alpha(\mathbb{R})$. That is, for any given $g \in \mathcal{S}_\alpha(\mathbb{R})$ and any time $t > 0$, the solution $T_t^\alpha g$ of the PDE* (3.5) *starting from g also belongs to $\mathcal{S}_\alpha(\mathbb{R})$.*

Definition 7. *Let $\Delta_\alpha : \mathcal{S}_\alpha(\mathbb{R}) \to \mathcal{S}_\alpha(\mathbb{R})$ be the Laplacian on $\mathcal{S}_\alpha(\mathbb{R})$, i.e., for any $f \in \mathcal{S}_\alpha(\mathbb{R})$,*

$$\Delta_\alpha f(u) \;=\; \begin{cases} \partial^2_{uu} f(u)\,, & \textit{if } u \neq 0\,, \\ \partial^2_{uu} f(0^+)\,, & \textit{if } u = 0\,. \end{cases} \tag{3.8}$$

The definition of the operator $\nabla_\alpha : \mathcal{S}_\alpha(\mathbb{R}) \to C^\infty[0,1]$ is analogous.

3.2.4 Discrete derivatives and covariance estimatives

Fix an initial measure μ_n on Ω. For $x \in \mathbb{Z}$ and $t \geq 0$, let

$$\rho_t^n(x) \;\stackrel{\text{def}}{=}\; \mathbb{E}_{\mu_n}\big[\eta_t(x)\big]\,. \tag{3.9}$$

A simple computation shows that $\rho_t^n(\cdot)$ is a solution of the discrete equation

$$\partial_t \rho_t^n(x) \;=\; \big(n^2 \mathcal{A}_n \rho_t^n\big)(x)\,, \quad x \in \mathbb{Z}\,, \quad t \geq 0\,, \tag{3.10}$$

where the operator $\mathcal{A}_n$ acts on functions $f : \mathbb{Z} \to \mathbb{R}$ as

$$(\mathcal{A}_n f)(x) \;:=\; \xi^n_{x,x+1}\Big(f(x+1) - f(x)\Big) + \xi^n_{x-1,x}\Big(f(x-1) - f(x)\Big)\,, \quad \forall\, x \in \mathbb{Z}\,, \tag{3.11}$$

with $\xi_{x,x+1}$ as defined in (3.3).

Definition 8. *For $x, y \in \mathbb{Z}$, and $t \in [0,T]$, define the two-point correlation function*

$$\varphi_t^n(x,y) \;\stackrel{\text{def}}{=}\; \mathbb{E}_{\mu_n}\big[\eta_t(x)\eta_t(y)\big] - \rho_t^n(x)\rho_t^n(y)\,. \tag{3.12}$$

We now state two results that are fundamental for the study of density fluctuations, which are interesting by themselves.

Theorem 3.2.4 (Discrete derivative estimate)**.** *Assume that there exists a constant $c > 0$ that does not depend on n such that*

$$\sup_{x \in \mathbb{Z}} |\rho_0^n(x) - \rho_0\big(\tfrac{x}{n}\big)| \;\leq\; \frac{c}{n}\,. \tag{3.13}$$

Then, there exists $\bar{c} > 0$ such that, for all $t \in [0,T]$, and all $n \in \mathbb{N}$,

$$\big|\rho_t^n(x+1) - \rho_t^n(x)\big| \;\leq\; \begin{cases} \frac{\bar{c}}{n}, & \textit{if } x \neq 0, \\ \bar{c}, & \textit{if } x = 0. \end{cases}$$

Note that the second inequality above is obvious, but we kept in the statement of the theorem for the sake of clarity.

Theorem 3.2.5 (Correlation estimate). *Assume that there exists a constant $c > 0$ that does not depend on n such that*

$$\sup_{(x,y)\in V} |\varphi_0^n(x,y)| \;\leq\; \frac{c}{n}\,. \tag{3.14}$$

Moreover, assume that (3.13) *is satisfied. Then, there exists $\hat{c} > 0$ such that for all $n \in \mathbb{N}$,*

$$\sup_{t\leq T}\sup_{(x,y)\in V} |\varphi_t^n(x,y)| \;\leq\; \frac{\hat{c}\log n}{n}\,, \tag{3.15}$$

where $V := \{(x,y) \in \mathbb{Z}\times\mathbb{Z} :\; y \geq x+1\}$.

Remark 3.2.6. *Note that by the symmetry of the correlation function, Theorem 3.2.5 immediately implies* (3.15) *for $x \neq y$.*

3.2.5 Ornstein-Uhlenbeck process

Let $\rho(t,\cdot)$ be the unique solution of the hydrodynamic equation (3.5). In what follows, $\mathcal{D}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ (resp. $\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$) denotes the space of càdlàg (resp. continuous) $\mathcal{S}'_\alpha(\mathbb{R})$ valued functions endowed with the Skohorod topology. We also denote by χ the *static compressibility* defined by $\chi(\rho) = \rho(1-\rho)$. Denote by $\langle\cdot,\cdot\rangle_{\rho_t(\cdot)}$ the inner product with respect to $L^2_{\Lambda_t}(\mathbb{R})$, where the measure $\Lambda_t(du)$ is given by

$$\Lambda_t(du) \;\overset{\text{def}}{=}\; 2\chi(\rho_t(u))du + \frac{1}{\alpha}\Big[\rho_t(0^-)(1-\rho_t(0^+)) + \rho_t(0^+)(1-\rho_t(0^-))\Big]\delta_0(du)\,, \tag{3.16}$$

where $\delta_0(du)$ denotes the Dirac measure at zero. More precisely, for $f,g \in \mathcal{S}_\alpha(\mathbb{R})$,

$$\begin{aligned}\big\langle f,g\big\rangle_{\rho_t(\cdot)} \;&=\; \int_{\mathbb{R}} 2\chi(\rho_t(u))\, f(u)g(u)\,du \\ &\quad+ \frac{1}{\alpha}\Big[\rho_t(0^-)(1-\rho_t(0^+)) + \rho_t(0^+)(1-\rho_t(0^-))\Big] f(0)g(0)\,.\end{aligned}$$

Proposition 3.2.7. *There exists a unique (in distribution) random element $\mathcal{Y}$ taking values in the space $\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ such that the following two conditions hold:*

i) *For every function $f \in \mathcal{S}_\alpha(\mathbb{R})$, the stochastic processes $\mathcal{M}_t(f)$ and $\mathcal{N}_t(f)$ given by*

$$\mathcal{M}_t(f) \;=\; \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s(\Delta_\alpha f)ds\,, \tag{3.17}$$

$$\mathcal{N}_t(f) \;=\; \big(\mathcal{M}_t(f)\big)^2 - \int_0^t \|\nabla_\alpha f\|^2_{\rho_s(\cdot)}\,ds \tag{3.18}$$

are $\mathcal{F}_t$-martingales, where for each $t \in [0,T]$, $\mathcal{F}_t := \sigma(\mathcal{Y}_s(f); s \leq t, f \in \mathcal{S}_\alpha(\mathbb{R}))$.

ii) *$\mathcal{Y}_0$ is a random element taking values in $\mathcal{S}'_\alpha(\mathbb{R})$ with a fixed distribution.*

*Moreover, if **i)** and **ii)** hold, then:*

• *for each $f \in \mathcal{S}_\alpha(\mathbb{R})$, conditionally to $\mathcal{F}_s$ with $s < t$, the distribution of $\mathcal{Y}_t(f)$ is normal of mean $\mathcal{Y}_s(T^\alpha_{t-s}f)$ and variance $\int_s^t \|\nabla_\alpha T^\alpha_r f\|^2_{\rho_r(\cdot)}\,dr$.*

• *If $\mathcal{Y}_0$ is a Gaussian field, then the stochastic process $\{\mathcal{Y}_t(f)\,;\, t \geq 0\}$ will be Gaussian indeed.*

In other words, if $\mathcal{Y}^1$ and $\mathcal{Y}^2$ are random elements taking values on $\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ and satisfying the martingale problem described above by **i)** and **ii)**, then $\mathcal{Y}^1$ and $\mathcal{Y}^2$ must have the same distribution.

It is common in the literature to write the martingale problem stated above as a *formal* solution of some generalized stochastic partial differential equation. We discuss it with no mathematical rigor, aiming only at giving some intuition on the fluctuations' global behavior.

We call the random element $\mathcal{Y}$ a generalized Ornstein-Uhlenbeck process, defined via Proposition 3.2.7, which takes values on $\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ and it is the *formal* solution of

$$d\mathcal{Y}_t \;=\; \Delta_\alpha \mathcal{Y}_t dt + \nabla_\alpha\, d\mathcal{W}_t\,, \tag{3.19}$$

where:

• The operators Δ_α and ∇_α have been given in Definition 7 and are usually referred to as the *characteristics* of the Ornstein-Uhlenbeck process.

• $\mathcal{W}$ is a space-time white noise with respect to the measure $\Lambda_s(du)$, i.e., $\mathcal{W}$ is a mean-zero Gaussian random element taking values in the dual space of $L^2_\Lambda([0,\infty)\times\mathbb{R})$ with covariances given by

$$\mathbb{E}\Big[\mathcal{W}(F)\mathcal{W}(G)\Big] \;=\; \int_0^\infty\int_{\mathbb{R}} F(s,u)G(s,u)\,d\Lambda(s,u)\,,\quad \forall\, F,G\in L^2_\Lambda([0,\infty)\times\mathbb{R})\,,$$

where $d\Lambda(s,u) = d\Lambda_s(u)\times ds$, and Λ_s has been defined in (3.16).

• For $f\in\mathcal{S}_\alpha(\mathbb{R})$, we define $\mathcal{W}_t(f) := \mathcal{W}(f\mathbf{1}_{[0,t]})$. In particular, $\{\mathcal{W}_t(f) : f\in\mathcal{S}_\alpha(\mathbb{R})\}$ is a Gaussian process with covariance given on $f,g\in\mathcal{S}_\alpha(\mathbb{R})$ by

$$\mathbb{E}\Big[\mathcal{W}_t(f)\mathcal{W}_t(g)\Big] \;=\; \int_0^t \langle f,g\rangle_{\rho_s(\cdot)}ds\,.$$

3.2.6 Non-equilibrium fluctuations

We define the density fluctuation field $\mathcal{Y}^n$ as the time-trajectory of a linear functional acting on functions $f\in\mathcal{S}_\alpha(\mathbb{R})$ via

$$\mathcal{Y}^n_t(f) \;\stackrel{\text{def}}{=}\; \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} f(\tfrac{x}{n})\Big(\eta_t(x)-\rho^n_t(x)\Big)\,. \tag{3.20}$$

For each $n\geq 1$, let Q_n be the probability measure on $\mathcal{D}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ induced by the density fluctuation field $\mathcal{Y}^n$ and a measure μ_n. We now state the main result of this paper:

Theorem 3.2.8 (Non-equilibrium fluctuations)**.** *Consider the Markov processes $\{\eta_t : t\geq 0\}$ starting from a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ associated with a profile as in* (3.4)*, and assume:*

(A) *Conditions* (3.13) *and* (3.14) *on mean and covariance, respectively.*

(B) *There exists a $\mathcal{S}'_\alpha(\mathbb{R})$-valued random variable $\mathcal{Y}_0$ such that $\mathcal{Y}_0^n$ converges in distribution to $\mathcal{Y}_0$, whose law we denote by $\mathfrak{L}$.*

Then, the sequence of processes $\{\mathcal{Y}_t^n\}_{n\in\mathbb{N}}$ converges in distribution, as $n \to +\infty$, with respect to the Skorohod topology of $\mathcal{D}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$ to a random element $\mathcal{Y}$ in $\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$, the generalized Ornstein-Uhlenbeck which is a solution of (3.19)*, and $\mathcal{Y}_0$ has law $\mathfrak{L}$.*

It is of worth to give examples of sequences $\{\mu_n\}_{n\in\mathbb{N}}$ of initial measures satisfying assumptions **(A)** and **(B)**. Next, we present two examples of such initial measures and we leave an open question on the subject.

The first example we present is the standard one for non-equilibrium fluctuations: take $\{\mu_n\}_{n\in\mathbb{N}}$ as the slowly varying Bernoulli product measure $\{\nu^n_{\rho_0(\cdot)}\}_{n\in\mathbb{N}}$ associated with a smooth profile $\rho_0 : \mathbb{R} \to [0,1]$, that is, $\nu^n_{\rho_0(\cdot)}$ is a product measure on $\{0,1\}^{\mathbb{Z}}$ such that

$$\nu^n_{\rho_0(\cdot)}\big\{\eta \in \{0,1\}^{\mathbb{Z}} \,:\, \eta(x) = 1\big\} \;=\; \rho_0\big(\tfrac{x}{n}\big)\,.$$

Obviously, **(A)** is satisfied. The proof that **(B)** holds is just an adaptation of the analogous result for the SSEP, being included in Proposition 3.7.1 for the sake of completeness.

The second example we discuss is somewhat artificial, but, in any case, illustrates the existence of a sequence of non-product measures satisfying **(A)** and **(B)**. Let μ_n be the measure on Ω induced by the distribution at the time rn^2, where $r > 0$ is fixed, of the (homogeneous) one-dimensional SSEP started from the slowly varying measure $\nu^n_{\rho_0(\cdot)}$ defined above.

From the *propagation of local equilibrium* for the SSEP (see [18] and references therein), one can check that condition (3.13) holds. Besides that, it is well known that the SSEP has longe range correlations of order $O(1/n)$, giving (3.14). Thus, assumption **(A)** is satisfied. From the non equilibrium fluctuations for the homogeneous SSEP (see [4, 22]) one can deduce that **(B)** is satisfied, where the law $\mathfrak{L}$ is determined by the distribution of the Ornstein-Uhlenbeck process at time $r > 0$.

We now debate the issue of which properties a sequence of initial measures should have in order to satisfy **(A)** and **(B)**. Assume, for the moment, that the initial measures $\{\mu_n\}_{n\in\mathbb{N}}$ for the Markov processes $\{\eta_t : t \geq 0\}$ satisfy:

(i) Condition (3.13) holds.

(ii) For each $n \in \mathbb{N}$, the correlation at the initial time is of order $O(1/n)$ times a bounded profile ζ^n, that is,

$$\varphi_0^n(x,y) \;=\; \frac{\zeta^n\big(\frac{x}{n},\frac{y}{n}\big)}{n}\,, \qquad \forall\, x,y \in \mathbb{Z}\,,\ \forall n \in \mathbb{N}\,,$$

where the sequence of functions $\zeta^n : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$ converge uniformly to a bounded continuous function $\zeta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$ as $n \to \infty$. Note that this implies (3.14).

Under **(i)** and **(ii)**, condition **(A)** holds. Moreover, under **(i)** and **(ii)**, and following the same steps of Subsection 3.5.2, one can obtain tightness of $\{\mathcal{Y}_0^n\}_{n\in\mathbb{N}}$. Hence, in order to achieve **(B)**, it is only missing the convergence in distribution of the sequence of initial density fields $\{\mathcal{Y}_0^n\}_{n\in\mathbb{N}}$. Let $f,g\in\mathcal{S}_\alpha(\mathbb{R})$. By simple calculations,

$$\mathbb{E}_{\mu_n}\Big[\mathcal{Y}_0^n(f)\mathcal{Y}_0^n(g)\Big] \;=\frac{1}{n}\sum_{x\in\mathbb{Z}} f(\tfrac{x}{n})g(\tfrac{x}{n})\,\mathbb{E}_{\mu_n}\Big[\big(\overline{\eta}_0(x)\big)^2\Big]+\frac{1}{n}\sum_{\substack{x\neq y\\ x,y\in\mathbb{Z}}} f(\tfrac{x}{n})g(\tfrac{y}{n})\varphi_0^n(x,y)\,.$$

Above $\bar{\eta}$ denotes the centered random variable η : $\bar{\eta}_t(x) := \eta_t(x)-\rho_t^n(x)$. Under **(i)** and **(ii)**, it is easy to check that expression above converges to

$$\int_{\mathbb{R}}\chi(\rho_0(u))f(u)g(u)\,du+\int_{\mathbb{R}}\int_{\mathbb{R}}\zeta(w,r)f(w)g(r)\,dw\,dr$$

as $n\to\infty$. Note that the limit above indicates that $1/n$ is the right order on the decay of correlations in order to exist a limiting non zero effect on the distribution of initial density field $\mathcal{Y}_0$. However, convergence of means and decay of correlations do not suffice to assure that $\mathcal{Y}_0^n(f)$ actually converges in distribution: some special central limit theorem is required here. This CLT is not an easy subject due to the slow decay of correlations and due to the fact that for each n, the random variables $\eta_0(x)$ may have different distributions. We therefore leave it as an open question:

Open Question 1. *Given assumption* **(A)** *of Theorem 3.2.8, which additional hypotheses are necessary for* **(B)** *to hold?*

Without going into details, we affirm that a natural strategy to prove current/tagged particle fluctuations relies in a decay of correlations of order $O(1/n)$, see [16]. However, the correlations of the non equilibrium SSEP with a slow bond here considered are of order $O(\log n/n)$, see Theorem 3.2.5. Moreover, the current/ tagged particle fluctuations for the equilibrium scenario with a slow bond are already understood, see [8]. This leads us to:

Open Question 2. *How to prove current/tagged particle fluctuations for the non equilibrium SSEP with a slow bond? May (or must) a different scaling be considered?*

Finally, naturally inspired by [8], we state:

Open Question 3. *Consider $\beta>0$ with $\beta\neq 1$. How to prove non-equilibrium fluctuations for the one-dimensional SSEP with a slow bond of rate $\alpha n^{-\beta}$?*

We believe that this last open problem shall be solved by the methods of this paper, and we leave it for a future work. For the first two problems, we have no clear strategy to solve it.

3.3 Estimates on local times

In this section we derive estimates on the local times of a random walk with inhomogeneous rates, which will be later used in Section 3.4 in the proofs of Theorems 3.2.4 and 3.2.5. In the sequel, given any Markov chain Z and a set A, we denote by $L_t(A)$ the local time of Z in A until time t:

$$L_t(A) \stackrel{\text{def}}{=} \int_0^t \mathbf{1}_{\{Z_s \in A\}} ds\,. \tag{3.21}$$

3.3.1 Estimates in dimension two

We denote by $\{(\mathbf{X}_t, \mathbf{Y}_t);\ t \geq 0\}$ the random walk on the set $V = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \geq x+1\}$ with generator B_n acting on local functions $f : V \to \mathbb{R}$ via

$$\big(\mathbf{B}_n f\big)(u) \stackrel{\text{def}}{=} \sum_{v \in V} c_n(u,v)\big[f(v) - f(u)\big]\,, \ \ \forall\, u \in V\,. \tag{3.22}$$

Here, the rates are defined as pictured in Figure 3.2. More precisely, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ such that the L^1-norm[1] satisfies $\|u - v\|_1 = 1$, we define

$$c_n(u,v) \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha}{n}, & \text{if } \ (u,v) \in \mathcal{U}, \\ 1, & \text{if } \ u \notin U \text{ or } v \notin U. \end{cases}$$

and $c_n(u, v) = 0$ if the L^1-distance of u and v is not equal to one. Here, U is the subset of V given by

$$U \stackrel{\text{def}}{=} \big\{(x,y) \in V :\ x \in \{0,1\} \text{ and } y \geq 2\big\} \cup \big\{(x,y) \in V :\ x \leq -1 \text{ and } y \in \{0,1\}\big\}\,,$$

and $\mathcal{U}$ is the subset of $U^{\otimes 2}$ defined via

$$\begin{aligned} \mathcal{U} \stackrel{\text{def}}{=} &\big\{(u,v) \in U^{\otimes 2} :\ \|u - v\| = 1,\ |u_1 - v_1| = 1 \text{ and } u_2, v_2 \geq 2\big\} \\ &\cup \big\{(u,v) \in U^{\otimes 2} :\ \|u - v\| = 1,\ |u_2 - v_2| = 1 \text{ and } u_1, v_1 \leq -1\big\}\,. \end{aligned} \tag{3.23}$$

We furthermore denote by D the "upper diagonal" defined by $D \stackrel{\text{def}}{=} \big\{(x,y) \in \mathbb{Z}^2 : y = x+1\big\}$, see Figure 3.2 below.

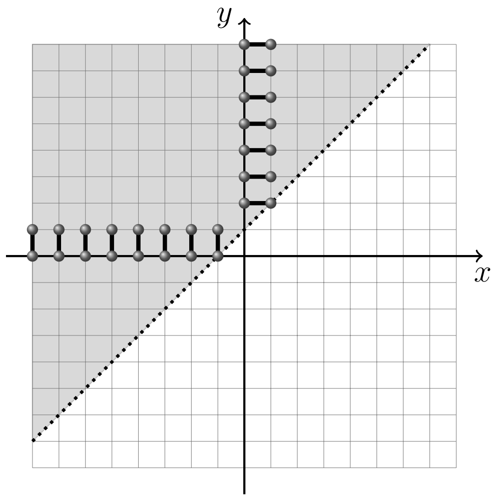


Figure 3.2: Sets V, D and U and $\mathcal{U}$. Sites of V are the ones laying on the light gray region. Sites in D lay on the dotted line and sites of U are marked as gray balls. Elements of $\mathcal{U}$ are edges marked with a thick black segment having jump rate equal to α/n (slow bonds). Any other edges have rate 1.

[1] We write $\|\cdot\|_1$ for the L^1-norm on $\mathbb{Z}^2$, that is, $\|(u_1,u_2)\|_1 = |u_1| + |u_2|$.

By $\mathbf{E}_{(\mathbf{x},\mathbf{y})}$, and $\mathbf{P}_{(\mathbf{x},\mathbf{y})}$ we denote the corresponding probability and expectation when starting from $(x, y) \in V$. The goal of this section is to prove the following result.

Proposition 3.3.1. *There exists a constant $c > 0$ such that for all $(x, y) \in V$, all $n \in \mathbb{N}$, and all $t \geq 0$,*

$$\begin{aligned} \mathbf{E}_{(x,y)}\Big[L_{tn^2}\Big(D\backslash\{(0,1)\}\Big)\Big] \;&\leq\; cn\sqrt{t}, \quad \textit{and} \\ \mathbf{E}_{(x,y)}\Big[L_{tn^2}\Big(\{(0,1)\}\Big)\Big] \;&\leq\; c\log(tn^2). \end{aligned} \tag{3.24}$$

To prove Proposition 3.3.1, we estimate first in Lemma 3.3.2 the local time of a simple random walk confined to the boundary of the set

$$W \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 : 0 \leq x \leq y\},$$

which is V intersected with the first quadrant shifted by the vector $(1, 2)$. In plain words we identified the vertex $(1, 2)$ in V with the origin, which is a change only of notational nature. Its proof consists on a comparison argument, which is the content of Proposition 3.3.3. Afterwards, in Lemma 3.3.5, we show that the expected number of jumps over the set of slow bonds (i.e., those with rates α/n) is finite. Finally, with all that at hand, we are able to finish the proof.

We denote by $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$ the continuous time simple random walk on W that jumps from a site $z_1 \in W$ to any fixed neighbouring site $z_2 \in W$ at rate 1, i.e., the simple random walk reflected at the boundary of W (which takes a triangular shape, see Figure 3.3). In particular the total jump rate out of $z_1 \in W$ is equal to the number of nearest neighbours of z_1 that lay inside W. Expectation with respect to $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$ conditioned to start at $(x, y) \in W$ is denoted by $\mathbf{E}^{\triangledown}_{(x,y)}$.

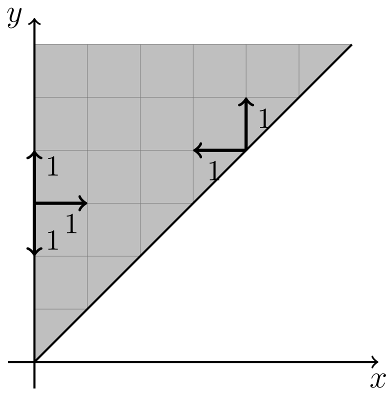


Figure 3.3: For $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$ any jump rate is equal to 1.

Lemma 3.3.2. *There exists a constant $c > 0$ such that for all $t \geq 0$ and all $(x, y) \in W$,*

$$\mathbf{E}^{\triangledown}_{(x,y)}\big[L_t(\partial W)\big] \;\leq\; c\sqrt{t}\,. \tag{3.25}$$

To prove the above lemma we will need two additional results. To introduce the first one, we remind the reader that a continuous time Markov chain on a countable set $\mathcal{E}$ can be constructed from a transition probability p on $\mathcal{E}$ and a bounded function $\lambda : \mathcal{E} \to (0, \infty)$ as follows:

(1) sample a discrete time Markov chain $(\xi_n)_{n\geq 0}$ with transition probability p;

(2) sample a sequence of independent random variables $(\tau_n)_{n\geq 0}$ such that τ_n is exponentially distributed with rate $\lambda(\xi_n)$ and define the successive sequence of jump times via $T_0 = 0$ and $T_n = \tau_n + T_{n-1}$ for $n \geq 1$;

(3) finally, define the continuous time Markov chain Z via

$$Z_t \;=\; \xi_n \mathbf{1}_{\{T_n \leq t < T_{n+1}\}} \,.$$

To continue, we fix a transition probability p on $\mathcal{E}$ and for any a, b such that $0 < a \leq b < \infty$, we denote by $Z^{[a,b]}$ the continuous time Markov chain with transition probability p and such that its field of rates $(\lambda_{[a,b]}(x))_{x\in G}$ is such that $\lambda_{[a,b]}(x) \in [a, b]$ for all $x \in G$. We denote the expectation with respect to $Z^{[a,b]}$ started in $z \in \mathcal{E}$ by $\mathbf{E}_z^{[a,b]}$.

Proposition 3.3.3. *Fix* $0 < a < b < c < d < \infty$*, and define* $\Lambda \overset{def}{=} \sup_{x\in\mathcal{E}} \frac{\lambda_{[c,d]}(x)}{\lambda_{[a,b]}(x)}$*. For any* $A \subseteq \mathcal{E}$ *and any* $z \in \mathcal{E}$*,*

$$\mathbf{E}_z^{[c,d]}\big[L_t(A)\big] \;\leq\; \mathbf{E}_z^{[a,b]}\big[L_{\Lambda t}(A)\big] \,. \tag{3.26}$$

The second result is about projections (also called lumping) of continuous time Markov chains.

Proposition 3.3.4. *Let* $\mathcal{E}$ *be a countable set, and consider a bounded function* $\zeta : \mathcal{E}\times\mathcal{E} \to [0,\infty)$*. Let* $(X_t)_{t\geq 0}$ *be the continuous time Markov chain with state space* $\mathcal{E}$ *and jump rates* $\{\zeta(x, y)\}_{x,y\in\Omega}$*. Fix an equivalence relation* $\sim$ *on* $\mathcal{E}$ *with equivalence classes* $\mathcal{E}^\sharp = \{[x] : x \in \mathcal{E}\}$ *and assume that* ξ *satisfies*

$$\sum_{y'\sim y} \zeta(x, y') \;=\; \sum_{y'\sim y} \zeta(x', y') \tag{3.27}$$

whenever $x \sim x'$*. Then,* $([X_t])_{t\geq 0}$ *is a Markov chain with state space* $\mathcal{E}^\sharp$ *and jump rates* $\zeta([x], [y]) = \sum_{y'\sim y} \zeta(x, y')$*.*

We first prove Proposition 3.3.3, afterwards Proposition 3.3.4 and finally we prove Lemma 3.3.2.

Proof of Proposition 3.3.3. To prove (3.26) we use a coupling argument. We do so by first sampling the discrete time Markov chain $(\xi_n)_{n\geq 0}$ as alluded above, and we intend to construct $Z^{[a,b]}$ and $Z^{[c,d]}$ both from the same realization of $(\xi_n)_{n\geq 0}$. To that end, we consider an independent field of Poisson clocks $(N_x^{[c,d]})_{x\in\mathcal{E}}$ such that for any $x \in \mathcal{E}$ the rate of $N_x^{[c,d]}$ equals $\lambda_{[c,d]}(x)$. We further define

$$N_x^{[a,b]}(t) \;\overset{\text{def}}{=}\; N_x^{[c,d]}\Big(\frac{t}{\lambda_{[c,d]}(x)}\lambda_{[a,b]}(x)\Big)$$

and it readily follows that for each $x \in \mathcal{E}$ the process $N_x^{[a,b]}$ is a Poisson process with rate $\lambda_{[a,b]}(x)$. Hence, it follows from the construction outlined before the statement of Proposition 3.3.3 that the construction above yields indeed a coupling of $Z^{[a,b]}$ and $Z^{[c,d]}$.

This coupling has the following two properties, which immediately proves (3.26). Denote by $Z^{[c,d]}_{[0,t]}$ the sequence of visited points by the process $Z_{[0,t]}$ until time t, with an analogous definition for $Z^{[a,b]}_{[0,\Lambda t]}$.

(1) There exists some $u \in [0, \Lambda t]$ such that $Z^{[c,d]}_{[0,t]} = Z^{[a,b]}_{[0,u]}$. That is, the sequence $Z^{[c,d]}_{[0,t]}$ is an initial piece of $Z^{[a,b]}_{[0,\Lambda t]}$.

(2) Given $x \in Z^{[c,d]}_{[0,t]}$, then at its k-th visit to x the holding time at that point of $Z^{[a,b]}$ is larger than the one of $Z^{[c,d]}$.

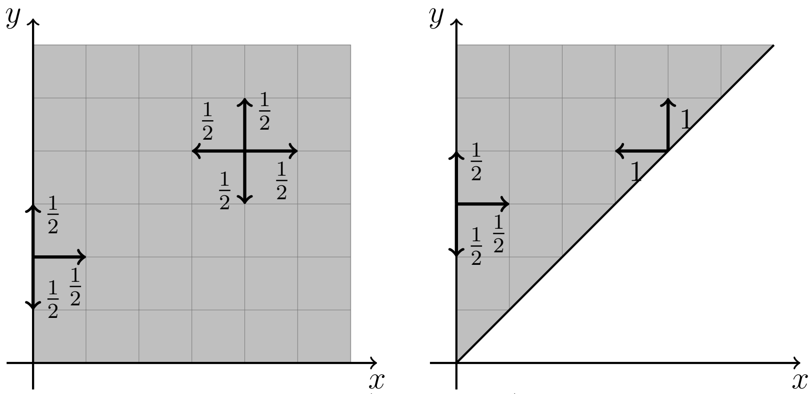
$\square$

Proof of Proposition 3.3.4. Let P be the transition matrix of the skeleton chain of $(X_t)_{t\geq 0}$ (i.e., of the underlying discrete time Markov chain). Assumption (3.27) implies that

$$P(x,[y]) \;=\; P(x',[y])$$

whenever $x \sim x'$. It then follows from [20, Lemma 2.5, pp. 25] that the skeleton chain of $([X_t])_{t\geq 0}$ is a discrete time Markov chain with transition matrix given by $P^\sharp([x],[y]) := P(x,[y])$. Thus, it remains to show that the holding times of $([X_t])_{t\geq 0}$ are exponentially distributed with rates $\left\{ \sum_{[y]} \zeta([x],[y]) \right\}_{[x]\in\mathcal{E}^\sharp}$. Yet, this is, as well, a consequence of (3.27). Hence, we can conclude the proof. $\square$

Proof of Lemma 3.3.2. The proof comes in two steps.



Rates of $(\mathbf{X}^\square, \mathbf{Y}^\square)$ are everywhere equal to $1/2$.

$([\mathbf{X}^\square], [\mathbf{Y}^\square])$ only lives on the upper triangle. The rates out of diagonal are doubled, but the rates *towards* the diagonal remain $1/2$.

Figure 3.4: Relation between $(\mathbf{X}^\square, \mathbf{Y}^\square)$ and $([\mathbf{X}^\square], [\mathbf{Y}^\square])$.

1st Step. In this step we show that it is sufficient to estimate the local time of a simple random walk on

$$\mathbb{Z}^2_{\geq 0} \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{Z}^2 : x, y \geq 0\},$$

and we refer the reader to Figure 3.3 and 3.4 for an illustration of the various random walks that will appear in this part of the proof. To that end, let $(\mathbf{X}^{\square}, \mathbf{Y}^{\square})$ be a simple random walk defined on $\mathbb{Z}^2_{\geq 0}$ that jumps from $z_1 \in \mathbb{Z}^2_{\geq 0}$ to a fixed neighbouring site $z_2 \in \mathbb{Z}^2_{\geq 0}$ at rate $\frac{1}{2}$. Write

$$\partial W_{\text{diag}} \stackrel{\text{def}}{=} \{(x,y) \in W : x = y\}. \tag{3.28}$$

Our aim is to show that for any $(x,y) \in W$,

$$\mathbf{E}^{\triangledown}_{(x,y)}\big[L_t(\partial W)\big] \leq \mathbf{E}^{\square}_{(x,y)}\big[L_{2t}(\partial \mathbb{Z}^2_{\geq 0})\big] + \mathbf{E}^{\square}_{(x,y)}\big[L_{2t}(\partial W_{\text{diag}})\big], \tag{3.29}$$

where the expectations on the right hand side of the display above denote the expectation with respect to $(\mathbf{X}^{\square}, \mathbf{Y}^{\square})$ started at (x,y). To see that (3.29) is true we consider the function $T : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}^2_{\geq 0}$ that maps each $z \in \mathbb{Z}^2_{\geq 0}$ to its reflection with respect to the diagonal ∂W_{diag}. Note in particular that T is its own inverse, so that we can define an equivalence relation on $\mathbb{Z}^2_{\geq 0}$ via

$$z_1 \sim z_2 \quad \Longleftrightarrow \quad \exists\, n \in \{1,2\} \text{ such that } T^n(z_1) = z_2. \tag{3.30}$$

Writing $\mathbf{P}^{\square}$ for the transition matrix corresponding to the underlying discrete time random walk of $(\mathbf{X}^{\square}, \mathbf{Y}^{\square})$, and by $\{\zeta(x,y)\}_{x,y \in \mathbb{Z}^2_{\geq 0}}$ its field of rates, it is easy to see that for any $z_1 \sim z_2$ and any $z_3 \in \mathbb{Z}^2_{\geq 0}$

$$\sum_{z_3' \sim z_3} \zeta(z_1, z_3') = \sum_{z_3' \sim z_3} \zeta(z_2, z_3'). \tag{3.31}$$

Hence, by Proposition 3.3.4 and the fact that the set $\mathbb{Z}^2_{\geq 0}/\!\sim$ equals W, the process $([\mathbf{X}^{\square}], [\mathbf{Y}^{\square}])$ can be identified with a simple random walk on W such that its jump rate out of each fixed edge equals $\frac{1}{2}$ except for those attached to ∂W_{diag}, where the jump rate is 1. Note in particular that this makes the set of edges directed. Indeed, the jump rate from $x \in \partial W_{\text{diag}}$ to any neighbour y is 1, whereas the jump rate from y to x is $\frac{1}{2}$.

Denoting by $\mathbf{E}^{[\square]}_{([x],[y])}$ the expectation with respect to $([\mathbf{X}^{\square}], [\mathbf{Y}^{\square}])$ when started at $([x],[y])$, we see that as a consequence of Proposition 3.3.3,

$$\mathbf{E}^{\triangledown}_{(x,y)}\big[L_t(\partial W)\big] \leq \mathbf{E}^{[\square]}_{([x],[y])}\big[L_{2t}([\partial W])\big]. \tag{3.32}$$

Thus, (3.29) readily follows from last inequality.

2nd Step. We now show that it is sufficient to estimate certain local times of a simple random walk (X,Y) on $\mathbb{Z}^2$ jumping at total rate 2 (i.e., the jump rate over any fixed edge is $\frac{1}{2}$), which will then yield the claim. To that end we define an

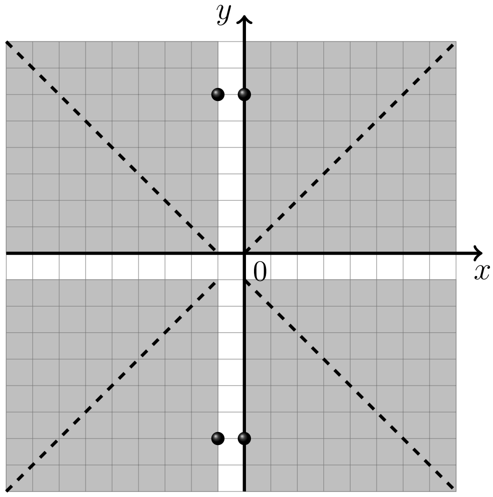


Figure 3.5: Ilustration of equivalence relation in the **2nd Step** of the proof of Lemma 3.3.2. ∂W_{diag} gets identified with the points on dashed lines. The four points marked with black balls compose a single equivalence class. Non-zero jump rates between any two equivalence classes are everywhere equal to $1/2$.

equivalence relation by imposing that $(x, y) \sim (x, -y-1)$ and $(x, y) \sim (-x-1, y)$, for any $x, y \in \mathbb{Z}$. We then note that in this way ∂W_{diag} gets identified with

$$\begin{aligned} \left[\partial W_{\text{diag}}\right] &\stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{Z}^2 : x = y\right\} \\ &\cup \left\{(x,y) \in \mathbb{Z}^2 : x = -y-1,\, y \geq 0 \text{ or } y = -x-1,\, x \geq 0\right\}, \end{aligned}$$

see Figure 3.5. Note that by Proposition 3.3.4 the random walk $(\mathbf{X}^{\square}, \mathbf{Y}^{\square})$ can be identified with $([X], [Y])$. This shows that it is sufficient to bound

$$\mathbf{E}^{(X,Y)}_{(x,y)}\left[L_{2t}(A)\right], \tag{3.33}$$

where $A = A_1 \cup A_2 \cup A_3$ with

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{Z}^2 : x = y\right\}, \\ A_2 &\stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{Z}^2 : x = -y-1,\, y \geq 0 \text{ or } y = -x-1,\, x \geq 0\right\}, \quad \text{and} \\ A_3 &\stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{Z}^2 : x \in \{0,-1\}, \text{ or } y \in \{0,-1\}\right\}. \end{aligned} \tag{3.34}$$

Since $X - Y$ has the same law as a one-dimensional symmetric simple random walk, we conclude that $L_{2t}(A_1)$ equals in law to the local time at zero of a one-dimensional symmetric simple random walk, for which the statement of this lemma is well known, and for completeness, we provide a short proof of it in Proposition 3.6.1. A similar argument may be used for A_2, and A_3. Therefore, we can finish the proof. □

We now come back to the original problem, i.e., estimating local times of the random walk $(\mathbf{X}, \mathbf{Y})$ defined on the set V. An important ingredient in the analysis will be an estimate on the number of jumps of $(\mathbf{X}, \mathbf{Y})$ over the set of slow edges, i.e., those that are depicted with thick black segments in Figure 3.2. We denote the set of these edges by $\mathcal{S}$, and we define a sequence of stopping times via

$$\begin{aligned} \tau_1 &= \inf\big\{t \geq 0 : (\mathbf{X}_t, \mathbf{Y}_t) \text{ crossed an edge in } \mathcal{S}\big\}, \quad \text{and for } i \geq 2, \\ \tau_i &= \inf\big\{t \geq \tau_{i-1} : (\mathbf{X}_t, \mathbf{Y}_t) \text{ crossed an edge in } \mathcal{S}\big\}. \end{aligned} \tag{3.35}$$

Finally, we define the number of crossings until the time tn^2 via

$$C_{tn^2} = \sup\big\{i \geq 0 : \tau_i \leq tn^2\big\}. \tag{3.36}$$

Lemma 3.3.5. *There exists a constant $c > 0$ such that uniformly over all starting points $(x, y) \in V$, all $t \geq 0$, and all $n \in \mathbb{N}$,*

$$\mathbf{E}_{(x,y)}[C_{tn^2}] \leq c. \tag{3.37}$$

Proof. We first show that for all $i \geq 1$,

$$\inf_{(x,y)} \mathbf{P}_{(x,y)}[\tau_i - \tau_{i-1} \geq tn^2] > 0. \tag{3.38}$$

To that end, assume without loss of generality that (x, y) is in the first quadrant. In this case τ_1 can be interpreted as a first success of the simple random walk $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$, which with a slight abuse of notation is now considered on the set W given by the intersection of V with the first quadrant, in the following way: whenever $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$ is on a vertex z that is attached to a slow bond it realizes the following experiment: besides its three (one if the vertex is $(1, 2)$) independent Poisson clocks N_z^1, N_z^2, and N_z^3 ringing at rate 1 that are needed for its graphical construction, it considers an additional independent Poisson clock $N_z(\alpha)$ ringing at rate α/n. We then say that the experiment is successful if $N_z(\alpha)$ rings before any of the other three clocks. It then follows from the construction that the time of the first success equals in law the time of the first jump of $(\mathbf{X}, \mathbf{Y})$ over a slow bond. Indeed, one may couple $(\mathbf{X}, \mathbf{Y})$ and $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$ such that they move together until the first time of success. Thus, using the fact that each experiment is independent of the evolution of $(\mathbf{X}^{\triangledown}, \mathbf{Y}^{\triangledown})$, and that the set of vertices that are attached to $\mathcal{S}$ is a subset of ∂W, we see that for any constant $c \in (0, 1)$,

$$\begin{aligned} \mathbf{P}_{(x,y)}[\tau_1 \geq tn^2] &\geq \mathbf{P}^{\triangledown}_{(x,y)}\big[L_{tn^2}(\partial W) \leq c\sqrt{t}n, \text{all experiments are unsuccessful}\big] \\ &\geq \mathbf{P}^{\triangledown}_{(x,y)}\big[L_{tn^2}(\partial W) \leq c\sqrt{t}n\big] \cdot \mathbf{P}\big[\exp(\alpha/\mathrm{n}) \geq c\sqrt{t}n\big], \end{aligned} \tag{3.39}$$

where $\exp(\alpha/\mathrm{n})$ denotes an exponentially distributed random variable with rate α/n. It now follows from Lemma 3.3.2 and Markov's inequality that there exists $c \in (0, 1)$ such that the right hand-side of (3.39) is strictly larger than zero, uniformly in (x, y). With similar arguments we may derive the same statement for all $i \geq 2$. We next introduce the random variable

$$N = \inf\big\{i \geq 1 : \tau_i - \tau_{i-1} \geq tn^2\big\}. \tag{3.40}$$

Then, using the strong Markov property at time τ_{i-1}, bounding the probability of the event $\{\tau_i - \tau_{i-1} \geq tn^2\}$ by 1, and then once again using the strong Markov property at time τ_{i-2}, we can estimate for any $i \geq 1$,

$$\begin{aligned} \mathbf{P}_{(x,y)}[N = i] &\leq \mathbf{P}_{(x,y)}\Big[\bigcap_{j=1}^{i-1}\{\tau_j - \tau_{j-1} < tn^2\}\Big] \\ &= \mathbf{E}_{(x,y)}\Big[\prod_{j=1}^{i-2}\mathbf{1}_{\{\tau_j-\tau_{j-1}<tn^2\}}\mathbf{E}_{(\mathbf{X}_{\tau_{i-2}},\mathbf{Y}_{\tau_{i-2}})}[\mathbf{1}_{\{\tau_1<tn^2\}}]\Big]\,. \end{aligned} \tag{3.41}$$

Using (3.38), we see that there exists $c \in [0,1)$ that is independent of the starting point (x,y), such that the latter term above is bounded from above by

$$c\,\mathbf{E}_{(x,y)}\Big[\prod_{j=1}^{i-2}\mathbf{1}_{\{\tau_j-\tau_{j-1}<tn^2\}}\Big]\,. \tag{3.42}$$

Iterating the above procedure we can get that

$$\sup_{(x,y)\in V}\mathbf{P}_{(x,y)}[N = i] \leq c^{i-1}\,, \tag{3.43}$$

which in turn implies the uniform boundedness in $(x,y) \in V$ of the expectation of N. Since $C_{tn^2} \leq N$, this implies the claim. □

We present now the proof of Proposition 3.3.1, and we focus first on the local time of the set $D\setminus\{(0,1)\}$. For definiteness we assume that $(\mathbf{X},\mathbf{Y})$ starts in $(x,y) \in W$, where we recall that, abusing of notation, W denotes V intersected with the first quadrant. All other cases follow by a straightforward adaptation of this proof. Note that the event $\{(\mathbf{X}_s,\mathbf{Y}_s) \in D\setminus\{(0,1)\}\}$ is only possible, if $s \in \cup_{i=0}^{\infty}[\tau_{2i},\tau_{2i+1})$, where $\tau_0 = 0$. Hence, we can write

$$\begin{aligned} \mathbf{E}_{(x,y)}\Big[L_{tn^2}\Big(D\setminus\{(0,1)\}\Big)\Big] &= \mathbf{E}_{(x,y)}\Big[\int_0^{tn^2}\mathbf{1}_{\{(\mathbf{X}_s,\mathbf{Y}_s)\in D\setminus\{(0,1)\}\}}\,ds\Big] \\ = \sum_{i=0}^{\infty}\mathbf{E}_{(x,y)}\Big[\int_{\tau_{2i}\wedge tn^2}^{\tau_{2i+1}\wedge tn^2}&\mathbf{1}_{\{(\mathbf{X}_s,\mathbf{Y}_s)\in D\setminus\{(0,1)\}\}}\,ds\Big]\,. \end{aligned} \tag{3.44}$$

Fix $i \in \mathbb{N}$. Applying the strong Markov property at time τ_{2i} we can rewrite each summand in the display above as

$$\mathbf{E}_{(x,y)}\Big[\mathbf{1}_{\{\tau_{2i}<tn^2\}}\mathbf{E}_{(\mathbf{X}_{\tau_{2i}},\mathbf{Y}_{\tau_{2i}})}\Big[\int_0^{\overline{\tau}_1\wedge tn^2-\tau_{2i}}\mathbf{1}_{\{(\overline{\mathbf{X}}_s,\overline{\mathbf{Y}}_s)\in D\setminus\{(0,1)\}\}}\,ds\Big]\Big]\,, \tag{3.45}$$

where $(\overline{\mathbf{X}},\overline{\mathbf{Y}})$ denotes an independent copy of $(\mathbf{X},\mathbf{Y})$ and $\overline{\tau}_1$ is the corresponding stopping time, defined in the same way as τ_1 in (3.35). We now recall that as a consequence of the proof of Lemma 3.3.5 until the time $\overline{\tau}_1$ the walk $(\mathbf{X},\mathbf{Y})$ can be coupled with $(\mathbf{X}^{\triangledown},\mathbf{Y}^{\triangledown})$. Hence, we see that (3.45) is at most

$$\mathbf{E}_{(x,y)}\big[\mathbf{1}_{\{\tau_{2i}<tn^2\}}\big]\sup_{(x,y)\in W}\mathbf{E}_{(x,y)}\Big[\int_0^{tn^2}\mathbf{1}_{\{(\mathbf{X}_s^{\triangledown},\mathbf{Y}_s^{\triangledown})\in D\setminus\{(0,1)\}\}}\,ds\Big]\,. \tag{3.46}$$

Making use of Lemma 3.3.2 we see that there exists a constant $c \in (0,\infty)$ such that for all starting points, all $t \geq 0$ and all $n \in \mathbb{N}$, the term on the left hand-side of (3.44) is bounded from above by

$$c\sqrt{t}n\,\mathbf{E}_{(x,y)}\big[C_{tn^2}\big]\,. \tag{3.47}$$

Hence, an application of Lemma 3.3.5 is enough to conclude the claim. To estimate the local time of the vertex $(0,1)$ we can proceed almost exactly as above, and we see that there exists a constant $c \in (0,+\infty)$ such that

$$\begin{aligned} \mathbf{E}_{(x,y)}\Big[L_{tn^2}\big(\{(0,1)\}\big)\Big] \;&=\; \int_0^{tn^2} \mathbf{P}_{(x,y)}\Big[(\mathbf{X}_s,\mathbf{Y}_s)=(0,1)\Big]\,ds \\ &\leq\; c\sum_{z\in\mathcal{A}}\int_0^{2tn^2} \mathbf{P}_{(x,y)}\Big[(X_s,Y_s)=z\Big]\,ds\,, \end{aligned} \tag{3.48}$$

where we recall that (X,Y) denotes the simple random walk on $\mathbb{Z}^2$ jumping at total rate 2, and $\mathcal{A}=\{(0,1),(1,1),(0,0),(1,0)\}$. The proof now follows from the local central limit theorem, see for instance [19, Theorem 2.5.6] (this result is stated for one-dimensional continuous time random walks, however using the fact that a d-dimensional continuous time random walk consists of d independent one-dimensional random walks, it may be easily adapted to our setting), or alternatively from Proposition 3.6.2.

3.3.2 Estimates in dimension one

We denote by $\{\mathbf{X}_t;\ t \geq 0\}$ the random walk on $\mathbb{Z}$ with a slow bond, that is, the random walk with infinitesimal generator $\mathcal{A}_n$ given in (3.11) and we use $\mathbf{E}_x, \mathbf{P}_x$ to denote the corresponding expectation and probability, starting from $x \in \mathbb{Z}$.

Lemma 3.3.6. *For all $x,y \in \mathbb{Z}$, and for all $t \geq 0$ we have the equality*

$$\mathbf{P}_x\big(\mathbf{X}_t = y\big) + \mathbf{P}_x\big(\mathbf{X}_t = -y+1\big) \;=\; \mathbf{P}_x\big(X_t = y\big) + \mathbf{P}_x\big(X_t = -y+1\big)\,, \tag{3.49}$$

where $(X_t)_{t\geq 0}$ denotes a one-dimensional symmetric simple random walk jumping at total rate 2.

Proof. The proof comes in two steps.

1st Step. In this step we rewrite the left hand-side in (3.49) in terms of the transition probabilities of a symmetric simple random walk that is reflected at 1. To that end, we define the following equivalence relation:

$$x \sim y \quad \Longleftrightarrow \quad y = -x+1 \text{ or } y = x\,. \tag{3.50}$$

Note that in particular in this way 0 gets identified with 1, so that jumps between these two vertices “do not count”. One may then readily check that condition (3.27) is satisfied, so that $([\mathbf{X}_t])_{t\geq 0}$ defines a continuous time Markov chain. It is then plain to see that for all $x \in \mathbb{Z}$ and all $t \geq 0$ the following relation holds:

$$[\mathbf{X}_t] = [x] \quad \Longleftrightarrow \quad \mathbf{X}_t \in \{x, -x+1\}\,. \tag{3.51}$$

Thus,

$$\mathbf{P}_x(\mathbf{X}_t \in \{y, -y+1\}) \;=\; \mathbf{P}_{[x]}([\mathbf{X}_t] = [y])\,. \tag{3.52}$$

Choosing only representants in the set $\mathbb{Z}_{\geq 1} \stackrel{\text{def}}{=} \{x \in \mathbb{Z} : x \geq 1\}$ we see that $([\mathbf{X}_t])_{t\geq 0}$ may be identified with a simple random walk $(X_t^R)_{t\geq 0}$ on $\mathbb{Z}_{\geq 1}$ that jumps from any vertex $x \in \mathbb{Z}_{\geq 1}$ to a fixed neighboring vertex in $\mathbb{Z}_{\geq 1}$ at rate 1. Thus, for any $x, y \geq 1$, (3.52) becomes

$$\mathbf{P}_x(\mathbf{X}_t \in \{y, -y+1\}) \;=\; \mathbf{P}_x(X_t^R = y)\,. \tag{3.53}$$

2nd Step. In this step we show that the right hand-side of (3.53) may be rewritten in terms of a symmetric simple random walk on $\mathbb{Z}$ jumping at total rate 2. To that end we use the same equivalence relation as above and we note that $(X_t)_{t\geq 0}$ can be, in the same way, identified with $(X_t^R)_{t\geq 0}$ as $(\mathbf{X}_t)_{t\geq 0}$ can be identified with $([\mathbf{X}_t])_{t\geq 0}$. This finishes the proof. $\square$

3.4 Estimates on the discrete derivative and correlations

In the next two subsections, we present the proofs of Theorems 3.2.4 and 3.2.5, respectively.

3.4.1 Estimate on the discrete derivative

This section is devoted to the proof of Theorem 3.2.4.

Proof. Recall that ρ_t^n is a solution of (3.10). Since the statement is clear for $x = 0$, we only need to deal with the case $x \neq 0$. Let ρ_t be the solution of the equation (3.5), and define $\gamma^n : [0,T] \times \mathbb{Z}^d \to \mathbb{R}$ via

$$\gamma_t^n(x) \;\stackrel{\text{def}}{=}\; \begin{cases} \rho_t^n(x) - \rho_t\big(\frac{x}{n}\big), & \text{if } x \neq 0, \\ \rho_t^n(0) - \rho_t\big(\frac{-1}{n^2}\big), & \text{otherwise.} \end{cases} \tag{3.54}$$

The reason for the previous definition is that it distinguishes two cases, since at $x = 0$ the time derivative of ρ is not related to its spatial derivatives in a way that is helpful for our purposes. However, with the above choice of γ^n we see that for all $x \in \mathbb{Z}$,

$$\partial_t \gamma_t^n(x) \;=\; n^2 \mathcal{A}_n \gamma_t^n(x) + F_t^n(x)\,, \tag{3.55}$$

where

$$F_t^n(x) \;\stackrel{\text{def}}{=}\; \begin{cases} \big(n^2\mathcal{A}_n - \partial_u^2\big)\rho_t\big(\frac{x}{n}\big) & \text{if } x \neq 0, \\ n^2\mathcal{A}_n\rho_t(0) - \partial_u^2\rho_t\big(\frac{-1}{n^2}\big) & \text{otherwise.} \end{cases} \tag{3.56}$$

Observe that, by the definition of $\mathcal{A}_n$ in (3.11), for $x \in \mathbb{Z}\backslash\{0,1\}$, F_t^n accounts for the difference between the discrete and the continuous Laplacian. To continue, we add and subtract $\rho_t\big(\frac{x}{n}\big)$ and $\rho_t\big(\frac{x+1}{n}\big)$ to $\big|\rho_t^n(x+1)-\rho_t^n(x)\big|$ and use the triangle inequality which yields

$$\big|\rho_t^n(x+1)-\rho_t^n(x)\big| \;\leq\; |\gamma_t^n(x+1)|+|\gamma_t^n(x)|+\big|\rho_t\big(\tfrac{x+1}{n}\big)-\rho_t\big(\tfrac{x}{n}\big)\big|\,. \tag{3.57}$$

We first treat the rightmost term above. Since $x\mapsto\rho_t(x)$ is differentiable in any neighborhood outside of zero, and ρ_t has one sided spatial derivatives at zero, we see that

$$\big|\rho_t\big(\tfrac{x+1}{n}\big)-\rho_t\big(\tfrac{x}{n}\big)\big| \;=\; O\big(\tfrac{1}{n}\big)\,.$$

Recall that $\{\mathbf{X}_t;\ t\geq 0\}$ denotes the random walk on $\mathbb{Z}$ generated by $\mathcal{A}_n$. Applying Duhamel's principle we see that we can write the solution of (3.55) as

$$\gamma_t^n(x) \;=\; \mathbf{E}_x\Big[\gamma_0^n(\mathbf{X}_{tn^2})+\int_0^t F_{t-s}^n(\mathbf{X}_{sn^2})\,ds\Big]\,.$$

Therefore,

$$\sup_{t\leq T}\,\sup_{x\in\mathbb{Z}}|\gamma_t^n(x)| \;\leq\; \sup_{x\in\mathbb{Z}}|\gamma_0^n(x)| \;+\; \sup_{t\leq T}\,\sup_{x\in\mathbb{Z}}\Big|\mathbf{E}_x\Big[\int_0^t F_{t-s}^n(\mathbf{X}_{sn^2})\,ds\Big]\Big|\,.$$

Since $|\gamma_0^n(x)|=|\rho_0^n(x)-\rho_0(x)|$, by Assumption (3.13) we only need to control the second term on the right hand-side of the previous expression. By Fubini's Theorem, we see that

$$\mathbf{E}_x\Big[\int_0^t F_{t-s}^n(\mathbf{X}_{sn^2})\,ds\Big] \;=\; \int_0^t\sum_{z\in\mathbb{Z}}\mathbf{P}_x\big[\mathbf{X}_{sn^2}=z\big]F_{t-s}^n(z)\,ds\,. \tag{3.58}$$

Since the discrete Laplacian approximates the continuous Laplacian, we conclude that $|F_t^n(x)|\leq C/n^2$ for any $x\in\mathbb{Z}\backslash\{0,1\}$ and for any $t\geq 0$. Therefore, we can bound the absolute value of (3.58) by

$$t\frac{C}{n^2} \;+\; \int_0^t\sum_{z\in\{0,1\}}\mathbf{P}_x\big[\mathbf{X}_{sn^2}=z\big]\big|F_{t-s}^n(z)\big|\,ds\,. \tag{3.59}$$

Moreover, we also have that

$$\begin{aligned} F_{t-s}^n(1) \;&=\; n^2\Big(\rho_t\big(\tfrac{2}{n}\big)-\rho_t\big(\tfrac{1}{n}\big)+\frac{\alpha}{n}\big(\rho_t\big(\tfrac{0}{n}\big)-\rho_t\big(\tfrac{1}{n}\big)\big)\Big)-\partial_u^2\rho_t\big(\tfrac{1}{n}\big)\\ &=\; n\Big(n\big(\rho_t\big(\tfrac{2}{n}\big)-\rho_t\big(\tfrac{1}{n}\big)\big)+\alpha\big(\rho_t\big(\tfrac{0}{n}\big)-\rho_t\big(\tfrac{1}{n}\big)\big)\Big)-\partial_u^2\rho_t\big(\tfrac{1}{n}\big)\,. \end{aligned}$$

Summing and subtracting $\alpha\rho(0^+)$, using the Robin boundary conditions and Taylor expansion, the last equation becomes bounded from above by

$$\Big|n\Big(\tfrac{1}{n}\partial_u^2\rho_t\big(\tfrac{1}{n}\big)+O(1/n^2)\Big)+\tfrac{1}{2}\partial_u^2\rho_t\big(\tfrac{1}{n}\big)-\alpha\partial_u\rho_t(0^+)+O(1/n)-\partial_u^2\rho_t\big(\tfrac{1}{n}\big)\Big|,$$

from where we get that $|F_t^n(1)|\leq C$ for any $t\geq 0$. For $z=0$ we obtain, in a similar way, a bound of the same order. Therefore, (3.59) is bounded from above by

$$t\frac{C}{n^2} \;+\; C\int_0^t\big(\mathbf{P}_x\big[\mathbf{X}_{sn^2}=0\big]+\mathbf{P}_x\big[\mathbf{X}_{sn^2}=1\big]\big)\,ds\,.$$

Thus, applying Lemma 3.4.1 below the result follows. □

Lemma 3.4.1. *Let* X *be as in Subsection 3.3.2. There exists a constant* $C > 0$ *such that the following estimate holds for all* $t \geq 0$*:*

$$\int_0^t \mathbf{P}_x\Big[\mathbf{X}_{sn^2} \in \{0,1\}\Big]\, ds \;\leq\; \frac{C\sqrt{t}}{n}\,.$$

Proof. Denote the symmetric simple random walk on $\mathbb{Z}$ jumping at rate 2 by $\{X_t;\, t \geq 0\}$. It is then well known that for all $t \geq 0$ the map $x \in \mathbb{Z} \mapsto \mathbf{P}_x[X_t = 0]$ is maximized at $x = 0$. Hence, Lemma 3.4.1 is a consequence of Lemma 3.3.6 together with Proposition 3.6.1. □

3.4.2 Estimate on the correlation function

In this section we prove Theorem 3.2.5. To that end, we show that the correlation function φ^n introduced in Definition 8 can be estimated from above by the local times of the random walk $\{(\mathbf{X}_t, \mathbf{Y}_t);\; t \geq 0\}$, introduced in Subsection 3.3.1. This is the content of Proposition 3.4.2. Proposition 3.3.1 then immediately yields the result. Given a set $A \subseteq V$, similarly as in Section 3.3 we denote by $L_t(A)$ the local time of $\{(\mathbf{X}_t, \mathbf{Y}_t);\, t \geq 0\}$ until time t in A, see (3.21).

Proposition 3.4.2. *There exists* $C > 0$ *such that*

$$\begin{aligned}&\sup_{t\leq T}|\varphi_t^n(x,y)|\\&\leq \frac{C}{n} + C\Big(\frac{1}{n^2}(\mathbf{E}_{(x,y)}[L_{n^2T}(D\setminus\{(0,1)\})] + \frac{1}{n}\mathbf{E}_{(x,y)}[L_{n^2T}(\{(0,1)\})]\Big).\end{aligned} \tag{3.60}$$

Proof. First, observe that from Kolmogorov's forward equation, we have that

$$\partial_t\varphi_t^n(x,y) \;=\; \mathbb{E}_{\mu_n}\big[n^2\mathcal{L}_n(\eta_t(x)\eta_t(y))\big] - \partial_t(\rho_t^n(x)\rho_t^n(y))\,.$$

Applying (3.2) and (3.9) and performing some long, but simple, calculations, one can deduce that φ_t^n solves the following equation:

$$\partial_t\varphi_t^n(x,y) \;=\; n^2\mathbf{B}_n\varphi_t^n(x,y) + g_t^n(x,y)\,,$$

where $\mathbf{B}_n$ was defined in (3.22) and

$$g_t^n(x,y) \;=\; -(\nabla_n^+\rho_t^n(x))^2\Big(\mathbf{1}_{\{D\setminus(0,1)\}} + \frac{\alpha}{n}\mathbf{1}_{\{(0,1)\}}\Big)\,. \tag{3.61}$$

Here, ∇_n^+ denotes the rescaled discrete right derivative which, for any function $f : \mathbb{Z} \to \mathbb{R}$, is defined via $\nabla_n^+ f(x) \stackrel{\text{def}}{=} n(f(x+1) - f(x))$. By Duhamel's Principle,

$$\varphi_t^n(x,y) \;=\; \mathbf{E}_{(x,y)}\Big[\varphi_0^n(\mathbf{X}_{tn^2}, \mathbf{Y}_{tn^2}) + \int_0^t g_{t-s}^n(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2})\, ds\Big],$$

where $\{(\mathbf{X}_t, \mathbf{Y}_t);\; t \geq 0\}$ is the random walk with generator $\mathbf{B}_n$. In order to prove the proposition we just have to estimate the right hand-side of the last equation. We see that

$$\sup_{t\leq T}|\varphi_t^n(x,y)| \leq |\varphi_0^n(x,y)| + \sup_{t\leq T}\Big|\mathbf{E}_{(x,y)}\Big[\int_0^t g_{t-s}^n(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2})\, ds\Big]\Big|. \tag{3.62}$$

By Assumption (3.14), the first term on the right hand-side of the last expression is bounded from above by C/n. Thus, to finish the proof we only need to estimate the rightmost term in the display above.

Applying the definition of g^n, and rewriting the expectation above in terms of transition probabilities, we see that for any $s \leq t$,

$$\mathbf{E}_{(x,y)}[g^n_{t-s}(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2})] = \sum_{z \neq 0} \big[-(\nabla_n^+ \rho^n_{t-s}(z))^2 \big] \mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (z, z+1)]$$
$$+ \frac{\alpha}{n} \big[-(\nabla_n^+ \rho^n_{t-s}(0))^2 \big] \mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (0,1)] \, .$$

Consequently, for all $(x, y) \in V$, the rightmost term in (3.62) is bounded from above by

$$S_n \int_0^t \Big(\mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) \in D \backslash \{(0,1)\}] + S_{n,0} \frac{\alpha}{n} \mathbf{P}_{(x,y)}[(\mathbf{X}_{sn^2}, \mathbf{Y}_{sn^2}) = (0,1)] \Big) ds, \quad (3.63)$$

where

$$S_n \;=\; \sup_{t \geq 0} \sup_{z \in \mathbb{Z} \backslash \{0\}} (\nabla_n^+ \rho^n_t(z))^2 \quad \text{and} \quad S_{n,0} \;=\; \sup_{t \geq 0} (\nabla_n^+ \rho^n_t(0))^2 \, . \quad (3.64)$$

Recalling Theorem 3.2.4, we easily deduce that $S_n \leq C$ and $S_{n,0} \leq Cn^2$. Substituting (3.64) into (3.63), together with a change of variables, the result follows. □

The proof of Theorem 3.2.5 is now an immediate consequence of Proposition 3.3.1.

3.4.3 Comments on the lower bound

In the usual symmetric simple exclusion process the correlation function is of order $O(\frac{1}{n})$. Since intuitively one could expect that the presence of the slow bond increases the correlation between sites which are located both on the positive half-line or both the negative half-line, the above result does not come as a total surprise.

However, for two sites x and y such that $x \leq 0 < 1 \leq y$, then at first sight it seems to be a reasonable guess that the correlations decrease, and they should be at most of order $O(\frac{1}{n})$. Yet, our proof yields the same bound as above when one restricts only to such kind of pairs of vertices (x, y). A natural question therefore is if a matching lower bound in (3.15) holds. Since our assumptions on the initial measure do not exclude the choice of a product Bernoulli measure with constant intensity, in which case at any time $t \geq 0$ the covariance between two distinct points is zero, such a lower bound certainly cannot hold in general.

Nevertheless, we argue that there are indeed choices of the various parameters in our model for which $|\varphi^n_t(x, y)|$ is bounded from below by a constant times $\log n/n$ uniformly in $t \in [0, T]$. We will not provide all the details, yet the gaps can be filled by an adaptation of the techniques used in Section **??**. We choose $\mu_n \sim \otimes_{x \in \mathbb{Z}} \text{Ber}(\rho_x)$, where

$$\rho_x \;=\; \begin{cases} \frac{1}{2}, & \text{if } x \leq 0, \\ \frac{1}{4}, & \text{otherwise.} \end{cases} \quad (3.65)$$

Analyzing carefully the proof of Theorem 3.2.5, we see that in order to establish the desired lower bound it is enough to show that there exists a constant $c > 0$ such that for all $t \in [0,T]$

$$|\rho_t^n(0) - \rho_t^n(1)| \;\geq\; c\,, \tag{3.66}$$

and that the rightmost local time term in (3.60) is bounded from below by a constant times $\log n$. We only provide a sketch of the argument for the former statement, the latter as mentioned above can be deduced by an application of the techniques developed in Section **??**. We note that it is possible to show that

$$\rho_t^n(0) \;=\; \sum_{x\in\mathbb{Z}} \mathbb{P}_0\big[\mathbf{X}_t = x\big]\rho_0^n(x) \quad \text{and} \quad \rho_t^n(1) \;=\; \sum_{x\in\mathbb{Z}} \mathbb{P}_1\big[\mathbf{X}_t = x\big]\rho_0^n(x), \tag{3.67}$$

where X denotes a random walk with generator $n^2\mathcal{A}_n$, and for $z \in \mathbb{Z}$ we denoted by $\mathbb{P}_z$ the distribution of X when started in z. Using that by symmetry $\mathbb{P}_1[\mathbf{X}_t \geq 1] = \mathbb{P}_0[\mathbf{X}_t \leq 0]$ and $\mathbb{P}_1[\mathbf{X}_t \leq 0] = \mathbb{P}_0[\mathbf{X}_t \geq 1]$, as well as our choice of μ_n, we see that

$$\rho_t^n(0) - \rho_t^n(1) \;=\; \tfrac{1}{4}\Big(\mathbb{P}_0\big[\mathbf{X}_t \leq 0\big] - \mathbb{P}_0\big[\mathbf{X}_t \geq 1\big]\Big)\,. \tag{3.68}$$

It is now possible to argue that a random walk that starts at zero, and that is reflected at zero has a local time of order n up to times of order n^2 at the origin. Using a coupling argument one may then show that one can choose α small enough so that the probability that X, when started at 0, crosses the bond $(0,1)$ before time Tn^2 becomes arbitrarily small. This readily yields that (3.68) is indeed strictly bounded away from zero uniformly in $t \in [0,T]$, and consequently we obtain a lower bound that matches the order of the upper bound in (3.15).

Remark 3.4.3. As argued above, at first sight it seems counterintuitive that $\varphi_t(x,y)$ is of order $\log n/n$ if $x \leq 0 < 1 \leq y$. Yet, an intuitive explanation for that phenomenon could be as follows: given an exclusion particle starting at $x \leq 0$, then up to time say $\frac{t}{2}n^2$ there is a strictly positive probability that it will cross the bond $\{0,1\}$, and afterwards it will have interaction with a particle started at $y \geq 1$ of same order as if it had started at a site $x \geq 1$.

3.5 Proof of density fluctuations

In this section we prove Theorem 3.2.8. We follow the usual procedure to establish such a result, i.e., first we establish tightness of the sequence of density fields $\{\mathcal{Y}_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ and afterwards we characterize the limit. Before proceeding, we introduce in the next subsection some martingales associated with the density fluctuation field defined in (3.20).

3.5.1 Associated martingales

Fix a test function $f \in \mathcal{S}_\alpha(\mathbb{R})$. By Dynkin's formula,

$$\mathcal{M}_t^n(f) \;:=\; \mathcal{Y}_t^n(f) - \mathcal{Y}_0^n(f) - \int_0^t (n^2\mathcal{L}_n + \partial_s)\,\mathcal{Y}_s^n(f)\,ds \tag{3.69}$$

is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s, s \le t)$. Our aim is to write this martingale in a more suitable form. Recall (3.3). Performing simple calculations,

$$
\begin{aligned}
&n^2\mathcal{L}_n\mathcal{Y}^n_s(f) = \\
&= n^2\sum_{x\in\mathbb{Z}}\xi^n_{x,x+1}\Big[\tfrac{1}{\sqrt{n}}\sum_{y\in\mathbb{Z}} f(\tfrac{y}{n})(\eta_s^{x,x+1}(y)-\rho^n_s(y)) - \tfrac{1}{\sqrt{n}}\sum_{y\in\mathbb{Z}} f(\tfrac{y}{n})(\eta_s(y)-\rho^n_s(y))\Big]\\
&= \tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\xi^n_{x,x+1}\Big\{\eta_s(x)\Big[f(\tfrac{x+1}{n})-f(\tfrac{x}{n})\Big]+\eta_s(x+1)\Big[f(\tfrac{x}{n})-f(\tfrac{x+1}{n})\Big]\Big\}\\
&= \tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\Big\{\xi^n_{x,x+1}\Big[f(\tfrac{x+1}{n})-f(\tfrac{x}{n})\Big]+\xi^n_{x-1,x}\Big[f(\tfrac{x-1}{n})-f(\tfrac{x}{n})\Big]\Big\}\eta_s(x)\\
&= \tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\mathcal{A}_n f(\tfrac{x}{n})\eta_s(x)\,,
\end{aligned}
$$

where the operator $\mathcal{A}_n$ has been defined in (3.11). Recalling (3.10) we get that

$$
\partial_s\,\mathcal{Y}^n_s(f) \;=\; -\tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} f(\tfrac{x}{n})\partial_s\rho^n_s(x) = -\tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\mathcal{A}_n f(\tfrac{x}{n})\rho^n_s(x)\,. \tag{3.70}
$$

Combining the previous equalities, we see that

$$
\mathcal{M}^n_t(f) \;=\; \mathcal{Y}^n_t(f)-\mathcal{Y}^n_0(f)-\int_0^t \tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\mathcal{A}_n f(\tfrac{x}{n})\overline{\eta}_s(x)\,ds. \tag{3.71}
$$

Adding and subtracting the term $\int_0^t \mathcal{Y}^n_s(\Delta_\alpha f)ds$, we can rewrite the martingale $\mathcal{M}^n_t(f)$ as

$$
\mathcal{M}^n_t(f) \;=\; \mathcal{Y}^n_t(f)-\mathcal{Y}^n_0(f)-\int_0^t \mathcal{Y}^n_s(\Delta_\alpha f)ds - R^n_t(f)\,, \tag{3.72}
$$

where

$$
R^n_t(f) \;:=\; \int_0^t \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}}\Big\{n^2\mathcal{A}_n f(\tfrac{x}{n})-(\Delta_\alpha f)(\tfrac{x}{n})\Big\}\overline{\eta}_s(x)\,ds\,.
$$

The next lemma allows us to control the error term $R^n_t(f)$ defined in the previous display, which is obtained by replacing the discrete operator $\mathcal{A}_n$ defined in (3.11) by the continuous Laplacian Δ_α defined in (3.8).

Lemma 3.5.1. *For any $f \in \mathcal{S}_\alpha(\mathbb{R})$, almost surely there exists a constant $c > 0$ such that for all $t \in [0,T]$ and all $n \in \mathbb{N}$ the estimate $|R^n_t(f)| \le \frac{ct}{\sqrt{n}}$ holds.*

Proof. We begin by splitting $R^n_t(f)$ as the sum

$$
R^n_t(f) \;=\; \int_0^t \frac{1}{\sqrt{n}}\sum_{x\neq 0,1}\Big\{n^2\mathcal{A}_n f(\tfrac{x}{n})-(\Delta_\alpha f)(\tfrac{x}{n})\Big\}\overline{\eta}_s(x)\,ds \tag{3.73}
$$

$$
+\int_0^t \frac{1}{\sqrt{n}}\Big\{n^2\mathcal{A}_n f(\tfrac{0}{n})-(\Delta_\alpha f)(\tfrac{0}{n})\Big\}\overline{\eta}_s(0)\,ds \tag{3.74}
$$

$$
+\int_0^t \frac{1}{\sqrt{n}}\Big\{n^2\mathcal{A}_n f(\tfrac{1}{n})-(\Delta_\alpha f)(\tfrac{1}{n})\Big\}\overline{\eta}_s(1)\,ds\,. \tag{3.75}
$$

We begin by dealing with (3.73). Recall that $f \in \mathcal{S}_\alpha(\mathbb{R})$ and note that $|\overline{\eta}_s(x)| \leq 2$. Thus, taking advantage of the fact that for $x \notin \{0,1\}$, the term $n^2\mathcal{A}_n f\big(\frac{x}{n}\big)$ is the discrete Laplacian, and applying a Taylor expansion up to second order with the Lagrangian form of the remainder, we see that (3.73) is bounded by

$$\begin{aligned}\frac{t}{\sqrt{n}}\sum_{x\neq 0,1}\Big|n^2\Big\{\Big[\tfrac{1}{n}f'\big(\tfrac{x}{n}\big)+\tfrac{1}{2n^2}f''\big(\tfrac{x}{n}\big)+\tfrac{f'''\big(\vartheta^+(\frac{x}{n})\big)}{3!n^3}\Big]\\ -\Big[\tfrac{1}{n}f'\big(\tfrac{x}{n}\big)-\tfrac{1}{2n^2}f''\big(\tfrac{x}{n}\big)+\tfrac{f'''\big(\vartheta^-(\frac{x}{n})\big)}{3!n^3}\Big]\Big\}-\big(\Delta_\alpha f\big)\big(\tfrac{x}{n}\big)\Big|\\ =\frac{t}{\sqrt{n}}\sum_{x\neq 0,1}\Big|\tfrac{f'''\big(\vartheta^+(\frac{x}{n})\big)}{3!n^3}-\tfrac{f'''\big(\vartheta^-(\frac{x}{n})\big)}{3!n^3}\Big\}\Big|,\end{aligned}$$

where $\vartheta^+(\frac{x}{n}) \in [\frac{x}{n},\frac{x+1}{n}]$ and $\vartheta^-(\frac{x}{n}) \in [\frac{x-1}{n},\frac{x}{n}]$. Since f''' is integrable, we conclude that (3.73) is of order $O(tn^{-5/2})$, and vanishes as n tends to infinity. Since $\Delta_\alpha f$ is bounded, we can see that the sum of (3.74) and (3.75) is equal to

$$\int_0^t \frac{1}{\sqrt{n}}\Big\{n^2\mathcal{A}_n f\big(\tfrac{0}{n}\big)\Big\}\overline{\eta}_s(0)\,ds+\int_0^t \frac{1}{\sqrt{n}}\Big\{n^2\mathcal{A}_n f\big(\tfrac{1}{n}\big)\Big\}\overline{\eta}_s(1)\,ds$$

plus a term of order $O(\frac{t}{\sqrt{n}})$. Applying the definition of $\mathcal{A}_n$, the expression above is equal to

$$\begin{aligned}&\int_0^t \frac{n^2}{\sqrt{n}}\Big\{\frac{\alpha}{n}\Big(f\big(\tfrac{1}{n}\big)-f\big(\tfrac{0}{n}\big)\Big)+\Big(f\big(\tfrac{-1}{n}\big)-f\big(\tfrac{0}{n}\big)\Big)\Big\}\overline{\eta}_s(0)\,ds\\ +&\int_0^t \frac{n^2}{\sqrt{n}}\Big\{\frac{\alpha}{n}\Big(f\big(\tfrac{0}{n}\big)-f\big(\tfrac{1}{n}\big)\Big)+\Big(f\big(\tfrac{2}{n}\big)-f\big(\tfrac{1}{n}\big)\Big)\Big\}\overline{\eta}_s(1)\,ds\,,\end{aligned}$$

and we can see that the absolute value of expression above is bounded by

$$\begin{aligned}t\sqrt{n}\Big\{\,&\Big|\alpha\Big(f\big(\tfrac{1}{n}\big)-f\big(\tfrac{0}{n}\big)\Big)+n\Big(f\big(\tfrac{-1}{n}\big)-f\big(\tfrac{0}{n}\big)\Big)\Big|\\ &+\Big|\alpha\Big(f\big(\tfrac{0}{n}\big)-f\big(\tfrac{1}{n}\big)\Big)+n\Big(f\big(\tfrac{2}{n}\big)-f\big(\tfrac{1}{n}\big)\Big)\Big|\,\Big\}.\end{aligned}\tag{3.76}$$

Since $f \in \mathcal{S}_\alpha(\mathbb{R})$, we have the boundary conditions $\alpha\big(f(0^+)-f(0^-)\big) = \partial_u f(0^+) = \partial_u f(0^-)$ and also that f is left continuous at zero, hence

$$\begin{aligned}f\big(\tfrac{1}{n}\big)-f\big(\tfrac{0}{n}\big) &= \Big[f(0^+)-f(0^-)\Big]+O(1/n)\,,\\ n\Big[f\big(\tfrac{-1}{n}\big)-f\big(\tfrac{0}{n}\big)\Big] &= -\partial_u f(0^-)+O(1/n)\,,\\ f\big(\tfrac{0}{n}\big)-f\big(\tfrac{1}{n}\big) &= -\Big[f(0^+)-f(0^-)\Big]+O(1/n)\,,\\ n\Big[f\big(\tfrac{2}{n}\big)-f\big(\tfrac{1}{n}\big)\Big] &= \partial_u f(0^+)+O(1/n)\,,\end{aligned}$$

which permits to conclude that (3.76) is of order $O(\frac{t}{\sqrt{n}})$, finishing the proof. □

Now we study the convergence of the sequence of martingales $\{\mathcal{M}_t^n(f) : t \in [0,T]\}_{n\in\mathbb{N}}$. This is the content of the next lemma.

Lemma 3.5.2. *For any $f \in \mathcal{S}_\alpha(\mathbb{R})$, the sequence of martingales $\{\mathcal{M}_t^n(f) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution under the topology of $\mathcal{D}([0,T],\mathbb{R})$, as $n \to \infty$, to a mean-zero Gaussian process $\{\mathcal{M}_t(f) : t \in [0,T]\}$ of quadratic variation given by*

$$\begin{aligned}\langle \mathcal{M}(f)\rangle_t \;&=\; \int_0^t \int_{\mathbb{R}} 2\chi(\rho_s(u))(\nabla_\alpha f(u))^2\, du\, ds \\ &+ \int_0^t \Big[\rho_s(0^-)(1-\rho_s(0^+)) + \rho_s(0^+)(1-\rho_s(0^-))\Big]\nabla_\alpha f(0^+)\, ds\,.\end{aligned} \tag{3.77}$$

Proof. The proof of this lemma consists on applying [15, Theorem VIII.3.12, page 473]. According to that theorem, we have to check:

i) condition (3.14), defined in [15, page 474],

ii) condition [$\hat{\delta}_5$-D], defined in [15, 3.4, page 470],

iii) condition [γ_5-D], defined in [15, 3.3, page 470].

By [15, Assertion VIII.3.5, page 470], both conditions [$\hat{\delta}_5$-D] and (3.14) are a consequence of

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\Big[\sup_{s\le t}\big|\mathcal{M}_s^n(f) - \mathcal{M}_{s-}^n(f)\big|\Big] \;=\; 0. \tag{3.78}$$

To show (3.78), note that only two sites of the configuration η change its values when a jump occurs. Therefore,

$$\sup_{s\le t}\big|\mathcal{M}_s^n(f) - \mathcal{M}_{s-}^n(f)\big| \;=\; \sup_{s\le t}\big|\mathcal{Y}_s^n(f) - \mathcal{Y}_{s-}^n(f)\big| \;\le\; \frac{2\|f\|_\infty}{\sqrt{n}}\,,$$

leading to (3.78). It remains to check Condition [γ_5-D], i.e., the convergence in probability of the quadratic variation of $\mathcal{M}_t(f)$, which is given by

$$\langle \mathcal{M}^n(f)\rangle_t \;=\; \int_0^t n^2\Big[\mathcal{L}_n\mathcal{Y}_s^n(f)^2 - 2\mathcal{Y}_s^n(f)\mathcal{L}_n\mathcal{Y}_s^n(f)\Big]\, ds\,.$$

After some elementary computations, the right hand-side of the display above can be rewritten as

$$\begin{aligned}&\int_0^t \frac{1}{n}\sum_{x\neq 0}(\eta_s(x)-\eta_s(x+1))^2\Big[n\Big(f\big(\tfrac{x+1}{n}\big) - f\big(\tfrac{x}{n}\big)\Big)\Big]^2 ds \\ &+\alpha\int_0^t (\eta_s(0)-\eta_s(1))^2\Big(f\big(\tfrac{1}{n}\big) - f\big(\tfrac{0}{n}\big)\Big)^2 ds\,.\end{aligned} \tag{3.79}$$

which is an *additive functional* of the exclusion process η_t. It is almost folklore in the literature that Theorem 3.2.1 together with a suitable *Replacement Lemma* and standard computations yield that (3.79) converges in distribution to the right hand-side of (3.77) as $n \to \infty$. Since this is not the main issue of the proof, and since such a Replacement Lemma under the slow bond's presence has been studied in previous works (as in [7, Lemma 5.4] for instance), we do not present the proof of this result with full details, but only a sketch instead.

By a *Replacement Lemma* we mean a result allowing to replace the time integral of the occupation number $\eta_t(x)$ by an average on a box around x. The only difference with respect to the usual Replacement Lemma (see [18]), is the fact that we should avoid an intersection between this box and the slow bond in our setting. Hence, we define

$$\eta^{\ell}(x) \;=\; \begin{cases} \dfrac{1}{\ell}\displaystyle\sum_{y=x}^{x+\ell-1}\eta(y)\,, & \text{for } x\geq 1\,,\\[2ex] \dfrac{1}{\ell}\displaystyle\sum_{y=x-\ell+1}^{x}\eta(y)\,, & \text{for } x\leq 0\,,\end{cases}$$

which is related to the side limits appearing in (3.77). Taking into account these definitions, the fact that $\eta_t(x)^2 = \eta_t(x)$, and the boundary condition of f at zero, one can show that the limit in distribution of (3.79) is in fact the right hand-side of (3.77).

Since the right hand-side of (3.77) is deterministic, the convergence in distribution implies the convergence in probability, and this finishes the proof of the lemma. □

3.5.2 Tightness

Let $\mathcal{S}$ be a Frechét space (see [23] for a definition of a Frechét space) and denote by $\mathcal{S}'$ its topological dual. We cite here the following useful criterion:

Proposition 3.5.3 (Mitoma's criterion, [21]). *A sequence of processes $\{x_t; t\in[0,T]\}_{n\in\mathbb{N}}$ in $\mathcal{D}([0,T],\mathcal{S}')$ is tight with respect to the Skorohod topology if, and only if, the sequence $\{x_t(f); t\in[0,T]\}_{n\in\mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0,T],\mathbb{R})$, for any $f\in\mathcal{S}$.*

Since $\mathcal{S}_\alpha(\mathbb{R})$ is a Frechét space (see [8]), tightness of the density field is reduced to showing tightness of a family of real-valued processes. For that purpose, let $f\in\mathcal{S}_\alpha(\mathbb{R})$. Since the sum of tight processes is also tight, in order to prove tightness of $\{\mathcal{Y}_t^n(f): t\in[0,T]\}_{n\in\mathbb{N}}$ it is enough to prove tightness of the remaining processes appearing in (3.72), namely $\{\mathcal{Y}_0^n(f)\}_{n\in\mathbb{N}}$, $\{\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f)\,ds : t\in[0,T]\}_{n\in\mathbb{N}}$, $\{\mathcal{M}_t^n(f): t\in[0,T]\}_{n\in\mathbb{N}}$ and $\{R_t^n(f): t\in[0,T]\}_{n\in\mathbb{N}}$. We deal with all of them separately.

Observe that

$$\mathbb{E}_{\nu^n_{\rho_0}(\cdot)}\Big[\Big(\mathcal{Y}_0^n(f)\Big)^2\Big] \;=\; \frac{1}{n}\sum_{x\in\mathbb{Z}} f^2\Big(\frac{x}{n}\Big)\chi(\rho_0^n(x)) + \frac{2}{n}\sum_{x<y} f\Big(\frac{x}{n}\Big)f\Big(\frac{y}{n}\Big)\varphi_0^n(x,y)$$

is bounded. As a consequence of Assumption **(B)** in Theorem 3.2.8 the sequence of initial conditions $\mathcal{Y}_0^n$ converges, therefore it is also tight.

By Lemma 3.5.1, the sequence of processes $\{R_t^n(f): t\in[0,T]\}_{n\in\mathbb{N}}$ is negligible, thus it is tight.

By Lemma 3.5.2 the sequence of martingales $\{\mathcal{M}_t^n(f): t\in[0,T]\}_{n\in\mathbb{N}}$ converges, hence it is tight as well.

It remains to prove tightness of the integral terms $\{\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f)\, ds : t \in [0,T]\}_{n\in\mathbb{N}}$. At this point we invoke *Aldous' criterion*:

Proposition 3.5.4 (Aldous' criterion)**.** *A sequence $\{x_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathcal{D}([0,T],\mathbb{R})$ if:*

i) $\lim\limits_{A\to+\infty} \limsup\limits_{n\to+\infty} \mathbb{P}\Big(\sup\limits_{0\le t\le T} |x_t^n| > A\Big) \;=\; 0\,,$

ii) *for any* $\varepsilon > 0\,, \lim\limits_{\delta\to 0} \limsup\limits_{n\to+\infty} \sup\limits_{\lambda\le\delta} \sup\limits_{\tau\in\mathcal{T}_T} \mathbb{P}(|x_{\tau+\lambda}^n - x_\tau^n| > \varepsilon) \;=\; 0\,,$

where $\mathcal{T}_T$ is the set of stopping times bounded by T.

We first check the first item of Aldous' criterion. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}_{\mu^n}\Big[\sup_{t\le T}\Big(\int_0^t \mathcal{Y}_s^n(\Delta_\alpha f)\, ds\Big)^2\Big]\\
&\le\; T\int_0^T \mathbb{E}_{\mu^n}\Big[\Big(\frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} \Delta_\alpha f(\tfrac{x}{n})(\eta_s(x) - \rho_s^n(x))\Big)^2\Big]\, ds\,.
\end{aligned}$$

Observe that the right hand-side of the display above is bounded by T^2 times

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\big(\Delta_\alpha f(\tfrac{x}{n})\big)^2 \sup_{t\le T}\chi(\rho_t^n(x)) + \frac{2}{n}\sum_{x<y}\Delta_\alpha f(\tfrac{x}{n})\Delta_\alpha f(\tfrac{y}{n})\sup_{t\le T}\varphi_t^n(x,y)\,, \tag{3.80}$$

where $\chi(\rho_t^n(x))$ was defined above (3.16) and $\varphi_t^n(x,y)$ is given in Definition 8. Since $f \in \mathcal{S}_\alpha(\mathbb{R})$, the first term in (3.80) may be easily shown to be bounded in n. As for the second term the estimate provided by Theorem 3.2.5 is unfortunately not quite enough. Yet, Proposition 3.4.2 in combination with Proposition 3.3.1 show that for some constants $c_1, c_2 > 0$ that do not depend on t, and (x,y) we have that for all $n \in \mathbb{N}$,

$$\varphi_t^n(x,y) \;\le\; \frac{c_1}{n} + \frac{c_2}{n}\int_0^{Tn^2} \mathbf{P}_{(x,y)}\Big[(\mathbf{X}_s, \mathbf{Y}_s) = (0,1)\Big]\, ds\,, \tag{3.81}$$

where $\{(\mathbf{X}_t, \mathbf{Y}_t);\ t \ge 0\}$ is defined in Subsection 3.3.1. Plugging the first term on the right hand-side of the display above into the second term in (3.80) gives the desired estimate. To deal with the second term on the right hand-side of (3.81) we use the fact that by (3.48) we can estimate the integral term from above by

$$c\sum_{z\in\mathcal{A}}\int_0^{2Tn^2} \mathbb{P}_{(x,y)}\Big[(X_s, Y_s) = z\Big]\, ds\,, \tag{3.82}$$

where (X,Y) denotes simple random walk on $\mathbb{Z}^2$ jumping at total rate 2, $\mathcal{A}$ denotes the set $\{(0,1),(1,1),(0,0),(1,0)\}$, and $c \in (0,+\infty)$ is some constant. Plugging this into the second term in (3.80), and using the reversibility of (X,Y) we see that we obtain a term that is bounded from above by a constant times

$$\frac{1}{n^2}\sum_{z\in\mathcal{A}}\int_0^{2Tn^2} \mathbb{E}_z\big[|\Delta_\alpha f(\tfrac{X_s}{n})\Delta_\alpha f(\tfrac{Y_s}{n})|\big]\, ds\,. \tag{3.83}$$

Since $|\Delta_\alpha f(\frac{x}{n})\Delta_\alpha f(\frac{y}{n})|$ is uniformly bounded in x and y we finally obtain that (3.80) is bounded by a constant, which implies condition **i)** of Aldous' criterion via Chebychev's inequality.

We now check **ii)**. For this purpose, fix a stopping time $\tau \in \mathcal{T}_T$. By Chebychev's inequality and repeating the argument above, we have that

$$\mathbb{P}_{\mu^n}\Big(\Big|\int_\tau^{\tau+\lambda} \mathcal{Y}_s^n(\Delta_\alpha f)\,ds\,\Big| > \varepsilon\Big) \;\leq\; \frac{1}{\varepsilon^2}\mathbb{E}_{\mu^n}\Big[\Big(\int_\tau^{\tau+\lambda} \mathcal{Y}_s^n(\Delta_\alpha f)\,ds\,\Big)^2\Big] \;\leq\; \frac{\delta^2 c}{\varepsilon^2}\,,$$

which vanishes as $\delta \to 0$, and yields tightness of the integral term, and concludes therefore the proof.

3.5.3 Uniqueness of the Ornstein-Uhlenbeck process

The existence of the Ornstein-Uhlenbeck process solution of (3.19) is a consequence of tightness proved in Subsection 3.5.2. This subsection is devoted to the proof of uniqueness of this process, as stated in Proposition 3.2.7. The guideline is mainly inspired by [14, 18].

In the proof of Proposition 3.2.7 we make use of the following result, which is a standard fact about local martingales.

Proposition 3.5.5. *If M_t is a local martingale with respect to a filtration $\mathcal{F}_t$ and*

$$\mathbb{E}\Big[\sup_{0\leq s\leq t}|M_s|\Big] \;<\; +\infty \tag{3.84}$$

for any $t \geq 0$, then M_t is a martingale.

Proof. Let τ_n be a sequence of stopping times such that $\tau_n \to \infty$ as $n \to \infty$ and such that the stopped process $(M_{t\wedge\tau_n})_{t\geq 0}$ is a martingale for each n. Let $s < t$, it then follows that for any $A \in \mathcal{F}_s$,

$$\mathbb{E}\big[M_{t\wedge\tau_n}\mathbf{1}_A\big] \;=\; \mathbb{E}\big[M_{s\wedge\tau_n}\mathbf{1}_A\big]\,.$$

Letting $n \to \infty$, using (3.84) and the Dominated Convergence Theorem, we conclude that

$$\mathbb{E}\big[M_t\mathbf{1}_A\big] \;=\; \mathbb{E}\big[M_s\mathbf{1}_A\big]\,,$$

thus finishing the proof. □

Proof of Proposition 3.2.7. Fix $f \in \mathcal{S}_\alpha(\mathbb{R})$ and $s > 0$. Recall the definition of the martingales $\mathcal{M}_t(f)$ and $\mathcal{N}_t(f)$ given in (3.17) and (3.18), respectively.

We claim that the process $\{X_t^s(f)\,:\,t \geq s\}$ defined by

$$X_t^s(f) \;=\; \exp\left\{\frac{1}{2}\int_s^t \|\nabla_\alpha f\|^2_{\rho_r(\cdot)}\,dr + i\Big(\mathcal{Y}_t(f) - \mathcal{Y}_s(f) - \int_s^t \mathcal{Y}_r(\Delta_\alpha f)\,dr\Big)\right\}$$

is a (complex) martingale. By [24, pp. 148, Proposition 3.4] it is immediate that $X_t^s(f)$ is a local martingale. Therefore, if we show that

$$\mathbb{E}\Big[\sup_{s\leq u\leq t}|X_u^s(f)|\Big] \;<\; +\infty\,, \tag{3.85}$$

then, by Proposition 3.5.5, we conclude that $X_t^s(f)$ is a martingale. But (3.85) is a simple consequence of the fact that the function $t\mapsto \frac{1}{2}\int_0^t \|\nabla_\alpha f\|^2_{\rho_s(\cdot)}ds$ is continuous, hence bounded on compact sets. Therefore, the claim is proved.

Fix $S>0$. We claim now that the process $\{Z_t\,:\,0\leq t\leq S\}$ defined by

$$Z_t(f) \;=\; \exp\Big\{\frac{1}{2}\int_0^t \|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_r(\cdot)}\,dr + i\,\mathcal{Y}_t(T^\alpha_{S-t}f)\Big\}$$

is also a martingale. To prove this second claim, consider two times $0\leq t_1<t_2\leq S$ and a partition of the interval $[t_1,t_2]$ in n intervals of equal size, that is, $t_1=s_0<s_1<\cdots<s_n=t_2$, with $s_{j+1}-s_j=(t_2-t_1)/n$. Observe now that

$$\begin{aligned}\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^\alpha_{S-s_j}f) \;=\; \exp\Bigg\{&\sum_{j=0}^{n-1}\frac{1}{2}\int_{s_j}^{s_{j+1}}\|\nabla_\alpha T^\alpha_{S-s_j}f\|^2_{\rho_s(\cdot)}\,ds\\ &+i\sum_{j=0}^{n-1}\Big(\mathcal{Y}_{s_{j+1}}(T^\alpha_{S-s_j}f)-\mathcal{Y}_{s_j}(T^\alpha_{S-s_j}f)-\int_{s_j}^{s_{j+1}}\mathcal{Y}_r(\Delta_\alpha T^\alpha_{S-s_j}f)\,dr\Big)\Bigg\}.\end{aligned}$$

Due to smoothness of $T_t^\alpha f$, the first sum in the exponential above converges to

$$\frac{1}{2}\int_{t_1}^{t_2}\|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_s(\cdot)}\,dr,$$

as $n\to+\infty$. The second sum inside the exponential is the same as

$$\begin{aligned}&\mathcal{Y}_{t_2}(T^\alpha_{S-t_2+\frac{1}{n}}f)-\mathcal{Y}_{t_1}(T^\alpha_{S-t_1}f)\\ &+\sum_{j=1}^{n-1}\Big(\mathcal{Y}_{s_j}(T^\alpha_{S-s_{j-1}}f-T^\alpha_{S-s_j}f)-\int_{s_j}^{s_{j+1}}\mathcal{Y}_r(\Delta_\alpha T^\alpha_{S-s_j}f)\,dr\Big).\end{aligned}$$

Since $\mathcal{Y}\in\mathcal{C}([0,T],\mathcal{S}'_\alpha(\mathbb{R}))$, since $T_t^\alpha f$ is continuous in time and applying the expansion $T^\alpha_{t+\varepsilon}f-T^\alpha_tf=\varepsilon\Delta_\alpha T^\alpha_tf+o(\varepsilon)$, one can show that the almost sure limit of the previous expression is $\mathcal{Y}_{t_2}(T^\alpha_{S-t_2}f)-\mathcal{Y}_{t_1}(T^\alpha_{S-t_1}f)$, see [8, 10] for more details. We have henceforth deduced that

$$\begin{aligned}&\lim_{n\to+\infty}\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^\alpha_{S-s_j}f)\\ &=\;\exp\Big\{\frac{1}{2}\int_{t_1}^{t_2}\|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_s(\cdot)}\,dr+i\Big(\mathcal{Y}_{t_2}(T^\alpha_{S-t_2}f)-\mathcal{Y}_{t_1}(T^\alpha_{S-t_1}f)\Big)\Big\}\;=\;\frac{Z_{t_2}}{Z_{t_1}}\,.\end{aligned}$$

Since the complex exponential is bounded, the Dominated Convergence Theorem ensures also the convergence in L^1. Thus,

$$\mathbb{E}\Big[g\,\frac{Z_{t_2}}{Z_{t_1}}\Big] \;=\; \lim_{n\to+\infty}\mathbb{E}\Big[g\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^\alpha_{S-s_j}f)\Big]\,,$$

for any bounded function g. Take g bounded and $\mathcal{F}_{t_1}$-measurable. For any $f\in\mathcal{S}_\alpha(\mathbb{R})$, the process $X^s_t(f)$ is a martingale. Thus, taking the conditional expectation with respect to $\mathcal{F}_{s_{n-1}}$, we get

$$\mathbb{E}\Big[g\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^\alpha_{S-s_j}f)\Big] \;=\; \mathbb{E}\Big[g\prod_{j=0}^{n-2}X^{s_j}_{s_{j+1}}(T^\alpha_{S-s_j}f)\Big]\,.$$

By induction, we conclude that

$$\mathbb{E}\Big[g\,\frac{Z_{t_2}}{Z_{t_1}}\Big] \;=\; \mathbb{E}\big[\,g\,\big]\,,$$

for any bounded and $\mathcal{F}_{t_1}$-measurable function g. This assures that $\{Z_t\,:\,t\geq 0\}$ is a martingale. From $\mathbb{E}[Z_t|\mathcal{F}_s]=Z_s$, we get

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\int_0^t\|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_r(\cdot)}\,dr+i\,\mathcal{Y}_t(T^\alpha_{S-t}f)\Big\}\Big|\mathcal{F}_s\Big]\\&\;=\;\exp\Big\{\frac{1}{2}\int_0^s\|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_r(\cdot)}\,dr+i\,\mathcal{Y}_s(T^\alpha_{S-s}f)\Big\}\,,\end{aligned}$$

which leads to

$$\mathbb{E}\Big[\exp\Big\{i\,\mathcal{Y}_t(T^\alpha_{S-t}f)\Big\}\Big|\mathcal{F}_s\Big] \;=\; \exp\Big\{-\frac{1}{2}\int_s^t\|\nabla_\alpha T^\alpha_{S-r}f\|^2_{\rho_r(\cdot)}\,dr+i\,\mathcal{Y}_s(T^\alpha_{S-s}f)\Big\}\,.$$

Choosing $S=t$ and replacing f by λf, we achieve

$$\mathbb{E}\Big[\exp\Big\{i\,\lambda\,\mathcal{Y}_t(f)\Big\}\Big|\mathcal{F}_s\Big] \;=\; \exp\Big\{-\frac{\lambda^2}{2}\int_s^t\|\nabla_\alpha T^\alpha_{t-r}f\|^2_{\rho_r(\cdot)}\,dr+i\,\lambda\,\mathcal{Y}_s(T^\alpha_{t-s}f)\Big\}\,,$$

meaning that, conditionally to $\mathcal{F}_s$, the random variable $\mathcal{Y}_t(f)$ has Gaussian distribution of mean $\mathcal{Y}_s(T^\alpha_{t-s}f)$ and variance $\int_s^t\|\nabla_\alpha T^\alpha_r f\|^2_{\rho_s(\cdot)}\,dr$.

We claim now that this last result implies the uniqueness of the finite dimensional distributions of the process $\{\mathcal{Y}_t(f)\,:\,t\in[0,T]\}$. For the sake of clarity, consider only two times, $t_0=0$ and $t_1>0$, two test functions $f_0,f_1\in\mathcal{S}_\alpha(\mathbb{R})$ and two Lebesgue measurable sets A_0 and A_1. By conditioning,

$$\mathbb{P}\Big[\mathcal{Y}_{t_1}(f_1)\in A_1,\mathcal{Y}_{t_0}(f_0)\in A_0\Big] \;=\; \mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{[\mathcal{Y}_{t_1}(f_1)\in A_1]}\big|\mathcal{F}_0\big]\cdot\big[\mathbf{1}_{[\mathcal{Y}_{t_0}(f_0)\in A_0]}\big]\Big]\,.$$

Since the conditional expectation $\mathbb{E}\big[\mathbf{1}_{[\mathcal{Y}_{t_1}(f_1)\in A_1]}\big|\mathcal{F}_0\big]$ is a function of $\mathcal{Y}_{t_0}(f_1)$ and $\mathcal{Y}_{t_0}$ is uniquely distributed as a random element of $\mathcal{S}'_\alpha(\mathbb{R})$ (by assumption **ii)** of Proposition 3.2.7), we get that the distribution of the vector $(\mathcal{Y}_{t_1}(f_1),\mathcal{Y}_{t_0}(f_0))$ is also uniquely distributed. The generalization for a general finite number of times is straightforward.

This proves the claim, implying the uniqueness in law of the random element $\mathcal{Y}$ and hence finishing the proof.

□

3.5.4 Characterization of limit points

From the results of the previous subsection we know that the sequence $\{\mathcal{Y}_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ has limit points. Let $\{\mathcal{Y}_t : t \in [0,T]\}$ be the limit in distribution of $\{\mathcal{Y}_t^n : t \in [0,T]\}_{n\in\mathbb{N}}$ along some subsequence n_k considering the uniform topology of $\mathcal{D}([0,T], \mathcal{S}'_\alpha(\mathbb{R}))$. Abusing of notation, we denote this subsequence simply by n. Our goal here is to prove that $\{\mathcal{Y}_t : t \in [0,T]\}$ satisfies the conditions **i)** and **ii)** of Proposition 3.2.7. Since Proposition 3.7.1 gives us condition **ii)**, it only remains to prove condition **i)**.

For $f \in \mathcal{S}_\alpha(\mathbb{R})$, let $\mathcal{M}_t$ and $\mathcal{N}_t$ be the processes *defined* by

$$\begin{aligned} \mathcal{M}_t(f) &= \mathcal{Y}_t(f) - \mathcal{Y}_0(f) - \int_0^t \mathcal{Y}_s(\Delta_\alpha f) ds\,, \\ \mathcal{N}_t(f) &= \big(\mathcal{M}_t(f)\big)^2 - \int_0^t \|\nabla_\alpha f\|^2_{\rho_s(\cdot)}\, ds\,. \end{aligned}$$

Since $\mathcal{Y}_t^n$ is assumed to converge in distribution to $\mathcal{Y}_t$ as $n \to +\infty$, by (3.72) and Lemma 3.5.1, we conclude that $\mathcal{M}_t(f)$ defined above coincides with the limit of $\mathcal{M}_t^n(f)$ as in Lemma 3.5.2, which was denoted by $\mathcal{M}_t(f)$ as well.

By Lemma 3.5.2, we already know that $\mathcal{M}_t(f)$ has quadratic variation given by $\int_0^t \|\nabla_\alpha f\|^2_{\rho_s(\cdot)}\, ds$. Therefore, if we show that $\mathcal{M}_t(f)$ is a martingale, then we will immediately get that $\mathcal{N}_t(f)$ is also a martingale.

Hence, we claim that $\mathcal{M}_t(f)$ is a martingale. First of all, we fix the filtration, which will be the natural one: $\mathcal{F}_t = \{\sigma(\mathcal{Y}_s(g)) : s \le t \text{ and } g \in \mathcal{S}_\alpha(\mathbb{R})\}$. Thus, $\mathcal{M}_t(f)$ is $\mathcal{F}_t$-measurable. The fact that $\mathcal{M}_t(f)$ is in L^1 for any time $t \in [0,T]$ is a consequence that $\mathcal{M}_t(f)$ is a Gaussian process, which was proved in Lemma 3.5.2. Thus, if we prove that

$$\mathbb{E}\big[\mathcal{M}_t(f)\mathbf{1}_U\big] = \mathbb{E}\big[\mathcal{M}_s(f)\mathbf{1}_U\big]\,, \quad \forall\, U \in \mathcal{F}_s\,, \tag{3.86}$$

we will conclude that $\mathcal{M}_t(f)$ is a martingale. To assure (3.86) it is enough to verify it for sets U of the form

$$U = \bigcap_{i=1}^k \big[\mathcal{Y}_{s_i}(f_i) \in A_i\big]$$

for $0 \le s_1 \le \cdots \le s_k \le s$, $f_i \in \mathcal{S}_\alpha(\mathbb{R})$ and A_i measurable sets of $\mathbb{R}$. Since $\mathcal{M}_t^n(f)$ is a martingale,

$$\mathbb{E}\big[\mathcal{M}_t^n(f)\mathbf{1}_{U_n}\big] = \mathbb{E}\big[\mathcal{M}_s^n(f)\mathbf{1}_{U_n}\big]\,, \quad \forall\, U \in \mathcal{F}_s\,, \tag{3.87}$$

where

$$U_n = \bigcap_{i=1}^k \big[\mathcal{Y}_{s_i}^n(f_i) \in A_i\big]$$

for $0 \le s_1 \le \cdots \le s_k \le s$, $f_i \in \mathcal{S}_\alpha(\mathbb{R})$ and A_i are measurable sets of $\mathbb{R}$. Therefore, in order to show (3.86) it is enough to prove the claim that the expectations in (3.87) converge to the respective expectations in (3.86).

Since $\mathcal{Y}_t^n(f)$ converges to $\mathcal{Y}_t(f)$ as $n \to +\infty$, which is concentrated on continuous paths, then $\mathcal{M}_t^n(f)\mathbf{1}_{U_n}$ converges in distribution to $\mathcal{M}_t(f)\mathbf{1}_U$. Thus, by [3, pp

32, Theorem 5.4] in order to get convergence of expectations, it is enough to assure that $\{\mathcal{M}_t^n(f)\mathbf{1}_{U_n}\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence. In its hand, the uniform integrability can be guaranteed by showing that the L^2 norm of $\mathcal{M}_t^n(f)\mathbf{1}_{U_n}$ is uniformly bounded in $n \in \mathbb{N}$. Since the indicator function is bounded by one, we can deal only with the L^2 norm of the martingale $\mathcal{M}_t^n(f)$. Now, applying the Minkowksi inequality to (3.71), we get

$$\begin{aligned}\mathbb{E}_{\mu_n}\Big[\big(\mathcal{M}_t^n(f)\big)^2\Big]^{1/2} \;\leq \mathbb{E}_{\mu_n}\Big[\big(\mathcal{Y}_t^n(f)\big)^2\Big]^{1/2} + \mathbb{E}_{\mu_n}\Big[\big(\mathcal{Y}_0^n(f)\big)^2\Big]^{1/2}\\ + \mathbb{E}_{\mu_n}\Big[\Big(\int_0^t \tfrac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} n^2\mathcal{A}_n f\big(\tfrac{x}{n}\big)\overline{\eta}_s(x)\,ds\Big)^2\Big]^{1/2}.\end{aligned} \tag{3.88}$$

The first term on the right hand-side of (3.88) is bounded by

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\Big(f\big(\tfrac{x}{n}\big)\Big)^2\chi(\rho_t^n(x)) + \frac{2}{n}\sum_{x<y} f\big(\tfrac{x}{n}\big)f\big(\tfrac{y}{n}\big)\varphi_t^n(x,y)\,.$$

Since $|\rho_t^n(x)| \leq 1$, the first parcel in the display above is uniformly bounded in n. To treat the second term of the last display, we use a similar argument to the one used below (3.80). The second term on the RHS of (3.88) is bounded by

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\big(f\big(\tfrac{x}{n}\big)\big)^2\chi(\rho_0^n(x)) + \frac{2}{n}\sum_{x<y} f\big(\tfrac{x}{n}\big)f\big(\tfrac{y}{n}\big)\varphi_0^n(x,y)\,,$$

which is uniformly bounded on $n \in \mathbb{N}$ due to conditions (3.13) and (3.14). Again by a similar argument to the one presented for tightness below (3.80), the third term on the right hand-side of (3.88) is bounded by t^2 times

$$\frac{1}{n}\sum_{x\in\mathbb{Z}}\big(f\big(\tfrac{x}{n}\big)\big)^2\sup_{t\leq T}\chi(\rho_t^n(x)) + \frac{2}{n}\sum_{x<y} f\big(\tfrac{x}{n}\big)f\big(\tfrac{y}{n}\big)\sup_{t\leq T}\varphi_t^n(x,y)\,,$$

thus concluding the characterization of limit points.

3.6 Auxiliary results on random walks

The next result is quite classical, but hard to find in the literature. It is included here for sake of completeness.

Proposition 3.6.1. *Let X be the symmetric simple one-dimensional continuous time random walk. Then,*

$$\int_0^t \mathbb{P}\big[\,X_s = 0\,\big]\,ds \;\leq\; c\sqrt{t}\,,$$

where $c > 0$ is a constant which does not depend on t.

Proof. Let $N := N_{2s}$ a Poisson distribution with parameter $2s$.

$$\begin{aligned}
\mathbb{P}\big[X_s = 0\big] &= \sum_{k=0}^{\infty} \mathbb{P}\big[X_k = 0 | N = k\big] \cdot \mathbb{P}\big[N = k\big] \\
&= \sum_{k=0}^{\infty} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \begin{pmatrix} k \\ k/2 \end{pmatrix} \mathbb{P}[N = k] \\
&= e^{-2s} + \sum_{k=1}^{\lfloor s \rfloor} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \begin{pmatrix} k \\ k/2 \end{pmatrix} \mathbb{P}[N = k] \\
&+ \sum_{k=\lfloor s \rfloor + 1}^{\infty} \mathbf{1}_{[k \text{ is even}]} \frac{1}{2^k} \begin{pmatrix} k \\ k/2 \end{pmatrix} \mathbb{P}[N = k]\,.
\end{aligned} \tag{3.89}$$

Using the Stirling Formula (seefor example Feller, Vol I.), it is easy to check that

$$\frac{1}{2^k}\binom{k}{k/2} \;\leq\; \frac{1}{\sqrt{\pi k}} \;\leq\; 1\,. \tag{3.90}$$

Applying the second inequality of (3.90) in the first sum of (3.89) and the first inequality of (3.90) in the second sum in (3.89), we obtain that $\mathbb{P}\big[X_s = 0\big]$ is bounded from above by

$$e^{-2s} + \mathbb{P}\big[N \leq \lfloor s \rfloor\,\big] + \frac{c_1}{\sqrt{s}} \sum_{k=\lfloor s \rfloor+1}^{\infty} \mathbb{P}[N = k] \;\leq\; e^{-2s} + \mathbb{P}\big[N \leq \lfloor s \rfloor\,\big] + \frac{c_1}{\sqrt{s}}\,. \tag{3.91}$$

In the sequel, we will get an exponential bound $\mathbb{P}\big[\,N \leq \lfloor s \rfloor\,\big]$ by a standard large deviations technique. In this way, note that, for any $\theta > 0$,

$$\begin{aligned}
\mathbb{P}\big[\,N \leq \lfloor s \rfloor\,\big] \;&=\; \mathbb{E}\big[\,\mathbf{1}_{[N \leq s]} e^{\theta N} e^{-\theta N}\,\big] \;\leq\; e^{\theta s}\, \mathbb{E}\big[\,\mathbf{1}_{[N \leq s]} e^{-\theta N}\,\big] \\
&\leq\; e^{\theta s}\, \mathbb{E}\big[\,e^{-\theta N}\,\big] \;=\; e^{\theta s} e^{2s(e^{-\theta}-1)} \;=\; e^{s(2e^{-\theta}-2+\theta)}\,.
\end{aligned}$$

Denote $f(\theta) = 2e^{-\theta} - 2 + \theta$ and note that f assumes its minimum at $\theta_0 = \log 2 > 0$, and $f(\theta_0) = \log 2 - 1 < 0$. Therefore, choosing $\theta = \theta_0$, we get

$$\mathbb{P}\big[\,N \leq \lfloor s \rfloor\,\big] \;\leq\; e^{s(\log 2 - 1)}\,.$$

Looking at (3.91) and then to (3.89), we conclude that

$$\mathbb{P}\big[\,X_s = 0\,\big] \;\leq\; e^{-2s} + e^{s(\log 2 - 1)} + \frac{c_1}{\sqrt{s}}\,.$$

Integrating, we get

$$\int_0^t \mathbb{P}\big[\,X_s = 0\,\big]\, ds \;\leq\; \int_0^t \Big(e^{-2s} + e^{s(\log 2 - 1)} + \frac{c_1}{\sqrt{s}}\Big)\, ds \;\leq\; c_2\sqrt{t}\,,$$

for some constant c_2 not depending on t.

□

Proposition 3.6.2. *Let (X,Y) be the symmetric simple two-dimensional continuous time random walk. Then,*

$$\int_0^t \mathbb{P}\big[\,(X_s,Y_s)=(0,0)\,\big]\,ds \;\leq\; c\log t\,,$$

where $c>0$ is a constant which does not depend on t.

The proof of the statement above can be adapted from the one of Proposition 3.6.1.

3.7 Fluctuations at the initial time

Proposition 3.7.1. *Let $\nu^n_{\rho_0(\cdot)}$ be the slowly varying Bernoulli product measure associated with a smooth profile ρ_0. Then, $\mathcal{Y}^n_0$ converges in distribution to $\mathcal{Y}_0$, where $\mathcal{Y}_0$ is a mean zero Gaussian field of covariance given by*

$$\mathbb{E}\Big[\mathcal{Y}_0(g)\mathcal{Y}_0(f)\Big] \;=\; \int_{\mathbb{R}} \chi\big(\rho_0(u)\big)\,g(u)\,f(u)\,du\,, \tag{3.92}$$

for any $f,g\in\mathcal{S}_\alpha(\mathbb{R})$.

Proof. As argued in Subsection 3.5.2, for each $f\in\mathcal{S}_\alpha(\mathbb{R})$, the sequence $\big\{\mathcal{Y}_0(f)\big\}_{n\in\mathbb{N}}$ is tight, hence $\big\{\mathcal{Y}_0\big\}_{n\in\mathbb{N}}$ is tight due to Mitoma's criterion (Proposition 3.5.3). Thus, it remains only to characterize the joint limit in distribution for the vectors of the form $\big(\mathcal{Y}_0(f_1),\dots,\mathcal{Y}_0(f_k)\big)$, with $f_i\in\mathcal{S}_\alpha(\mathbb{R})$, for $i=1,\cdots,k$. Since $\nu^n_{\rho_0}$ is a product measure,

$$\log\mathbb{E}_{\nu^n_{\rho_0}(\cdot)}\Big[\exp\Big\{i\theta\mathcal{Y}^n_0(f)\Big\}\Big] \;=\; \sum_{x\in\mathbb{Z}}\log\mathbb{E}_{\nu^n_{\rho_0}(\cdot)}\Big[\exp\Big\{\frac{i\theta}{\sqrt{n}}\,\bar{\eta}_0(x)f\Big(\frac{x}{n}\Big)\Big\}\Big]$$
$$=\sum_{x\in\mathbb{Z}}\log\Big[\rho_0(\tfrac{x}{n})\exp\big\{\tfrac{i\theta}{\sqrt{n}}f(\tfrac{x}{n})\big(1-\rho_0(\tfrac{x}{n})\big)\big\}+\big(1-\rho_0(\tfrac{x}{n})\big)\exp\big\{-\tfrac{i\theta}{\sqrt{n}}f(\tfrac{x}{n})\rho_0(\tfrac{x}{n})\big\}\Big].$$

Since $f\in\mathcal{S}_\alpha(\mathbb{R})$, we have smoothness of f except possibly at $x=0$, together with fast decaying. Keeping this in mind, Taylor's expansion on the exponential function permits to conclude that the expression above is equal to

$$-\frac{\theta^2}{2n}\sum_{x\in\mathbb{Z}}f^2\Big(\frac{x}{n}\Big)\chi(\rho_0(\tfrac{x}{n}))+O(\tfrac{1}{\sqrt{n}})\,,$$

which gives us that

$$\lim_{n\to+\infty}\log\mathbb{E}_{\nu^n_{\rho_0}(\cdot)}\Big[\exp\Big\{i\theta\mathcal{Y}^n_0(f)\Big\}\Big] \;=\; -\frac{\theta^2}{2}\int_{\mathbb{R}}\chi\big(\rho_0(u)\big)\,f^2(u)\,du\,.$$

Replacing f by a linear combination of functions and then applying the Crámer-Wold device, the proof ends. □

Bibliography

[1] R. A. Adams and J. Fournier. *Sobolev spaces*. Pure and Applied Mathematics. Academic Press, 2 edition, 2003.

[2] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[3] P. Billingsley. *Convergence of Probability Measures*. John Wiley and Sons, 2nd edition, 1999.

[4] A. De Masi, N. Ianiro, A. Pellegrinotti, and E. Presutti. A survey of the hydrodynamical behavior of many-particle systems. In *Nonequilibrium phenomena, II*, Stud. Statist. Mech., XI, pages 123–294. North-Holland, Amsterdam, 1984.

[5] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[6] J. Farfan, A. B. Simas, and F. J. Valentim. Equilibrium fluctuations for exclusion processes with conductances in random environments. *Stochastic Processes and their Applications*, 120(8):1535 – 1562, 2010.

[7] T. Franco, P. Gonçalves, and A. Neumann. Hydrodynamical behavior of symmetric exclusion with slow bonds. *Ann. Inst. H. Poincaré Probab. Statist.*, 49(2):402–427, 05 2013.

[8] T. Franco, P. Gonçalves, and A. Neumann. Phase transition in equilibrium fluctuations of symmetric slowed exclusion. *Stoch. Proc. Appl.*, 123(12):4156–4185, 2013.

[9] T. Franco, P. Gonçalves, and A. Neumann. Phase transition of a heat equation with Robin's boundary conditions and exclusion process. *Trans. Amer. Math. Soc.*, 367:6131–6158, 2015.

[10] T. Franco, P. Gonçalves, and A. Neumann. Corrigendum to: Phase transition in equilibrium fluctuations of symmetric slowed exclusion. *Stoch. Proc. Appl.*, 126(10):3235–3242, 2016.

[11] T. Franco, P. Gonçalves, and A. Neumann. Non-equilibrium and stationary fluctuations of a slowed boundary symmetric exclusion. *Stochastic Processes and their Applications (to appear)*, 2018.

[12] T. Franco and C. Landim. Hydrodynamic limit of gradient exclusion processes with conductances. *Archive for Rational Mechanics and Analysis*, 195(2):409–439, Feb 2010.

[13] T. Franco, A. Neumann, and G. Valle. Hydrodynamic limit for a type of exclusion process with slow bonds in dimension $d \geq 2$. *J. Appl. Probab.*, 48(2):333–351, 06 2011.

[14] R. A. Holley and D. W. Stroock. Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. *Publ. Res. Inst. Math. Sci.*, 14(3):741–788, 1978.

[15] J. Jacod and A. N. Shiryaev. Limit theorems for stochastic processes. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Second edition.*, 147(288):519–528, 2003.

[16] M. D. Jara and C. Landim. Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(5):567–577, 2006.

[17] M. D. Jara and C. Landim. Quenched non-equilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder. *Ann. Inst. H. Poincaré Probab. Statist.*, 44(2):341–361, 04 2008.

[18] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1st edition, 1999.

[19] G. Lawler and V. Limic. *Random Walk: A Modern Introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2010.

[20] D. A. Levin and Y. Peres. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2017.

[21] I. Mitoma. Tightness of probabilities on $C([0,1];\mathcal{Y}')$ and $D([0,1];\mathcal{Y}')$. *Ann. Probab.*, 11(4):989–999, 1983.

[22] K. Ravishankar. Fluctuations from the hydrodynamical limit for the symmetric simple exclusion in $\mathbb{Z}^d$. *Stoch. Proc. Appl.*, 42(1):31–37, 1992.

[23] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, first edition, 1981.

[24] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[25] F. J. Valentim. Hydrodynamic limit of a d-dimensional exclusion process with conductances. *Ann. Inst. H. PoincarÃ© Probab. Statist.*, 48(1):188–211, 02 2012.

[26] E. Zeidler. *Applied functional analysis*, volume 108 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1995. Applications to mathematical physics.

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