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Cociclos lineares e otimização ergódica para fluxos hiperbólicos

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Salvador-Bahia
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Tese apresentada ao Colegiado do Programa de Pós-Graduação em Matemática UFBA/UFAL como requisito parcial para obtenção do título de Doutor em Matemática, aprovada em 13 de Julho de 2018.

Orientador: Prof. Dr. Paulo César Rodrigues Pinto Varandas

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"Que ninguém se engane, só se consegue a simplicidade através de muito trabalho."

Clarice Lispector.

Resumo

O conceito de hiperbolicidade uniforme introduzido por Smale e os modelos hiperbólicos são ainda fonte de inúmeros problemas em aberto. No contexto de dinâmica a tempo contínuo, as contribuições de Bowen, Ruelle e Ratner nos mostram que fluxos hiperbólicos são semi-conjugados a fluxos de suspensão sobre um *shift*. Este resultado nos permite estudar propriedades de um sistema a tempo contínuo a partir do shift associado. Nesta tese abordamos duas questões, nomeadamente os expoentes de Lyapunov de cociclos lineares e otimização ergódica.

No contexto de cociclos lineares, mostramos a simplicidade do espectro de Lyapunov para cociclos sobre fluxos hiperbólicos que preservam uma medida hiperbólica ergódica com estrutura de produto local. Mais precisamente, mostramos que existe um conjunto aberto e denso de geradores infinitesimais que geram cociclos com esta propriedade. Aqui usamos a topologia no espaço dos geradores infinitesimais com regularidade pelo menos Hölder.

No contexto de otimização ergódica, provamos que, para um fluxo hiperbólico, funções Hölder contínuas genéricas possuem uma única medida maximizante, a qual é suportada em uma órbita periódica. No contexto de funções contínuas, mostramos que para um fluxo hiperbólico funções contínuas genéricas possuem uma única medida maximizante com suporte total e entropia zero, em contraponto com o caso mais regular.

Palavras chaves: Fluxos hiperbólicos; Cociclos lineares; Expoentes de Lyapunov; Otimização ergódica.

Abstract

The concept of uniform hyperbolicity introduced by Smale and the hyperbolic models are still the source of numerous open problems. In the context of continuous-time dynamics, the contributions of Bowen, Ruelle and Ratner show that hyperbolic flows are semi-conjugated to suspension flows over a shift. This result allows us to study properties of a continuous time system from the associated shift. In this thesis we address two questions, namely the Lyapunov exponents of linear cocycles and ergodic optimization.

In the context of linear cocycles, we show the simplicity of the Lyapunov spectrum for cocycles on hyperbolic flows that preserve an ergodic hyperbolic measure with local product structure. More precisely, we show that there is an open and dense set of infinitesimal generators that generate cocycles with this property. Here we use the topology in the space of infinitesimal generators with at least Hölder regularity.

In the context of ergodic optimization, we prove that for a hyperbolic flow, generic Hölder continuous functions have a single maximizing measure, which is supported in a periodic orbit. In the context of continuous functions, we show that for a hyperbolic flow generic continuous functions have a single maximizing measure with full support and zero entropy, in counterpoint to the more regular case.

Keywords: Hyperbolic flows; Linear cocycles; Lyapunov exponents; Ergodic optimization.

Contents

Chapter 1

Introduction

Uniform hyperbolic dynamical systems were introduced by Smale in the 1960's. Since then the theory has been developed in many directions, one of them being the area of nonuniformly hyperbolicity, with the study of the Lyapunov exponents. Non-zero Lyapunov exponents assure asymptotic exponential rate of divergence or convergence of two neighboring trajectories, whereas zero exponents give us the lack of any kind of asymptotic exponential behavior. A central question in dynamical systems is to determine whether we have non-zero Lyapunov exponents for a given dynamics and some or the majority of nearby systems. An answer for that usually depends on the smoothness and richness of the dynamical system, among the other aspects.

The ergodic theory of hyperbolic systems has also been developed. Hyperbolic flows have been studied since the 1970's and, in particular, its geometric structure is very well understood. By the works of Bowen, Sinai and Ruelle [15, 13, 40] hyperbolic flows admit finite Markov partitions and are semi-conjugated to suspension flows over hyperbolic maps. The Markov structure was strongly used to study the thermodynamic formalism for hyperbolic flows and it was established in the 1970's (see [15]) that there is a unique equilibrium state μ_ξ with respect to any Hölder continuous potential $\xi : \Lambda \to \mathbb{R}$. In addition, μ_ξ is obtained as a suspension of a $\sigma_{\mathbf{R}}$-invariant measure ν with the usual local product structure, where $\sigma_{\mathbf{R}}$ is a subshift of finite type. It is known that the set of invariant measures for hyperbolic dynamical systems is large (see for example [39]), which is the best scenario for the problems of ergodic optimization, where, roughly speaking, we are interested in maximizing (or minimizing) integral of functions under invariant measures.

In this work we give contributions for both study of Lyapunov exponents and ergodic optimization for uniformly hyperbolic flows.

First we deal with Lyapunov exponents for hyperbolic flows. Given a linear differential system $A : M \to \mathfrak{sl}(2, \mathbb{K})$ over a flow $(X^t)_{t\in\mathbb{R}} : M \to M$ (see Subsection 2.1.3), the Lyapunov exponents associated to A detect if there are any exponential asymptotic behavior on the evolution of the time-continuous cocycle $(\Phi_A^t)_t$ along orbits (cf. [5]). Under certain measure preserving assumptions on $(X^t)_t$ and integrability of $\log \|\Phi_A^t\|$, the existence of Lyapunov exponents for almost every point is guaranteed by Oseledets' theorem ([36]).

For discrete-time dominated cocycles over uniformly hyperbolic maps Bonatti and Viana [11] proved that for the majority of cocycles all Lyapunov exponents have multiplicity 1. Avila and Viana [1] exhibited an explicit sufficient condition for the Lyapunov exponents of a linear cocycle over a Markov map to have multiplicity 1. More recently, in [2] Backes, Poletti, Varandas, and Lima proved that generic fiber-bunched and Hölder continuous linear cocycles over a non-uniformly hyperbolic system endowed with a u-Gibbs measure have simple Lyapunov spectrum. In the context of continuous flows over compact Hausdorff spaces Bessa ([7, 8]) proved the existence of a residual set $\mathcal{R}$, i.e. a C^0-dense G_δ (a countable intersection of open sets), such that for any conservative linear differential system in $\mathcal{R}$ either the Oseledets' decomposition along the orbit of almost every point has dominated splitting or else the spectrum is trivial, meaning that all the Lyapunov exponents vanish. Considering the L^p topologies, Bessa and Vilarinho proved in [10] the abundance of trivial spectrum for a large class of linear differential systems.

In this work we are interested in proving abundance of non-zero Lyapunov exponents for more regular time continuous cocycles (at least Hölder continuous) taking values in $SL(2,\mathbb{K})$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Our purpose here is to contribute to the better understanding of the ergodic theory cocycles over a hyperbolic flow and to answer some of the questions raised by Viana, namely part of Problem 6 of [43]. First we address the case of suspension flows (as a model to flows that admit a global cross-section) and then deal with the uniformly hyperbolic flows. The strategy used to prove the result for $SL(2,\mathbb{K})$-cocycle over suspension flows is to make a reduction to the discrete-time case by considering an induced cocycle in the fiber that also depends on the roof function. We perform perturbations on the space $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ of the infinitesimal generators so the induced discrete-time cocycle satisfies the hypothesis of the criterion of [1]. Here we point out that instead of making perturbations on the discrete-time cocycles, our perturbations are on the space of its infinitesimal generators, which demands extra work.

Our strategy uses [11, Proposition 9.1], where more general $SL(d,\mathbb{K})$-cocycles, with $d \geq 2$, are considered. But in their proof they constructed a dominated splitting to find, by perturbation, periodic points without complex eingenvalues. This splitting needs to be Hölder with respect to the point on the basis. This fact is not proved in [11, Proposition 9.1], what makes the proof incomplete.

Our second point of view is about *ergodic optimization*. Let $T : X \to X$ be a continuous map, where X is a compact metric space, and let $\mathcal{M}_T$ be the collection of those Borel probability measures on X which are preserved by T. The objects of interest in the field of ergodic optimization are those T-invariant probability measures which maximize, or minimize, the space average $\int f d\mu$, for $f : X \to \mathbb{R}$, over all $\mu \in \mathcal{M}_T$. These are the maximizing measures and minimizing measures for the function f (with respect to the dynamical system $T : X \to X$). Since maximizing measures do exist, the fundamental question of ergodic optimization is: what can we say about the maximizing measures? For example, is there only one maximizing measure for typical observables? Can we describe the support of a maximizing measure?

The problem of describing the set

$$\mathcal{P}(E) := \{\phi \in E : \text{there is a single } \phi\text{-maximizing measure supported on a periodic orbit}\},$$

where E is some suitable set of continuous observables has been studied by several authors in the discrete-time context. See for instance, Contreras [19], Contreras, Lopes and Thieullen [20] for the expanding case, Fathi [22] for an approach using KAM theory, Bousch [12]. Quas and Siefken [37] for the one-sided shift. See also Yuan and Hunt [45], Morris [34] and references therein. For continuous-time, Mañé [33] conjectured that for a generic Lagrangian, there exits a unique minimizing measure, and it is supported by a periodic orbit. See also Garibaldi, Lopes and Thieullen [24] for a relation with lagrangian systems in the context of ergodic optimization. Based on various approaches utilized in the literature, we can emphasize that the regularity of the observables plays an important role in the proofs: for Lipschitz potentials, one can obtain maximizing measures supported in periodic orbits, whereas for continuous potentials, the support of the maximizing measure is the whole space. We shall consider ergodic optimization for flows with respect to both continuous and Hölder continuous observables.

We prove that for a hyperbolic suspension flow there is a open and dense set of Hölder observable with a single maximizing measure, which is supported on a periodic orbit. We also prove that for a hyperbolic suspension flow there is a dense set of continuous observables with a single maximizing measure which has full support.

This thesis is organized as follows. In Chapter 2 we given necessary definitions and state the main results. In Chapter 3 we give preliminary results that will be used in Chapters 4 and 5. In Chapter 4 we deal with Lyapunov exponents for hyperbolic flows. In Chapter 5 we deal with ergodic optimization. Finally in Chapter 6 we comment on some open question on both themes of the thesis.

Chapter 2

Main results

2.1 Some definitions

2.1.1 Hyperbolic flows

Let M be a closed Riemannian manifold and $d : M \times M \to [0, \infty)$ distance function given by the arc length of a minimizing geodesic. Let $(X^t)_t \colon M \to M$ be a smooth flow generated by a C^1 vector field $X : M \to TM$. Let $x \in M$ be a critical point for the field X, that is, $X(x) = 0$, and let J denote the Jacobian matrix of X at x. If the matrix J has no eigenvalues with zero real parts then x is called *hyperbolic critical point.* Note that if x is a critical point for the field X, than x is a fixed point for the flow $(X^t)_t$, that is, $X^t(x) = x$ for all $t \in \mathbb{R}$. So a hyperbolic critical points may also be called *hyperbolic fixed points.*

Now let $\Lambda \subseteq M$ be a compact and $(X^t)_t$-invariant set. We say that a flow $(X^t)_t \colon \Lambda \to \Lambda$ is *uniformly hyperbolic* if for every $x \in \Lambda$ there exist a DX^t-invariant and continuous splitting $T_xM = E^s_x \oplus E^X_x \oplus E^u_x$ and constants $C > 0$ and $0 < \theta_1 < 1$ such that

$$\|DX^t \mid E^s_x\| \leq C\theta_1^t \quad \text{and} \quad \|(DX^t)^{-1} \mid E^u_x\| \leq C\theta_1^t, \tag{2.1.1}$$

for every $t \geq 0$. We say that $(X^t)_t$ is an Anosov flow if $(X^t)_t \colon M \to M$ is uniformly hyperbolic. It has been shown by Gourmelon in [25] that there exists an adapted metric which allows us to take $C = 1$.

Now let Λ be a hyperbolic set for $(X^t)_{t\in\mathbb{R}}$. For each $x \in \Lambda$, we consider the sets

$$W^s(x) = \{y \in M : d(X^t(y), X^t(x)) \xrightarrow[t\to+\infty]{} 0\}$$

and

$$W^u(x) = \{y \in M : d(X^t(y), X^t(x)) \xrightarrow[t\to-\infty]{} 0\}.$$

And for any sufficiently small $\epsilon > 0$, we consider the sets

$$W^s_\epsilon(x) = \{y \in M : d(X^t(y), X^t(x)) \leq \epsilon \text{ for } t \geq 0\}$$

and

$$W^u_\epsilon(x) = \{y \in M : d(X^t(y), X^t(x)) \leq \epsilon \text{ for } t \leq 0\}$$

These are smooth manifolds, called respectively local stable and unstable manifolds (of size ϵ) at the point x. Moreover:

1. $T_x W^s_\epsilon(x) = E^s_x$ and $T_x W^u_\epsilon(x) = E^u_x$;
2. for each $t > 0$ we have

$$X^t(W^s_\epsilon(x)) \subset W^s_\epsilon(X^t(x)) \quad \text{and} \quad X^{-t}(W^u_\epsilon(x)) \subset W^u_\epsilon(X^{-t}(x));$$

3. there exist $\kappa > 0$ and $\gamma \in (0,1)$ such that for each $t > 0$ we have

$$d(X^t(y), X^t(x)) \leq \kappa\gamma^t d(y,x) \quad \text{for} \quad y \in W^s_\epsilon(x),$$

   and

$$d(X^{-t}(y), X^{-t}(x)) \leq \kappa\gamma^t d(y,x) \quad \text{for} \quad y \in W^u_\epsilon(x).$$

We define the weak local stable and unstable manifolds as

$$W^{ws}_\epsilon(x) = \bigcup_{t\in\mathbb{R}} W^s_\epsilon(X^t(x))$$

and

$$W^{wu}_\epsilon(x) = \bigcup_{t\in\mathbb{R}} W^u_\epsilon(X^t(x)),$$

respectively. These sets are invariant manifolds tangents to $E^s_x \oplus E^X_x$ and $E^X_x \oplus E^u_x$ em x, respectively.

We also introduce the notion of a locally maximal hyperbolic set.

Definition 2.1.1. A set Λ is said to be *locally maximal* (with respect to a flow $(X^t)_{t\in\mathbb{R}}$) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} X^t(U).$$

Now let Λ be a locally maximal hyperbolic set. For any sufficiently small ϵ, there exists a $\delta > 0$ such that if $x, y \in \Lambda$ are at a distance $d(x,y) \leq \delta$, then there exists a unique $t = t(x,y) \in [-\epsilon, \epsilon]$ for which the set

$$[x,y] = W^s_\epsilon(X^t(x)) \cap W^u_\epsilon(y)$$

is a single point in Λ (see [28, Proposition 7.2]).

Definition 2.1.2. We say that Λ is a *hyperbolic basic* set if

1. Λ contains no fixed points and is hyperbolic;
2. the periodic orbits of $(X^t)_t|\Lambda$ are dense in Λ;
3. $(X^t)_t|\Lambda$ is a topologically transitive flow, that is, $(X^t)_t|\Lambda$ has a dense orbit;
4. Λ is locally maximal.

Definition 2.1.3. The *nonwandering set* Ω for the flow $(X^t)_t$ is defined by

$$\begin{aligned}\Omega =\{x \in M : & \text{ for every open neighborhood } V \text{ of } x \text{ and every } t_0 > 0 \\ & \text{there exists } t > t_0 \text{ so that } X^t(V) \cap V \neq \emptyset\}.\end{aligned}$$

The flow $(X^t)_t$ is said to satisfy *Axiom A* if its nonwandering set Ω is the disjoint union of hyperbolic sets and a finite number of hyperbolic fixed points.

2.1.2 Local Product Structure

Given a regular point $x \in M$ for the C^1 vector field $X : M \to TM$, that is, $X(x) \neq 0$, the *Tubular Neighborhood Theorem* (see for instance [31, Chapter 3]) ensures the existence of a positive number $\delta = \delta_x > 0$, an open neighborhood U_x^δ of x, and a diffeomorphism $\Psi_x : U_x^\delta \to (-\delta, \delta) \times B(x, \delta) \subset \mathbb{R} \times \mathbb{R}^d$, where $B(x, \delta)$ is identified with the ball $B(\vec{0}, \delta) \cap \langle (1, 0, \dots, 0)^\perp \rangle$, and $\langle (1, 0, \dots, 0)^\perp \rangle$ denotes the hyper-space orthogonal to the vector $(1, 0, \dots, 0)$, such that the vector field X in U_x^δ is the pull-back of the vector field $Y := (1, 0, \dots, 0)$ in $(-\delta, \delta) \times B(x, \delta)$. More precisely, $Y = (\Psi_x)_* X := D(\Psi_x)_{\Psi_x^{-1}} X(\Psi_x^{-1})$. In this case the associated flows are conjugated, that is, $Y^t(\cdot) = \Psi_x(X^t(\Psi_x^{-1}(\cdot)))$ for every t sufficiently small.

Let Λ be a hyperbolic set. Given $x \in \Lambda$ and $\epsilon > 0$ small enough, both invariant manifolds $W_\epsilon^s(X^t(y))$ and $W_\epsilon^u(X^t(y))$ have size of at least ϵ for all $y \in \Lambda \cap U_x^\delta$ and all t such that $X^t(y) \in U_x^\delta$. As a consequence, if we consider the section $\Sigma_x = \Psi_x^{-1}(\{0\} \times B(x, \delta))$ at point x, then for any $y \in \Lambda \cap U_x^\delta$ the intersection $\mathcal{F}_y^s = W_\epsilon^{ws}(y) \cap \Sigma_x$ (respectively $\mathcal{F}_y^u = W_\epsilon^{wu}(y) \cap \Sigma_x$) defines a smooth and long stable (respectively unstable) submanifold in Σ_x (see Figure 2.1.1).

Since the angles between stable and unstable foliations are bounded away from zero in hyperbolic sets, it is not difficult to verify that for all $y, z \in \Lambda \cap U_x^\delta$ the intersection $[y, z]_{\Sigma_x} := \mathcal{F}_y^u \pitchfork \mathcal{F}_z^s$ consists of a unique point, since δ is small (see [30] and Figure 2.1.2).

Define

$$\mathcal{N}_x^u(\delta) = \{[x, y]_{\Sigma_x} : y \in \Lambda \cap U_x^\delta\} \subset \Sigma_x \cap \mathcal{F}_x^u$$

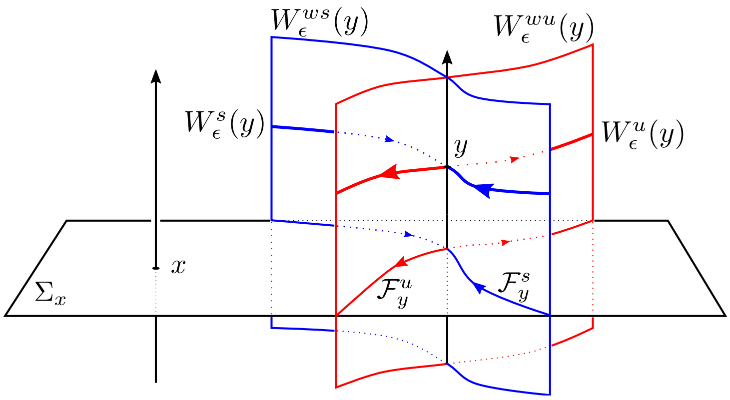

Figure 2.1.1: Stable and unstable leaves.

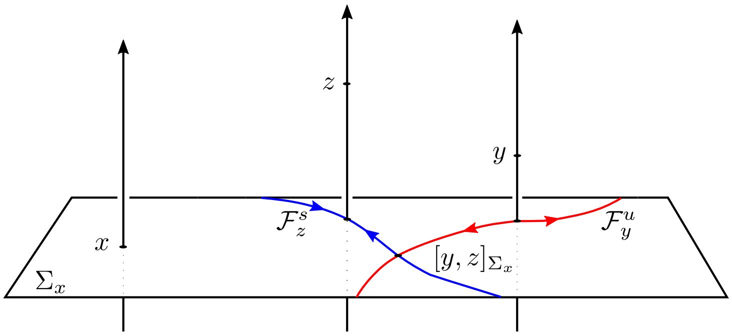

Figure 2.1.2: Intersection of unstable and stable leaves.

as a u-neighborhood of x in Σ_x and

$$\mathcal{N}_x^s(\delta) = \{[y,x]_{\Sigma_x} : y \in \Lambda \cap U_x^\delta\} \subset \Sigma_x \cap \mathcal{F}_x^s$$

a s-neighborhood of x in Σ_x. The set $\mathcal{N}_x(\delta) := \Lambda \cap U_x^\delta$ is a neighborhood x in Λ. Then the transformation

$$\begin{aligned} \Upsilon_x : \mathcal{N}_x(\delta) &\longrightarrow \mathcal{N}_x^u(\delta) \times \mathcal{N}_x^s(\delta) \times (-\delta,\delta) \\ y &\longmapsto ([x,y]_{\Sigma_x}, [y,x]_{\Sigma_x}, t(y)), \end{aligned} \tag{2.1.2}$$

with $t(\cdot)$ uniquely determined by $X^{t(y)}(y) \in \Sigma_x$, is a homeomorphism.

A Borel $(X^t)_t$-invariant probability measure μ on M is called *hyperbolic* if for μ-almost $x \in M$ and $v \in T_xM \backslash \{\mathbb{R} \cdot X(x)\}$ we have that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|DX^t(x) \cdot v\| \neq 0.$$

Recall that given measurable spaces (X_1, Σ_1) and (X_2, Σ_2), a measurable mapping $f : X_1 \to X_2$ and a measure $\mu : \Sigma_1 \to [0, +\infty]$, the *pushforward* of μ is defined to be the measure $f_*(\mu) \colon \Sigma_2 \to [0, +\infty]$ given by

$$(f_*(\mu))(B) = \mu\left(f^{-1}(B)\right) \text{ for } B \in \Sigma_2.$$

We can now define a local product structure for flow invariant measures.

Definition 2.1.4. A hyperbolic measure μ has *local product structure on* Λ if for all $x \in \mathrm{supp}(\mu) \cap \Lambda$ and a small $\delta > 0$ the measure $(\Upsilon_x)_* \mu|_{\mathcal{N}_x(\delta)}$ is equivalent to the product measure $\mu_x^u \times \mu_x^s \times Leb$, where μ_x^i denotes the marginal measure of $(\Upsilon_x)_*(\mu|_{\mathcal{N}_x(\delta)})$ in $\mathcal{N}_x^i(\delta)$, for $i = u, s$, and Leb is the Lebesgue measure on the interval $(-\delta, \delta)$ identified with a segment of the trajectory through x and Υ_x is given by (2.1.2). We denote by μ_Σ the marginal measure $\mu|_{\mathcal{N}_x(\delta)}$ in Σ obtained via projection along the direction of the flow.

2.1.3 Linear differential systems and infinitesimal generators

We now describe the set of time-continuous linear differential systems associated to an *infinitesimal generator* $A : M \to \mathfrak{sl}(2, \mathbb{K})$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $\mathfrak{sl}(2, \mathbb{K})$ is the special linear Lie algebra of 2×2 matrices with trace zero and with the Lie bracket $[X, Y] := XY - YX$. Given $r \geq 0$ and $\nu \in [0, 1]$, with $r + \nu > 0$, denote by $C^{r,\nu}(M, \mathfrak{sl}(2, \mathbb{K}))$ the Banach space of $C^{r+\nu}$ linear differential systems with values on the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$, endowed with the topology $C^{r,\nu}$ defined by the norm

$$\|A\|_{r,\nu} = \sup_{0 \leq j \leq r} \sup_{x \in M} \|D^j A(x)\| + \sup_{\substack{x,y \in M \\ x \neq y}} \frac{\|D^r A(x) - D^r A(y)\|}{\|x - y\|^\nu}. \tag{2.1.3}$$

Given $A \in C^{r,\nu}(M, \mathfrak{sl}(2, \mathbb{K}))$ and a $C^{1+\alpha}$ flow $(X^t)_t : M \to M$ the dynamics on the fibers is given by a cocycle in continuous time $\Phi_A^t : M \to SL(2, \mathbb{K})$. For each $x \in M$ we obtain $\Phi_A^t(x)$ as a solution of the equation

$$\partial_t u(s)\Big|_{s=t} = A(X^t(x)) \cdot u(t), \quad u(0) = u_0 \in SL(2, \mathbb{K}) \tag{2.1.4}$$

known as *linear variational equation* (or equation of the first variation). The unique solution of (2.1.4) with $u(0) = \mathbf{1}_d$ is called *fundamental solution* related to the system A. This solution is a curve of linear applications $(\Phi_A^t(x))_{t \in \mathbb{R}}$ in $SL(2, \mathbb{K})$ which can be seen as a skew product flow

$$\begin{aligned} F_A^t \ : \ M \times \mathbb{K}^2 &\longrightarrow M \times \mathbb{K}^2 \\ (x, v) &\longmapsto (X^t(x), \Phi_A^t(x) v), \end{aligned}$$

for all $t \in \mathbb{R}$. The cocycle identity holds for the fundamental solution of (2.1.4), that is, $\Phi_A^{t+s}(x) = \Phi_A^s(X^t(x)) \circ \Phi_A^t(x)$ holds for all $x \in M$ and $t, s \in \mathbb{R}$ and, clearly, $A(x) = \partial_t \Phi_A^t(x)|_{t=0}$ for all $x \in M$ (see Figure 2.1.3). It follows from the previous cocycle identity that, for all $x \in M$ and $t \in \mathbb{R}$, $(\Phi_A^t(x))^{-1} = \Phi_A^{-t}(X^t(x))$ and $(\Phi_A^t(x))^{-1}$ coincides with the solution of the differential equation associated with the infinitesimal generator $-A$, that is,

$$\partial_t u(s)\Big|_{s=t} = -A(X^t(x)) \cdot u(t), \tag{2.1.5}$$

because of the time reversal.

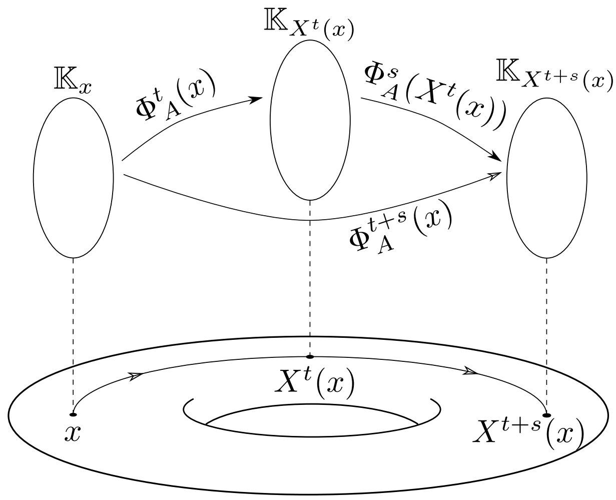

Figure 2.1.3: Action of the cocycle on the fibers.

For the proofs of ours main results (see statement in Section 2.2) it is enough to consider $\nu = 1$, that is, we can assume A to be Lipschitz. In fact, if A is ν-Hölder continuous with respect to the metric $d(\cdot, \cdot)$, then A is Lipschitz with respect to the metric $d(\cdot, \cdot)^\nu$. Therefore, up to a the change of metric, we can assume that A is Lipschitz.

We now recall Oseledets' Theorem, which guaranties the existence of Lyapunov exponents. If μ is a $(X^t)_t$-invariant probability measure such that $\log \|\Phi^t_A(\cdot)^{\pm 1}\| \in L^1(\mu)$, for each $t \in \mathbb{R}$, then it follows from the Oseledets theorem ([36]) that for μ-almost every x there is a decomposition $\mathbb{K}^2 = E^1_x \oplus E^2_x$ (which can be trivial), called *Oseledets decomposition*, and for $1 \le i \le 2$ there are well defined real numbers

$$\lambda_i(A, X^t, x) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|\Phi^t_A(x) v_i\|, \quad \forall v_i \in E^i_x \setminus \{\vec{0}\}$$

called *Lyapunov exponents* associated to A, $(X^t)_t$ and x. The Oseledets decomposition is $(\Phi^t_A)_t$-invariant, that is, $\Phi^t_A(x) \cdot E^i_x = E^i_{X^t(x)}$ for every $t \in \mathbb{R}$. Moreover, if μ is ergodic, then the Lyapunov exponents are constant for almost every point. We say that $\lambda_i(A, X^t, x)$ has multiplicity 1 if $\dim E^i = 1$, and that the cocycle Φ^t_A has *simple spectrum* on x if $\lambda_i(A, X^t, x)$ has multiplicity 1 for all $1 \le i \le 2$, in other words, if all Lyapunov exponents associated with A, $(X^t)_t$ and x are distinct. Since we deal with cocycles taking values in the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$, this implies that $\lambda_1(A, X^t, x) = -\lambda_2(A, X^t, x)$ (see, for example, [44, Subsection 4.3.3]). If $\lambda_1(A, X^t, x) = 0$ we have a trivial decomposition, that is, we define $E^1_x = \{0\}$ and $\mathbb{K}^2 = E^2_x$.

Remark 2.1.5. For more general $GL(2, \mathbb{K})$-cocycles the sum of the exponents may not be zero, so simple spectrum means that two exponents are different, but not necessary symmetrical.

Definition 2.1.6. Let $(X^t)_t : M \to M$ be a smooth flow and let $\Lambda \subset M$ be a hyperbolic set for $(X^t)_t$. Let $\theta_1 > 0$ be the constant of hyperbolicity of $(X^t)_t$ in (2.1.1). We say that the cocycle $(\Phi_A^t)_t$ associated with A is *fiber-bunched* if there is $0 < \theta_2 < 1$ such that

$$\|\Phi_A^t(x)\| \cdot \|(\Phi_A^t(x))^{-1}\| \cdot \theta_1^{t \cdot \beta} < \theta_2, \tag{2.1.6}$$

for all $t \geq 0$ and all $x \in \Lambda$.

Note that the latter defines a C^0-open set in the space of linear cocycles.

2.1.4 Ergodic optimization

Let M be a closed Riemannian manifold and $(X^t)_t \colon M \to M$ a smooth flow. Denote by $\mathcal{M}_1(M, (X^t)_t)$ the set of all $(X^t)_t$-invariant Borel probability measures in M. Given a continuous function $\Phi : M \to \mathbb{R}$, a *maximizing measure* for $(X^t)_t$ with respect to Φ is a measure $\mu \in \mathcal{M}_1(M, (X^t)_t)$ which maximizes the integral of Φ among all $(X^t)_t$-invariant Borel probabilities, that is

$$\int \Phi d\mu = \max \left\{ \int \Phi d\nu \middle| \nu \in \mathcal{M}_1(M, (X^t)_t) \right\}.$$

Note that this maximum always exists because $\mathcal{M}_1(M, (X^t)_t)$ is compact in the weak* topology and $\nu \mapsto \int \Phi d\mu$ is continuous. We denote

$$M(\Phi, (X^t)_t) = \max \left\{ \int \Phi d\nu \middle| \nu \in \mathcal{M}_1(M, (X^t)_t) \right\}.$$

Analogously, if $f : \Sigma \to \Sigma$ is a continuous map and $\varphi : \Sigma \to \mathbb{R}$ is continuous, a *maximizing measure* for f and φ is a f-invariant Borel probability measure $\bar{\mu}$ which maximizes the integral of φ among all f-invariant Borel probabilities, that is

$$\int \varphi d\bar{\mu} = \max \left\{ \int \varphi d\bar{\nu} \middle| \bar{\nu} \in \mathcal{M}_1(\Sigma, f) \right\}.$$

We denote $M(\varphi, f) = \max \left\{ \int \varphi d\bar{\nu} \middle| \bar{\nu} \in \mathcal{M}_1(\Sigma, f) \right\}$.

2.2 Statements

2.2.1 Simplicity of Lyapunov spectra

Our main result about simplicity of Lyapunov spectra for cocycles over hyperbolic flows is the following (definitions will be given in Chapter 3).

Theorem A. *Let $(X^t)_t$ be a smooth flow on a compact Riemannian manifold M, and let Λ be a hyperbolic set for $(X^t)_t$. Assume that μ is an ergodic, hyperbolic measure and has local product structure on Λ. Then there exists an open and dense subset $\mathscr{O}$ of infinitesimal generators in $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ with fiber-bunched associated cocycles, such that for any $A \in \mathscr{O}$ the cocycle Φ_A^t has simple Lyapunov spectrum for μ-almost every point.*

Let us mention that Fanaee [21] proved that there exists an open and dense set of *fiber-bunched* $SL(d,\mathbb{K})$-cocycles over Lorenz flows that have simple spectrum. In comparison with [21], our theorem is stated with respect to open and dense set of infinitesimal generators while in [21] the author uses a stronger topology on the space of linear differential systems with a much strong domination condition and does not characterize fiber bunching.

2.2.2 Ergodic optimization

For ergodic optimization our first result concerns suspension flows over a two-sided subshift of finite type and Hölder observables.

Let $C^\alpha(\Sigma, \mathbb{R})$ denote the space of α-Hölder observables $\phi : \Sigma \to \mathbb{R}$: there are constants $C, \alpha > 0$ so that

$$|\phi(x) - \phi(y)| \leq Cd(x,y)^\alpha \tag{2.2.1}$$

for all $x, y \in \Sigma$.

Theorem B. *Let $\sigma : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ be a two-sided subshift of finite type. Given a Hölder continuous function $r : \Sigma_{\mathbf{R}} \to \mathbb{R}$, let $(X^t)_{t\in\mathbb{R}}$ be the suspension flow over σ with height function r. There exists an open and dense set $\mathcal{R}_r \subset C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R})$ of observables $\Phi : \Sigma^r_{\mathbf{R}} \to \mathbb{R}$ such that, for every $\Phi \in \mathcal{R}_r$, there is a single $(X^t)_t$-maximizing measure with respect to Φ, and it is supported on a periodic orbit.*

Using Theorem B and semi-conjugation we extend this result for a flow with a hyperbolic basic set. More precisely:

Theorem C. *Let M be a d-dimension compact boundaryless Riemannian manifold and $(X^t)_{t\in\mathbb{R}}$ be a C^1-flow in M. If there is a hyperbolic basic set $\Lambda \subset M$ for $(X^t)_{t\in\mathbb{R}}$ embedding on a suspension flow over a subshift of finite type, then there exists an open and dense set $\mathcal{R} \subset C^\alpha(M, \mathbb{R})$ of observables $\Phi : M \to \mathbb{R}$ such that, for every $\Phi \in \mathcal{R}$, there is a single $(X^t)_{t\in\mathbb{R}}$-maximizing measure, with respect to Φ, and it is supported on a periodic orbit.*

We also achieve the following result for continuous observables:

Theorem D. *Let $f : M \to M$ be a continuous transformation of a compact metric space satisfying Bowen's specification property. Given a Hölder continuous function $r : M \to \mathbb{R}$, let $(X^t)_{t\in\mathbb{R}}$ be the suspension flow over f with height function r. Then there exists a dense G_δ set $Z \subset C^0(M^r, \mathbb{R})$ such that for every $\varphi \in Z$, there is a single $(X^t)_{t\in\mathbb{R}}$-maximizing measure, it has zero entropy and support equal to M^r.*

Chapter 3

Background material on hyperbolic flows

In this chapter we recall necessary results on hyperbolic flows which will be used in the remaining chapters. More precisely, we will see how to use Markov systems constructed by Bowen and Ratner for basic hyperbolic sets for flows to associate symbolic dynamic to these sets. This will be used to semi-conjugate a hyperbolic flow to a suspension flows over a shift map.

3.1 Symbolic dynamics

Let $\Sigma_n = \{1, \ldots, n\}^{\mathbb{Z}}$ be the space of all sequences $\underline{x} = \{x_i\}_{i=-\infty}^{\infty}$ with $x_i \in \{1, \ldots, n\}$ for all $i \in \mathbb{Z}$. We define the *(left) shift* homeomorphism $\sigma : \Sigma_n \to \Sigma_n$ by $\sigma\left(\{x_i\}_{i=-\infty}^{\infty}\right) = \{x_{i+1}\}_{i=-\infty}^{\infty}$. If $\mathbf{R}$ is an $n \times n$ matrix of 0's and 1's, let

$$\Sigma_{\mathbf{R}} = \left\{\underline{x} \in \Sigma_n : \mathbf{R}_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\right\},$$

we call the restriction $\sigma_{\mathbf{R}} = \sigma|\Sigma_{\mathbf{R}} : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ the *subshift of finite type map.* Now let $\Sigma_n^+ = \{1, \ldots, n\}^{\mathbb{N}}$ be the space of all sequences $\underline{x} = \{x_i\}_{i=0}^{\infty}$ with $x_i \in \{1, \ldots, n\}$ for all $i \in \mathbb{N}$. We define the *one-sided (left) shift* homeomorphism $\sigma^+ : \Sigma_n^+ \to \Sigma_n^+$ by $\sigma^+\left(\{x_i\}_{i=0}^{\infty}\right) = \{x_{i+1}\}_{i=0}^{\infty}$. If $\mathbf{R}$ is an $n \times n$ matrix of 0's and 1's, the one-sided subshift of finite type determined by $\mathbf{R}$ is given by

$$\Sigma_{\mathbf{R}}^+ = \left\{\underline{x} \in \Sigma_n^+ : \mathbf{R}_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\right\}$$

and we call the restriction $\sigma_{\mathbf{R}}^+ = \sigma^+|\Sigma_{\mathbf{R}}^+ : \Sigma_{\mathbf{R}}^+ \to \Sigma_{\mathbf{R}}^+$ the *one-sided (left) subshift of finite type map.*

For $\phi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ continuous we define the variation of ϕ on k-cylinders by

$$\mathrm{var}_k\phi = \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : x_i = y_i \text{ for all } |i| \leq k\}.$$

Let $\mathscr{F}_{\mathbf{R}}$ be the family of all continuous $\phi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ for which $\mathrm{var}_k\phi \leq bc^k$ (for all $k \geq 0$) for some positive constants b and $c \in (0, 1)$.

Remark 3.1.1. For any $\beta \in (0, 1)$ one can define the metric d_β on $\Sigma_{\mathbf{R}}$ by $d_\beta(\underline{x}, \underline{y}) = \beta^N$ where N is the largest non-negative integer with $x_i = y_i$ for every $|i| < N$. Then $\mathscr{F}_{\mathbf{R}}$ is the set of functions which have a positive Hölder exponent with respect to d_β. In fact, for $\underline{x}, \underline{y} \in \Sigma_{\mathbf{R}}$ there is $N \in \mathbb{N}$ such that $d_\beta(x, y) = \beta^N$, this means that $\underline{x}$ and $\underline{y}$ are in a N-cylinder and for $\phi \in \mathscr{F}_{\mathbf{R}}$ we have that

$$\mathrm{var}_N\phi \leq bc^N$$

and this implies that

$$|\phi(\underline{x}) - \phi(\underline{y})| \leq bc^N.$$

So choosing $\alpha \in (0, 1)$ such that $c \leq \beta^\alpha$ we have

$$\begin{aligned} |\phi(\underline{x}) - \phi(\underline{y})| &\leq b\beta^{\alpha N} \\ &= b(\beta^N)^\alpha \\ &= bd_\beta(\underline{x}, \underline{y})^\alpha. \end{aligned}$$

By (2.2.1) this means that ϕ is α-Hölder in the metric d_β. From now on will consider the metric d_β for some fixed $\beta \in (0, 1)$.

Remark 3.1.2. We have that $\sigma^+_{\mathbf{R}} : \Sigma^+_{\mathbf{R}} \to \Sigma^+_{\mathbf{R}}$ is an expanding transformation. If $\rho \in (\frac{1}{2}, 1)$ the ball of radius ρ around of any point $(p_n)_n \in \Sigma^+_{\mathbf{R}}$ is the cylinder $[0; p_0]_{\mathbf{R}}$ that contains this point. We have that

$$d(\sigma^+_{\mathbf{R}}(x_n)_n, \sigma^+_{\mathbf{R}}(y_n)_n) = d((x_{n+1})_n, (y_{n+1})_n) = \beta d((x_n)_n, (y_n)_n)$$

for any $(x_n)_n$ and $(y_n)_n$ in the cylinder $[0; p_0]_{\mathbf{R}}$. Moreover, $\sigma^+_{\mathbf{R}}([0; p_0]_{\mathbf{R}})$ is the union of all cylinders $[0; q]$ such that $\mathbf{R}_{p_0,q} = 1$. In particular, it contains the cylinder $[0; p_1]_{\mathbf{R}}$. Since cylinders are open and closed sets of $\Sigma^+_{\mathbf{R}}$, this shows us that the image of the ball of radius ρ around $(p_n)_n$ contains a neighborhood of the closure of the ball of radius ρ around $(p_{n+1})_n$. This shows that $\sigma^+_{\mathbf{R}} : \Sigma^+_{\mathbf{R}} \to \Sigma^+_{\mathbf{R}}$ is a Ruelle expanding transformation.

3.2 Suspension Flows

Here we introduce the notions of suspension flows and the Bowen–Walters distance following [4]. Let $f : \Sigma \to \Sigma$ be a homeomorphism of a compact metric space (Σ, d_Σ)

and let $r : \Sigma \to (0, \infty)$ be a continuous function bounded away from zero. Consider the quotient space

$$\Sigma^r = \{(x,t) : 0 \le t \le r(x), x \in \Sigma\} / \sim \tag{3.2.1}$$

where $(x, r(x)) \sim (f(x), 0)$.

Definition 3.2.1. The *suspension flow over f with height function r* is the flow $(X^t)_{t \in \mathbb{R}}$ in Σ^r with $X^t : \Sigma^r \to \Sigma^r$ defined by $X^t(x,s) = (f^n(x), s')$, where n and s' are uniquely determined by

$$\sum_{i=0}^{n-1} r(f^i(x)) + s' = t + s, \quad 0 \le s' < r(f^n(x)). \tag{3.2.2}$$

Example 3.2.2. Consider $\Sigma = [0,1]$, $f : [0,1] \to [0,1]$, $x \mapsto 2x \pmod 1$ and take the height function $r : [0,1] \to \mathbb{R}$, $x \mapsto \sin(\frac{5\pi x}{2}) + 3$. See Figure 3.2.1. We have that $f(0.2) = 0.4$, $f^2(0.2) = 0.8$ and $f^3(0.2) = 0.6$. For the height function we have In this case we have $r(0.2) = 4$, $r(0.4) = 3$, $r(0,8) = 3$. Therefore $X^9(0.2, 2) = (0.6, 1)$. In fact, taking $s = 2$, $t = 9$ and $s' = 1$ the equation in 3.2.2 becomes

$$\begin{aligned} r(0.2) + r(0.4) + r(0,8) + s' &= t + s \\ 4 + 3 + 3 + 1 &= 9 + 2. \end{aligned}$$

What is true.

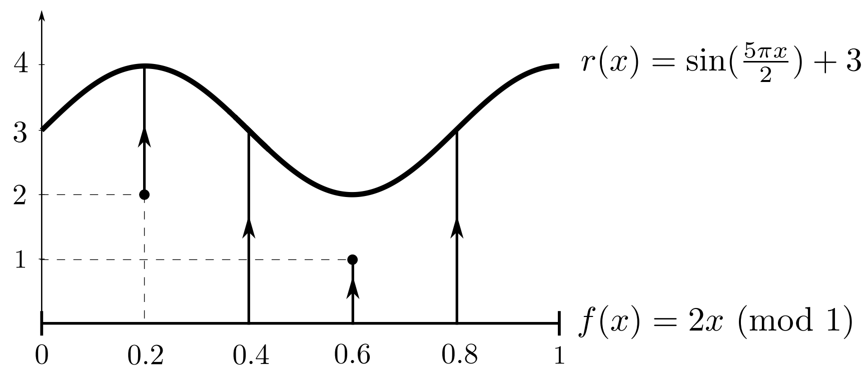

Figure 3.2.1: Suspension flow over $f : [0,1] \to [0,1]$, $x \mapsto 2x \pmod 1$ with height function $r : [0,1] \to \mathbb{R}$, $x \mapsto \sin(\frac{5\pi x}{2}) + 3$.

Now we describe a distance introduced by Bowen and Walters in [16] for suspension flows. Without loss of generality, one can always assume that the diameter $diam\Sigma$ of the space Σ is at most 1. When this is not the case, since Σ is compact, one can simply consider the new distance $d_\Sigma / diam\Sigma$ in Σ.

We first assume that the height function r is constant equal to 1. Given $x, y \in \Sigma$ and $t \in [0,1]$, we define the length of the horizontal segment $[(x,t),(y,t)]$ by

$$\rho_h((x,t),(y,t)) = (1-t)d_\Sigma(x,y) + t d_\Sigma(f(x),f(y)).$$

Clearly,

$$\rho_h((x,0),(y,0)) = d_\Sigma(x,y) \quad \text{and} \quad \rho_h((x,1),(y,1)) = d_\Sigma(f(x),f(y)).$$

Moreover, given points $(x,t),(y,s) \in \Sigma^r$ in the same orbit, we define the length of the vertical segment $[(x,t),(y,s)]$ by

$$\rho_v((x,t),(y,s)) = \inf\left\{|q| : X^q(x,t) = (y,s) \text{ e } q \in \mathbb{R}\right\}.$$

For the height function $r = 1$, the Bowen–Walters distance $d((x,t),(y,s))$ between two points $(x,t),(y,s) \in \Sigma^r$ is defined as the infimum of the lengths of all paths between (x,t) and (y,s) that are composed of finitely many horizontal and vertical segments.

Now we consider an arbitrary continuous height function $r : \Sigma \to (0,\infty)$ and we introduce the Bowen–Walters distance d_{Σ^r} in Σ^r.

Definition 3.2.3. Given $(x,t),(y,s) \in \Sigma^r$, we define

$$d_{\Sigma^r}((x,t),(y,s)) = d((x,t/r(x)),(y,s/r(y))),$$

where d is the Bowen–Walters distance for the height function $r = 1$.

For an arbitrary function r, a horizontal segment takes the form

$$w = [(x, t\cdot r(x)),(y, t\cdot r(y))],$$

and its length is given by

$$\ell_h(w) = (1-t)d_\Sigma(x,y) + t d_\Sigma(f(x),f(y)).$$

Moreover, the length of a vertical segment $w = [(x,t),(x,s)]$ is now

$$\ell_v(w) = |t-s|/r(x),$$

for any sufficiently close t and s.

It is sometimes convenient to measure distances in another manner. Namely, given $(x,t),(y,s) \in \Sigma^r$, let

$$d_\pi((x,t),(y,s)) = \min\left\{\begin{array}{c} d_\Sigma(x,y) + |t-s|, \\ d_{\Sigma^r}(f(x),y) + r(x) - t + s, \\ d_{\Sigma^r}(x,f(y)) + r(y) - s + t, \end{array}\right\}. \tag{3.2.3}$$

We note that d_π may not be a distance. Nevertheless, the following result relates d_π to the Bowen–Walters distance d_{Σ^r}. The proof of the next proposition can be found in [4, Proposition 2.1].

Proposition 3.2.4 ([4]). *If f is an invertible Lipschitz map with Lipschitz inverse, then there exists a constant $c \geq 1$ such that*

$$c^{-1} d_\pi(p,q) \leq d_{\Sigma^r}(p,q) \leq c d_\pi(p,q) \tag{3.2.4}$$

for every $p, q \in \Sigma^r$.

Let ν be a measure in Σ invariant by f. We denote by *Leb* the Lebesgue measure in $\mathbb{R}$. The measure $(\nu \times Leb)|_{\Sigma^r}$ is invariant by the suspension flow $(X^t)_t$. We call $\mu = (\nu \times Leb)|_{\Sigma^r}$ the *suspension of* ν. We that for every mensurable function $\psi : \Sigma^r \to \mathbb{R}$

$$\int \psi d\mu = \int d\nu(x) \int_0^{r(x)} \psi(X^s(x)) ds.$$

In particular

$$\mu(\Sigma^r) = \int 1 d\mu = \int r(x) d\nu(x).$$

Given a $(X^t)_t$-invariant measure μ, we will build a f-invariant measure $\tilde{\mu}$ on Σ from μ following [35, Section 3.4.2]. For each $\rho > 0$, we denote $\Sigma_\rho = \{x \in \Sigma : r(x) \geq \rho\}$. Given $V \subset \Sigma_\rho$ and $\delta \in (0, \rho]$, we denote $V_\delta = \{X^t(x) : x \in V \text{ e } 0 \leq t \leq \delta\}$. Observe that the application $(x,t) \mapsto X^t(x)$ is a bijection from $V \times (0, \delta]$ in V_δ. We shall assume that Σ is endowed with a σ-algebra of measurable subsets for which

1. the function r and the maps f and f^{-1} are measurable;
2. if $V \subset \Sigma_\rho$ is measurable then $V_\delta \subset \Sigma^r$ is measurable, for all $\delta \in (0, \rho]$.

Lemma 3.2.5 ([35]). *Let V be a measurable subset of Σ_ρ, for some $\rho > 0$. Then the function $\delta \mapsto \dfrac{\mu(V_\delta)}{\delta}$ is constant in the interval $(0, \rho]$.*

Proof. Consider any $\delta \in (0, \rho]$ and $\ell \geq 1$. It is clear that $V_\delta = \cup_{i=0}^{\ell-1} X^{\frac{i\delta}{\ell}}(V_{\frac{\delta}{\ell}})$ and this union is disjoint. Using that μ is invariant under the flow $(X^t)_t$, $t \in \mathbb{R}$, we conclude that $\mu(V_\delta) = \ell \mu(V_{\frac{\delta}{\ell}})$ for all $\delta \in (0, \rho]$ and all $\ell \geq 1$. Then, $\mu(V_{s\delta}) = s\mu(V_\delta)$ for all $\delta \in (0, \rho]$ and all rational number $s \in (0,1)$. Using that the two sides of this relation vary monotonously with s, we conclude that the equality remains valid for all real number $s \in (0,1)$. This implies the conclusion of the lemma. $\square$

For any measurable subset V of Σ_ρ, $\rho > 0$, we define $\tilde{\mu}(V) = \dfrac{\mu(V_\delta)}{\delta}$ for any $\delta \in (0, \rho]$. Then given any measurable set $V \subset \Sigma$, we define $\tilde{\mu}(V) = \sup_\rho \tilde{\mu}(V \cap \Sigma_\rho)$.

Lemma 3.2.6 ([35]). *The measure $\tilde{\mu}$ in Σ is invariant by the map f.*

Proof. We begin by observing that the complement of the image $f(\Sigma)$ has zero measure. Indeed, suppose that there exits a set $F \subset \Sigma \backslash f(\Sigma)$ with $\tilde{\mu}(F) > 0$. It is not restriction to assume that $F \subset \Sigma_\rho$ for some $\rho > 0$. Then, $\mu(F_\rho) > 0$. Since μ is finite, by hypothesis, we can apply the Poincaré's recurrence theorem in the flow $(X^{-t})_{t\in\mathbb{R}}$. We obtain that there is $z \in F_\rho$ such that $X^{-s}(z) \in F_\rho$ for values of $s > 0$ arbitrarily big. By definition, $z = X^t(y)$ for some $y \in F$ and some $t \in (0, \rho]$. By construction, the past trajectory of y intersects Σ and therefore there $x \in \Sigma$ such that $f(x) = y$. This contradicts the choice of F. Therefore our assertion is proved.

Given a measurable set $F \subset \Sigma$, we denote $E = f^{-1}(F)$. Furthermore, given $\epsilon > 0$, we consider a measurable partition of F in measurable subsets F^i satisfying the following conditions: for each i there exists $\rho_i > 0$ such that

1. F^i and $E^i = f^{-1}(F^i)$ are contained in Σ_{ρ_i};
2. $\sup(r|E^i) - \inf(r|E^i) < \epsilon\rho_i$.

Then choose $t_i < \inf(r|E^i) \leq \sup(r|E^i) < s_i$ such that $s_i - t_i < \epsilon\rho_i$. Fix $\delta_i = \rho_i/2$. Then, using the fact that f is surjective,

$$X^{t_i}(E^i_{\delta_i}) \supset F^i_{\delta_i-(s_i-t_i)} \quad \text{and} \quad X^{s_i}(E^i_{\delta_i}) \subset F^i_{\delta_i+(s_i-t_i)}.$$

Therefore, using the hypothesis that μ is invariant,

$$\mu(E^i_{\delta_i}) = \mu(X^{t_i}(E^i_{\delta_i})) \geq \mu(F^i_{\delta_i-(s_i-t_i)})$$

and

$$\mu(E^i_{\delta_i}) = \mu(X^{s_i}(E^i_{\delta_i})) \geq \mu(F^i_{\delta_i+(s_i-t_i)}).$$

Dividing by δ_i we obtain that

$$\tilde{\mu}(E^i) \geq 1 - \frac{(s_i - t_i)}{\delta_i}\tilde{\mu}(F^i) > (1 - 2\epsilon)\mu(F^i)$$

and

$$\tilde{\mu}(E^i) \leq 1 + \frac{(s_i - t_i)}{\delta_i}\tilde{\mu}(F^i) > (1 + 2\epsilon)\mu(F^i).$$

Finally, summing over all values of i, we conclude that

$$(1 - 2\epsilon)\tilde{\mu}(E) \leq \tilde{\mu}(F) \leq (1 + 2\epsilon)\tilde{\mu}(E).$$

Since ϵ is arbitrary, this proves that the measure $\tilde{\mu}$ is invariant under f. $\square$

Remark 3.2.7. It is easy to see that $\mu \to \tilde{\mu}$ is onto and one-to-one. In fact, every f-invariant measure ν is of the form $\tilde{\mu}$ for $\mu = (\nu \times Leb)|_{\Sigma^r} / \int r d\nu$ on Σ^r, which means that $\mu \to \tilde{\mu}$ is onto. And if μ_1 and μ_2 are two different $(X^t)_t$-invariant measures, there must be some set $E \subset \Sigma$ such that $\mu_1(E_\delta) \neq \mu_2(E_\delta)$ which implies that $\tilde{\mu}_1 \neq \tilde{\mu}_2$ and $\mu \to \tilde{\mu}$ is one-to-one. Moreover, we have a bijection between the set $\mathcal{M}_1(\Sigma^r, (X^t)_t)$ of the $(X^t)_t$-invariant probabilities measures and the set $\mathcal{M}_1(\Sigma, f)$ of the f-invariant probabilities. For this, we just consider $\mu \mapsto \bar{\mu}$, where

$$\bar{\mu} = \frac{\tilde{\mu}}{\tilde{\mu}(\Sigma)}.$$

3.3 Hyperbolic flows

The results in this section follow [6, Chapter 3], which was based on the works of Bowen [13] and Ratner [38]. Let $(X^t)_{t\in\mathbb{R}}$ be a C^1 flow in a smooth manifold M. This means that $X^0 = id$,

$$X^t \circ X^s = X^{t+s} \quad \text{for} \quad t, s \in \mathbb{R},$$

and that the map $(t, x) \mapsto X^t(x)$ is of class C^1.

Let $(X^t)_{t\in\mathbb{R}}$ be a C^1 flow with a locally maximal hyperbolic set Λ. Consider an open smooth disk $D \subset M$ of dimension $\dim M - 1$ that is transverse to the flow $(X^t)_{t\in\mathbb{R}}$, and take $x \in D$. Let also $U(x)$ be an open neighborhood of x diffeomorphic to the product $D \times (-\epsilon, \epsilon)$. The projection $\pi_D : U(x) \to D$ defined by $\pi_D(X^t(y)) = y$ is differentiable.

Definition 3.3.1. A closed set $R \subset \Lambda \cap D$ is said to be a rectangle if $R = \text{int}R$ (with the interior computed with respect to the induced topology on $\Lambda \cap D$) and $\pi_D([x, y]) \in R$ for $x, y \in R$.

Now we consider a collection of rectangles $R_1, ..., R_k \subset \Lambda$ (each contained in some open disk transverse to the flow) such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for} \quad i \neq j.$$

Let $\Gamma = \bigcup_{i=1}^k R_i$. We assume that there exists an ϵ such that:

1. $\Lambda = \bigcup_{t\in[0,\epsilon]} X^t(\Gamma)$;
2. for each $i \neq j$ either

$$X^t(R_i) \cap R_j = \emptyset \quad \text{for every} \quad t \in [0, \epsilon],$$

or

$$X^t(R_j) \cap R_i = \emptyset \quad \text{for every} \quad t \in [0, \epsilon].$$

We define the *transfer function* $\tau : \Lambda \to [0, \infty)$ by

$$\tau(x) = \min \left\{ t > 0 : X^t(x) \in \Gamma \right\},$$

and the transfer map $P : \Lambda \to \Gamma$ by

$$P(x) = X^{\tau(x)}(x). \tag{3.3.1}$$

The set Γ is a Poincaré section for the flow $(X^t)_{t\in\mathbb{R}}$. One can easily verify that the restriction of the map P to Γ is invertible. We also have $P^n(x) = X^{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(P^i(x)).$$

Now we introduce the notion of a Markov system.

Definition 3.3.2. The collection of rectangles $R_1, ..., R_k$ is said to be a Markov system for $(X^t)_{t\in\mathbb{R}}$ on Λ if

$$P\left(\mathrm{int}\left(W^s_\epsilon(x) \cap R_i\right)\right) \subset \mathrm{int}\left(W^s_\epsilon(P(x)) \cap R_j\right)$$

and

$$P^{-1}\left(\mathrm{int}\left(W^u_\epsilon(P(x))\right) \cap R_j\right) \subset \mathrm{int}\left(W^u_\epsilon(x) \cap R_i\right)$$

for every $x \in \mathrm{int}P(R_i) \cap \mathrm{int}R_j$.

It follows from work of Bowen [13] and Ratner [38] that any locally maximal hyperbolic set Λ has a Markov system of arbitrary small diameter. Furthermore, the map τ is Hölder continuous on each domain of continuity, and

$$0 < \inf\left\{\tau(x) : x \in \Gamma\right\} \leq \sup\left\{\tau(x) : x \in \Lambda\right\} < \infty.$$

Now we describe how a Markov system for a hyperbolic set gives rise to a symbolic dynamics.

Given a Markov system $R_1, ..., R_k$ for a flow $(X^t)_{t\in\mathbb{R}}$ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix $\mathbf{R}$ with entries

$$r_{ij} = \begin{cases} 1 & \text{if } \mathrm{int}P(R_i) \cap \mathrm{int}R_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where P is the transfer map in (3.3.1). We also consider the set $\Sigma_{\mathbf{R}} \subset \{1, ..., k\}^{\mathbb{Z}}$ given by

$$\Sigma_{\mathbf{R}} = \left\{ (\cdots i_{-1} i_0 i_1 \cdots) : r_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z} \right\},$$

and the shift map $\sigma_{\mathbf{R}} : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ defined by $\sigma_{\mathbf{R}}(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$.

Definition 3.3.3. The map $\sigma_{\mathbf{R}}$ is said to be a *(two-sided) topological Markov chain* with *transition matrix* $\mathbf{R}$.

We define a coding map $\rho : \Sigma_{\mathbf{R}} \to \bigcup_{i=1}^{k} R_i$ for the hyperbolic set by

$$\rho(\cdots i_0 \cdots) = \bigcap_{j\in\mathbb{Z}} \overline{(P|\Gamma)^{-j}(\mathrm{int} R_{i_j})}.$$

One can easily verify that

$$\rho \circ \sigma_{\mathbf{R}} = P \circ \rho. \tag{3.3.2}$$

Given $\beta > 1$, we equip $\Sigma_{\mathbf{R}}$ with the distance d given by

$$d\left((\cdots i_{-1} i_0 i_1 \cdots), (\cdots j_{-1} j_0 j_1 \cdots)\right) = \sum_{n=-\infty}^{\infty} \beta^{-|n|}|i_n - j_n|.$$

As observed in [13, Lemma 2.2], it is always possible to choose the constant β so that the function $\tau \circ \rho : \Sigma_{\mathbf{R}} \to [0, \infty)$ is Lipschitz. By (3.3.2), the restriction of a smooth flow to a locally maximal hyperbolic set is a factor of a suspension flow over a topological Markov chain. Namely, to each Markov system one can associate the suspension flow $(Y^t)_{t\in\mathbb{R}}$ over $\sigma_{\mathbf{R}}|\Sigma_{\mathbf{R}}$ with (Lipschitz) height function $r = \tau \circ \rho$. We extend ρ to a finite-to-one onto map $\pi : \Sigma_{\mathbf{R}}^r \to \Lambda$ by

$$\pi(x, s) = (X^s \circ \rho)(x)$$

for

$$(x, s) \in \Sigma_{\mathbf{R}}^r = \{(x, t) : 0 \le t \le r(x), x \in \Sigma_{\mathbf{R}}\} / \sim$$

where $(x, 0) \sim (\sigma_{\mathbf{R}}, r(x))$. Then

$$\pi \circ Y^t = X^t \circ \pi \tag{3.3.3}$$

for every $t \in \mathbb{R}$. We denote by $\Sigma_{\mathbf{R}}^+$ the set of (one-sided) sequences $(i_0 i_1 \cdots)$ such that

$$(i_0 i_1 \cdots) = (j_0 j_1 \cdots) \text{ for some } (\cdots j_{-1} j_0 j_1 \cdots) \in \Sigma_{\mathbf{R}},$$

and by $\Sigma_{\mathbf{R}}^-$ the set of (one-sided) sequences $(\cdots i_{-1} i_0)$ such that

$$(\cdots i_{-1} i_0) = (\cdots j_{-1} j_0) \text{ for some } (\cdots j_{-1} j_0 j_1 \cdots) \in \Sigma_{\mathbf{R}}.$$

The set $\Sigma_{\mathbf{R}}^-$ can be identified with $\Sigma_{\mathbf{R}^*}^+$, where $\mathbf{R}^*$ is the transpose of $\mathbf{R}$, by the map

$$\Sigma_{\mathbf{R}}^- \ni (\cdots i_{-1} i_0) \mapsto (i_0 i_{-1} \cdots) \in \Sigma_{\mathbf{R}^*}^+.$$

We also consider the shift maps $\sigma_{\mathbf{R}}^+ : \Sigma_{\mathbf{R}}^+ \to \Sigma_{\mathbf{R}}^+$ and $\sigma_{\mathbf{R}}^- : \Sigma_{\mathbf{R}}^- \to \Sigma_{\mathbf{R}}^-$ defined by

$$\sigma_{\mathbf{R}}^+(i_0 i_1 \cdots) = (i_1 i2 \cdots) \text{ and } \sigma_{\mathbf{R}}^-(\cdots i_{-1} i_0) = (\cdots i - 2 i_{-1}).$$

Now we describe how distinct points in a stable or unstable manifold can be characterized in terms of the symbolic dynamics. Given $x \in \Lambda$, take $\omega \in \Sigma_{\mathbf{R}}$ such that $\rho(\omega) = x$. Let $R(x)$ be a rectangle of the Markov system that contains x. For each $\omega' \in \Sigma_{\mathbf{R}}$, we have

$$\rho(\omega') \in W^u_\epsilon(x) \cap R(x) \quad \text{whenever} \quad \rho_-(\omega') = \rho_-(\omega),$$

and

$$\rho(\omega') \in W^s_\epsilon(x) \cap R(x) \quad \text{whenever} \quad \rho_+(\omega') = \rho_+(\omega),$$

where $\rho_+ : \Sigma_{\mathbf{R}} \to \Sigma^+_{\mathbf{R}}$ and $\rho_- : \Sigma_{\mathbf{R}} \to \Sigma^-_{\mathbf{R}}$ are the projections defined by

$$\rho_+(\cdots i_{-1} i_0 i_1 \cdots) = (i_0 i_1 \cdots)$$

and

$$\rho_-(\cdots i_{-1} i_0 i_1 \cdots) = (\cdots i_{-1} i_0).$$

Therefore, writing $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$, the set $W^u_\epsilon(x) \cap R(x)$ can be identified with the cylinder set

$$C^+_{i_0} = \left\{ (j_0 j_1 \cdots) \in \Sigma^+_{\mathbf{R}} : j_0 = i_0 \right\} \subset \Sigma^+_{\mathbf{R}},$$

and the set $W^s_\epsilon(x) \cap R(x)$ can be identified with the cylinder set

$$C^-_{i_0} = \left\{ (\cdots j_{-1} j_0) \in \Sigma^-_{\mathbf{R}} : j_0 = i_0 \right\} \subset \Sigma^-_{\mathbf{R}}.$$

Chapter 4

Cocycles over hyperbolic flows

In this chapter we show that for an open and dense set with respect to $SL(2,\mathbb{K})$ fiber-bunched cocycles (cf Definition 2.1.6), the Lyapunov exponents are non-zero almost everywhere.

Let $(X^t)_{t\in\mathbb{R}}$ be a C^1 flow on M with a locally maximal hyperbolic set $\Lambda \subset M$. Assume that μ is an invariant probability measure of $(X^t)_t$ that is ergodic, hyperbolic and satisfies the local product structure on Λ. Let also $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator with fiber-bunched associated cocycle Φ_A^t.

4.1 Lipschitz continuity

We start by showing that the cocycle $\Phi_A^t(x)$ is also Lipschitz continuous with respect to variable x, for each $t \in \mathbb{R}$.

Lemma 4.1.1. *Given any $t \in \mathbb{R}$ and $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$, there is $C_1 = C_1(t, A) > 0$ such that, for all $y, z \in M$, we have $\|\Phi_A^t(y) - \Phi_A^t(z)\| \le C_1 d(y, z)$.*

Proof. Fix $x \in M$ and $t \in \mathbb{R}$. Since $\Phi_A^t(x)$ is the solution of the differential equation $\partial_t u(t) = A(X^t(x)) \cdot u(t)$, we obtain that

$$\Phi_A^t(x)v = v + \int_0^t A(X^s(x))\Phi_A^s(x)vds.$$

Similarly we have

$$(\Phi_A^t(x))^{-1}v = v - \int_0^t A(X^s(x))(\Phi_A^s(x))^{-1}vds.$$

Hence we have

$$\|\Phi_A^t(x)v\| \le \|v\| + \int_0^t \|A(X^s(x))\| \cdot \|\Phi_A^s(x)v\|ds \tag{4.1.1}$$

and

$$\|(\Phi_A^t(x))^{-1}v\| \leq \|v\| + \int_0^t \|A(X^s(x))\| \cdot \|(\Phi_A^s(x))^{-1}v\| ds. \tag{4.1.2}$$

We make use the Grönwall's inequality, that is given by the following

Lemma 4.1.2 (Grönwall's inequality)**.** *Let $u, v : [a, b] \to \mathbb{R}$ be non-negative continuous functions that, for some $\alpha \geq 0$, satisfy*

$$u(t) \leq \alpha + \int_a^t u(s)v(s)ds$$

for all $t \in [a, b]$. Then

$$u(t) \leq \alpha \exp\left[\int_a^t v(s)ds\right]$$

for all $t \in [a, b]$.

Proof. See [26]. □

Hence, from (4.1.1) we have

$$\|\Phi_A^t(x)v\| \leq \|v\| \exp\left[\int_0^t \|A(X^s(x))\|ds\right] \tag{4.1.3}$$

and, thus, $\|\Phi_A^t(x)v\| \leq e^{\|A\||t|}\|v\|$ for all $t \in \mathbb{R}$ and $v \in \mathbb{R}^2$. Since A is Lipschitz, there is a constant $K > 0$ such that $\|A(x) - A(y)\| \leq Kd(x, y)$. Applying Gronwall's inequality to $(X^t)_t$ we have that

$$\begin{aligned}
\|\Phi_A^t(y)v - \Phi_A^t(z)v\| &\leq \int_0^t \left[\|A(X^s(y)) - A(X^s(z))\|\|\Phi_A^t(y)v\| + \|A\|\|\Phi_A^s(y)v - \Phi_A^s(z)v\|\right] ds \\
&\leq e^{\|X\||t|}\|v\|K \int_0^t e^s d(y, z)ds + \int_0^t \|A\|\|\Phi_A^s(y)v - \Phi_A^s(z)v\|ds \\
&\leq e^{\|X\||t|}\|v\|Kd(y, z) + \int_0^t \|A\|\|\Phi_A^s(y)v - \Phi_A^s(z)v\|ds.
\end{aligned}$$

Applying again Grönwall's inequality to Φ_A^t, it follows that

$$\|\Phi_A^t(y) - \Phi_A^t(z)\| \leq e^{|t|(\|A\|+\|X\|)}Kd(y, z),$$

which proves the lemma. □

The next lemma tells us that fixing $x \in M$ and $t \in \mathbb{R}$ the matrix $\Phi_A^t(x)$ varies continuously with respect to the infinitesimal generator A.

Lemma 4.1.3. *Given $t \in \mathbb{R}$ and $A, B \in C^{r,\nu}(M, \mathfrak{sl}(2, \mathbb{K}))$, there exists $C_1 = C_1(t, A, B) > 0$ such that for all $x \in M$ and all $v \in \mathbb{K}^2$, we have $\|\Phi_A^t(x)v - \Phi_B^t(x)v\| \leq C_1\|A - B\|\|v\|$.*

Proof. We have that

$$\Phi_A^t(x)v = v + \int_0^t A(X^s(x))\Phi_A^s(x)vds$$

and

$$\Phi_B^t(x)v = v + \int_0^t B(X^s(x))\Phi_B^s(x)vds$$

for all $t \in \mathbb{R}$. So

$$\begin{aligned}
\|\Phi_A^t(x)v - \Phi_B^t(x)v\| &\leq \int_0^t \|A(X^s(x)) - B(X^s(x))\|\|\Phi_B^s(x)v\| + \|A\|\|\Phi_A^s(x)v - \Phi_B^s(x)v\|ds \\
&\leq e^{\|B\||t|}\|v\| \int_0^t \|A - B\|ds + \int_0^t \|A\|\|\Phi_A^s(x)v - \Phi_B^s(x)v\|ds \\
&\leq e^{\|B\||t|}\|v\||t| + \int_0^t \|A\|\|\Phi_A^s(x)v - \Phi_B^s(x)v\|ds \\
\text{(Grönwall inequality)} &\leq |t|\|v\|e^{\|B\||t|}e^{\|A\||t|}\|A - B\| \\
&\leq |t|\|v\|e^{(\|A\|+\|B\|)|t|}\|A - B\|.
\end{aligned}$$

So we have $\|\Phi_A^t(x)v - \Phi_B^t(x)v\| \leq C_1\|A - B\|\|v\|$ for all $x \in M$ with $C_1 = |t| \cdot e^{(\|A\|+\|B\|)|t|}$, which proves the lemma. □

Note that there exists a C^0-open set in the space of fiber-bunched linear cocycles.

Lemma 4.1.4. *Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator and $(X^t)_t\colon M \to M$ be a smooth flow on M. Let $\Lambda \subset M$ be a hyperbolic set for $(X^t)_t$ and $\theta_1 \in (0,1)$ the hyperbolicity constant as in (2.1.1). If $\|A(x)\| < \frac{\alpha}{2}\log\theta_1^{-1}$ for some $\alpha > 0$ and all $x \in \Lambda$, then*

$$\|\Phi_A^t(x)\| \cdot \|(\Phi_A^t(x))^{-1}\| < \frac{1}{\theta_1^{t\alpha}}$$

for all $x \in \Lambda$ e all $t \in \mathbb{R}$.

Proof. Since, by hypothesis, $\|A(x)\| < \dfrac{\alpha}{2}\log\theta_1^{-1}$, for all $x \in \Lambda$, we have $\|A(X^s(x))\| < \dfrac{\alpha}{2}\log\theta_1^{-1}$, for all $s \in \mathbb{R}$. By (4.1.3), we have

$$\frac{\|\Phi_A^t(x)v\|}{\|v\|} < \sqrt{\frac{1}{\theta_1^{t\alpha}}},$$

for all $v \neq 0$. Hence $\|\Phi_A^t(x)\| < \sqrt{\frac{1}{\theta_1^{t\alpha}}}$ for every $x \in \Lambda$. Similarly, using (4.1.2), we find that $\|(\Phi_A^t(x))^{-1}\| < \sqrt{\frac{1}{\theta_1^{t\alpha}}}$. Hence, it follows that $\|\Phi_A^t(x)\| \cdot \|(\Phi_A^t(x))^{-1}\| < \frac{1}{\theta_1^{t\alpha}}$ for all $t \in \mathbb{R}$. □

4.2 Proof of generic simplicity of Lyapunov spectra

We begin by describing the criterion for simplicity of the Lyapunov spectra presented in [1] for discrete-time cocycles.

Let $f : \Sigma \to \Sigma$ be an invertible measurable map and $A : \Sigma \to GL(d, \mathbb{C})$ be a measurable function with values in the group of invertible $d \times d$ complex matrices. These data define a linear cocycle F_A over the map f,

$$F_A : \Sigma \times \mathbb{C}^d \to \Sigma \times \mathbb{C}^d, \quad F_A(x, v) = (f(x), A(x)v)\,.$$

Note that $F_A^n(x, v) = (f^n(x), A^n(x) \cdot v)$, where $A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))A(x)$ and $A^n(x)$ is the inverse of $A^{-n}(f^n(x))$ if $n < 0$.

Symbolic dynamics

Let $\hat{\Sigma} = \mathbb{N}^{\mathbb{Z}}$ be the full shift space with countable many symbols, and let $\sigma : \hat{\Sigma} \to \hat{\Sigma}$ shift map:

$$\sigma\left((x_n)_{n\in\mathbb{Z}}\right) = (x_{n+1})_{n\in\mathbb{Z}}.$$

Let us call *cylinder* of $\hat{\Sigma}$ any set of the form

$$[\imath_m, \ldots, \imath_{-1}; \imath_0; \imath_1, \ldots, \imath_n] = \{\hat{x} : x_j = \imath_j \text{ for } j = m, \ldots, n\}\,.$$

Cylinders of $\Sigma^u = \mathbb{N}^{\{n\geq 0\}}$ and $\Sigma^s = \mathbb{N}^{\{n<0\}}$ are defined similarly, corresponding to

$$[\imath_0; \imath_1, \ldots, \imath_n] = \{\hat{x} : x_j = \imath_j \text{ for } j = 0, \ldots, n\} \subset \Sigma^u$$

and

$$[\imath_m, \ldots, \imath_{-1}; \imath_0] = \{\hat{x} : x_j = \imath_j \text{ for } j = m, \ldots, -1\} \subset \Sigma^s.$$

We endow $\hat{\Sigma}$, Σ^u, Σ^u with the topologies generated by the corresponding cylinders. Let $Q^u : \hat{\Sigma} \to \Sigma^u$ and $Q^s : \hat{\Sigma} \to \Sigma^s$ be the natural projections. We also consider the one-sided shift maps $\sigma^u : \Sigma^u \to \Sigma^u$ and $\sigma^s : \Sigma^s \to \Sigma^s$ defined by

$$\sigma^u \circ Q^u = Q^u \circ \sigma \text{ and } \sigma^s \circ Q^s = Q^s \circ \sigma^{-1}.$$

For each $\hat{x} = (x_n)_{n\in\mathbb{Z}}$ in $\hat{\Sigma}$, we denote $x^u = Q^u(\hat{x})$ and $x^s = Q^s(\hat{x})$. Then $\hat{x} \mapsto (x^s, x^u)$ is a homeomorphism from $\hat{\Sigma}$ to the product $\Sigma^s \times \Sigma^u$. In what follows we often identify the two sets through this homeomorphism. When there is no risk of ambiguity, we also identify the *local stable set*

$$W^s_\epsilon(x^u) = W^s_\epsilon(\hat{x}) = \{(y_n)_{n\in\mathbb{Z}} : x_n = y_n \text{ for all } n \geq 0\} \text{ with } \Sigma^s$$

and the *local unstable set*

$$W^u_\epsilon(x^s) = W^u_\epsilon(\hat{x}) = \{(y_n)_{n\in\mathbb{Z}} : x_n = y_n \text{ for all } n < 0\} \text{ with } \Sigma^u,$$

via the projections Q^s and Q^u.

Local product structure

Let $\hat{\mu}$ be an ergodic σ-invariant probability measure and let $\mu^u = Q^u_* \hat{\mu}$ and $\mu^s = Q^s_* \hat{\mu}$ be the images of $\hat{\mu}$ under the natural projections. It is easy to see that these are ergodic invariant probabilities for σ^u and σ^s, respectively. We assume μ^s and μ^u to be positive on cylinders. Moreover, we assume $\hat{\mu}$ to be equivalent to their product, meaning there exists a measurable function $\rho : \hat{\Sigma} \to (0, \infty)$, bounded away from zero and infinity, such that

$$\hat{\mu} = \rho(\hat{x})(\mu^s \times \mu^u), \quad \hat{x} \in \hat{\Sigma}.$$

Stable and unstable holonomies

Definition 4.2.1. Let $A : \hat{\Sigma} \to GL(d, \mathbb{C})$ be a measurable function and $F_A : \hat{\Sigma} \times \mathbb{C}^d \to \Sigma \times \mathbb{C}^d$ be a linear cocycle over the map $\sigma : \hat{\Sigma} \to \hat{\Sigma}$. A *stable holonomy* for A is a continuous map $H^s_A : (x, y) \mapsto H^s_{A,x,y}$, where $x \in \hat{\Sigma}$, $y \in W^s(x)$, and $H^s_{A,x,y} \in GL(d, \mathbb{C})$, is such that

(i) $H^s_{A,x,y}$ is a linear map from $\mathcal{E}_x = \{x\} \times \mathbb{C}^d$ in $\mathcal{E}_y = \{y\} \times \mathbb{C}^d$;

(ii) $H^s_{A,x,x} = Id$ and $H^s_{A,y,z} \circ H^s_{A,x,y} = H^s_{A,x,z}$, for every $y, z \in W^s(x)$

(iii) $H^s_{A,x,y} = (A^n(y))^{-1} \circ H^s_{A,f^n(x),f^n(y)} \circ A^n(x)$ for all $n \in \mathbb{N}$ and $y, z \in W^s(x)$.

Unstable holonomies $H^u_{A,x,y}$ are defined similarly as the stable for holonomies for f^{-1}.

As an easy example, if A is constant on each cylinder $[i]$, $i \in \mathbb{N}$, then we can define $H^s_{A,x,y} \equiv \text{id}$ and $H^u_{A,x,y} \equiv \text{id}$. We will see that fiber-bunched cocycles also admit stable and unstable holonomies.

Statement of the criterion

Let $\widehat{\Psi} : \hat{\Sigma} \to GL(d, \mathbb{C})$ be a continuous cocycle over the full shift map $\sigma : \hat{\Sigma} \to \hat{\Sigma}$. Let $\hat{p} \in \hat{\Sigma}$ be a periodic point for σ and $q(\hat{p}) \geq 1$ be its period. We call $\hat{z} \in \hat{\Sigma}$ a *homoclinic point* of $\hat{p}$ if $\hat{z} \in W^u_\epsilon(\hat{p})$ and there exists some multiple $l \geq 1$ of $q(\hat{p})$ such that $\sigma^l(\hat{z}) \in W^s_\epsilon(\hat{p})$. We assume that $\widehat{\Psi}$ admits stable and unstable holonomies, respectively, $H^s_{\widehat{\Psi}}$ and $H^u_{\widehat{\Psi}}$. Then we define the *transition map* (see Figure 4.2.1)

$$\zeta_{\widehat{\Psi},\hat{p},\hat{z}} : \mathbb{C}^d_{\hat{p}} \to \mathbb{C}^d_{\hat{p}}, \quad \zeta_{\widehat{\Psi},\hat{p},\hat{z}} = H^s_{\widehat{\Psi},\sigma^l(\hat{z}),\hat{p}} \circ \widehat{\Psi}^l(\hat{z}) \circ H^u_{\widehat{\Psi},\hat{p},\hat{z}}.$$

Theorem 4.2.2 ([1, Theorem A])**.** *Let $\widehat{\Psi} : \hat{\Sigma} \to GL(d, \mathbb{C})$ be a continuous cocycle over the full shift map σ, such that $\hat{\Psi}$ admits stable and unstable holonomies. Suppose that $\hat{\mu}$*

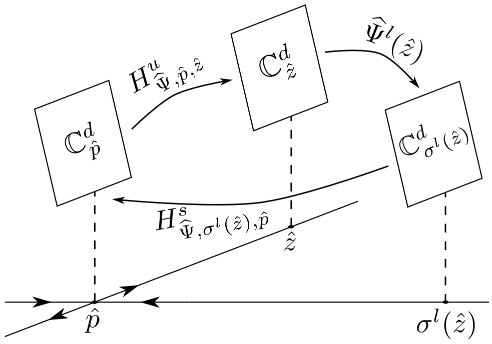

Figure 4.2.1: Transition map.

is an ergodic σ-invariant probability measure with local product structure. Suppose also that there exists a periodic point $\hat{p} \in \hat{\Sigma}$ of σ and some homoclinic point $\hat{z} \in \hat{\Sigma}$ of $\hat{p}$ such that

(p) All the eigenvalues of $\widehat{\Psi}^{q(\hat{p})}(\hat{p})$ have distinct absolute values.

(t) For any invariant subspaces (sums of eigenspaces) E and F of $\widehat{\Psi}^{q(\hat{p})}(\hat{p})$ with $dimE + dimF = d$, we have $\zeta_{\widehat{\Psi},\hat{p},\hat{z}}(E) \cap F = \{0\}$.

Then all the Lyapunov exponents of the cocycle $\widehat{\Psi}$ for the measure $\hat{\mu}$ have multiplicity 1.

We refer to (p) as the *pinching property* and to (t) as the *twisting property*.

Remark 4.2.3. Let E_j, $j = 1, \dots, d$, represent the eigenspaces of $\widehat{\Psi}^{q(\hat{p})}(\hat{p})$. For $d = 2$ the twisting condition means that $\zeta_{\widehat{\Psi},\hat{p},\hat{z}}(E_i) \neq E_j$ for all $1 \leq i, j \leq 2$. For $d = 3$ it means that $\zeta_{\widehat{\Psi},\hat{p},\hat{z}}(E_i)$ is outside the plane $E_j \oplus E_k$ and E_i is outside the plane $\zeta_{\widehat{\Psi},\hat{p},\hat{z}}(E_j \oplus E_k)$, for all choices of $1 \leq i, j, k \leq 3$. In general, this condition is equivalent to saying that the matrix of the transition map in a basis of eigenvectors of $\widehat{\Psi}^{q(\hat{p})}(\hat{p})$ has all its algebraic minors different from zero. Indeed, it may be restated as saying that the determinant of the square matrix

$$\begin{pmatrix} B_{1,i_1} & \cdots & B_{1,i_r} & \delta_{1,j_1} & \cdots & \delta_{1,j_s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{d,i_1} & \cdots & B_{d,i_r} & \delta_{d,j_1} & \cdots & \delta_{d,j_s} \end{pmatrix} \tag{4.2.1}$$

is non-zero for any $I = \{i_1, \dots, i_r\}$ and $J = \{i_1, \dots, i_s\}$ with $r + s = d$, where the $\delta_{i,j}$ are Dirac symbols and the $B_{i,j}$ are the entries of the matrix of $\zeta_{\widehat{\Psi},\hat{p},\hat{z}}$ in the basis of eigenvectors. Up to sign, this determinant is the algebraic minor $B[J^c \times I]$ corresponding to the lines $j \notin J$ and columns $i \in I$.

Remark 4.2.4. As pointed out in [1, Appendix A] the simplicity criterion extends directly to cocycles over any subshift of countable type $\sigma_{\mathbf{R}} : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$. This will be important for the rest of the present work.

4.2.1 Reduction to a cocycle over a Poincaré map

Let Λ be a hyperbolic set for the flow $(X^t)_{t\in\mathbb{R}}$. Let Γ be a Poincaré section for the flow $(X^t)_{t\in\mathbb{R}}$ such that $\Lambda = \bigcup_{t\in[0,\epsilon]} X^t(\Gamma)$. The transfer function $\tau : \Lambda \to [0,\infty)$ is given by

$$\tau(x) = \min\left\{t > 0 : X^t(x) \in \Gamma\right\},$$

and the transfer map $P : \Lambda \to \Gamma$ is given by

$$P(x) = X^{\tau(x)}(x). \tag{4.2.2}$$

as in Section 3.3. By defining

$$\Psi_A(x) : \mathbb{K}^2_x \to \mathbb{K}^2_{P(x)}, \quad \Psi_A(x) = \Phi_A^{\tau(x)}(x),$$

for $x \in \Gamma$, we obtain a cocycle over P.

One can build a $P|_\Gamma$-invariant measure μ_P on Γ from μ (see [35, Section 3.4.2]).

Lemma 4.2.5. *If μ is ergodic and has local product structure, then μ_P is ergodic and has local product structure.*

Proof. Suppose that μ is ergodic and has local product structure. We recall that

$$\mu = \frac{\mu_P \times Leb}{\int_\Gamma \tau d\mu_P}$$

and, given a measurable set $A \subset \Gamma$,

$$\mu(A) = \int \chi_A d\mu = \frac{1}{\int_\Gamma \tau d\mu_P} \int_\Gamma d\mu_P(x) \int_0^{\tau(x)} \chi_A(X^s(x)) ds,$$

where ds indicates integration with respect to the Lebesgue measure Leb and χ_A is the characteristic function of set A.

If $B \subset \Gamma$ is invariant under P, then the set $\widehat{B} := \cup_{t\in\mathbb{R}} X^t(B)$ is invariant under the flow $(X^t)_t$. Hence, since μ is ergodic, we have that $\mu(\widehat{B}) = 0$ or $\mu(\widehat{B}) = 1$. Suppose that $\mu_P(B) > 0$, this implies that $\mu(\widehat{B}) > 0$, therefore $\mu(\widehat{B}) = 1$. By the other, if $\mu_P(B) < 1$ we have that the complementary set B^c is such that $\mu_P(B^c) > 0$, what implies that $\mu(\widehat{B^c}) = \mu(\cup_{t\in\mathbb{R}} X^t(B^c)) > 0$, which is an absurd since $\widehat{B} \cap \widehat{B^c} = \emptyset$ and $\mu(\widehat{B}) = 1$. Therefore $\mu_P(B) = 1$ and μ_P is ergodic.

By Definition 2.1.4, local product structure for μ means that, up to a chenge of coordinates, $\mu = \mu^u \times \mu^s \times Leb$. In [27, Section 6] Haydn shows that the local product structure of μ passes to μ_P through projection along the weak stable and weak unstable leaves, that is, $\mu_P = \mu_P^u \times \mu_P^s$, up to a change of coordinates. $\square$

Let $I_{xy} : \{x\} \times \mathbb{K}^2 \mapsto \{y\} \times \mathbb{K}^2$ be the natural identification given by

$$\begin{aligned} I_{xy} : \{\mathrm{x}\}\times\mathbb{K}^2 &\longrightarrow \{\mathrm{y}\}\times\mathbb{K}^2 \\ (x,v) &\longmapsto (y,v). \end{aligned}$$

The identifications $\{I_{xy}\}$ are β-Hölder on a neighborhood of the diagonal in $M \times M$ and satisfy for some constant C and any unit vector $u \in \{x\} \times \mathbb{K}^2$ (see [29, Proposition 4.2]),

$$I_{xy} = I_{yx}^{-1}, \quad \|I_{xy}u - u\| \le Cd(x,y)^\beta, \quad \text{and hence } |\|I_{xy}\| - 1| \le Cd(x,y)^\beta. \tag{4.2.3}$$

The cocycle Ψ_A is said to be β-Hölder, if $\Psi_A(x)$ is β-Hölder with x, specifically, if there is C such that for all close points $x, y \in \Gamma$

$$\|\Psi_A(x) - I_{P(x)P(y)}^{-1} \circ \Psi_A(y) \circ I_{xy}\| \le Cd(x,y)^\beta. \tag{4.2.4}$$

Definition 4.2.6. We say that the reduced cocycle Ψ_A is *fiber-bunched* if there is $\theta_2 < 1$ such that

$$\|\Psi_A(x)\| \cdot \|(\Psi_A(x))^{-1}\| \cdot \theta_1^{\tau(x)\cdot\beta} < \theta_2, \tag{4.2.5}$$

for all $x \in \Gamma$.

Note that if Φ_A is fiber-bunched, then so is Ψ_A as a direct consequence of Definition 2.1.6. Moreover, their Lyapunov exponents differ by a multiplicative constant.

Lemma 4.2.7. *The Lyapunov exponents of Ψ_A relative to the measure μ_P coincide with the Lyapunov exponents of Φ_A relative to the measure μ, up to the multiplicative factor $\int_\Gamma \tau d\mu_P$.*

Proof. Since μ_P is ergodic, by the Oseledets theorem, for μ_P-almost every $x \in \Gamma$ there are a $\Psi_A(x)$-invariant decomposition $T_x\Gamma = E_x^1 \oplus E_x^2$ and Lyapunov exponents well defined by

$$\lambda_i(\Psi_A, P, \mu_P) = \lim_{n\to\infty} \frac{1}{n} \log \|\Psi_A^n(x)v_i\|$$

for all $v_i \in E_x^i \backslash \{0\}$ and $i = 1, 2$. Note that we take one of the E_x^i as trivial if the Lyapunov exponents are equal to zero. On the other hand, since μ_P is ergodic, it follows from the ergodic theorem of Birkhoff that $\lim_{n\to+\infty} \frac{\tau^{(n)}(x)}{n} = \int_\Gamma \tau d\mu_P$, where we denote $\tau^{(n)}(x) = \sum_{j=0}^{n-1} \tau(f^j(x))$. In particular, if $\tau_0(x)$ denotes the first time that a point $x \in \Lambda$ reaches Γ, then for μ-almost every $x \in \Lambda$ we have that $X^{\tau_0(x)}(x) \in \Gamma$ and we define the spaces $\hat{E}_x^i := \Phi_A^{-\tau_0(x)}(X^{\tau_0(x)}(x)) \cdot E^i_{X^{\tau_0(x)}(x)}$ for all $i = 1, 2$. By construction, for μ-almost every point $x \in \Lambda$, if $\hat{E}_x^0 = \mathbb{R} \cdot X(x)$, the decomposition $T_x\Lambda = \hat{E}_x^0 \oplus \hat{E}_x^1 \oplus \hat{E}_x^2$ is $\Phi_A^t(x)$-invariant. Moreover, for μ-almost every x and any $v_i \in \hat{E}_x^i \backslash \{0\}$

$$\begin{aligned} \lambda_i(\Phi_A^t, X^t, \mu) &= \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi_A^n(x)v_i\| \\ &= \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi_A^{n-\tau_0(x)}(X^{\tau_0(x)}(x))\Phi_A^{\tau_0(x)}(x)v_i\|. \end{aligned}$$

If we write $x_0 = X^{\tau_0(x)}(x) \in \Gamma$, $w_i = \Phi_A^{\tau_0(x)}(x)v_i$ and $n - \tau_0(x) = \tau^{(\ell-1)}(x_0) + s(\ell, x_0)$ for any $\ell \geq 1$ and $0 \leq s < \tau(P^\ell(x_0))$, then

$$\begin{aligned}
\lambda_i(\Phi_A^t, X^t, \mu) &= \lim_{n\to+\infty} \frac{1}{n} \log \|\Phi_A^{s(\ell,x_0)}(P^\ell(x_0))\Psi_A^\ell(x_0)w_i\| \\
&= \lim_{\ell\to+\infty} \frac{1}{\ell} \frac{\ell}{\tau^{(\ell-1)}(x_0) + s(\ell, x_0)} \log \|\Phi_A^{s(\ell,x_0)}(P^\ell(x_0))\Psi_A^\ell(x_0)w_i\| \\
&= \lim_{\ell\to+\infty} \frac{1}{\ell} \frac{\ell}{\tau^{(\ell-1)}(x_0) + s(\ell, x_0)} \log \|\Psi_A^\ell(x_0)w_i\| \\
&\quad + \lim_{\ell\to+\infty} \frac{1}{\ell} \frac{\ell}{\tau^{(\ell-1)}(x_0) + s(\ell, x_0)} \log \frac{\|\Phi_A^{s(\ell,x_0)}(P^\ell(x_0))\Psi_A^\ell(x_0)w_i\|}{\|\Psi_A^\ell(x_0)w_i\|} \\
&= \frac{1}{\int_\Gamma \tau d\mu_P} \lambda_i(\Psi_A, P, \mu_P),
\end{aligned}$$

since

$$\lim_{\ell\to+\infty} \frac{1}{\ell} \frac{\ell}{\tau^{(\ell-1)}(x_0) + s(\ell, x_0)} \log \frac{\|\Phi_A^{s(\ell,x_0)}(P^\ell(x_0))\Psi_A^\ell(x_0)w_i\|}{\|\Psi_A^\ell(x_0)w_i\|} = 0.$$

This proves the lemma. □

4.2.2 Existence of holonomies for the reduced cocycle Ψ_A

The following proposition, proved in [29], establishes existence and some properties of the stable and unstable holonomies. We include here for the reader convenience.

Proposition 4.2.8 ([29, Proposition 4.2])**.** *Suppose that the cocycle Ψ_A is* fiber-bunched. *Then there exists $C > 0$ such that for any $x \in \Gamma$ and $y \in W_\epsilon^s(x)$,*

(a) $\|(\Psi_A^n(y))^{-1} \circ I_{P^n(x)P^n(y)} \circ \Psi_A^n(x) - I_{xy}\| \leq Cd(x,y)^\beta$ *for all $n \in \mathbb{N}$;*

(b) The limit $H_{A,x,y}^s = \lim_{n\to\infty} (\Psi_A^n(y))^{-1} \circ I_{P^n(x)P^n(y)} \circ \Psi_A^n(x)$ *exists and is a linear map satisfying* (i), (ii) and (iii) *of Definition 4.2.1 and*

(iv) $\|H_{A,x,y}^s - I_{xy}\| \leq Cd(x,y)^\beta$*;*

(c) The holonomy satisfying (iv) *is unique.*

Furthermore, $H_{A,x,y}^s$ can be extended to any $y \in W^s(x)$ using (iii) *of Definition 4.2.1. Similarly, for $y \in W^u(x)$ the unstable holonomy $H_{A,x,y}^u$ can be defined as*

$$H_{A,x,y}^u = \lim_{n\to-\infty} (\Psi_A^n(y))^{-1} \circ I_{P^n(x)P^n(y)} \circ \Psi_A^n(x).$$

Proof. Fix $x \in \Gamma$ and denote $x_i = P^i(x)$. Then for any $y \in W^s_\epsilon(x)$, if $y_i = P^i(y)$, we have

$$\begin{aligned}
(\Psi^n_A(y))^{-1} \circ I_{x_n y_n} \circ \Psi^n_A(x) &= (\Psi^{n-1}_A(y))^{-1} \circ \big((\Psi_A(y_{n_1}))^{-1} \circ I_{x_n y_n} \circ \Psi_A(x_{n-1})\big) \circ \Psi^{n-1}_A(x) \\
&= (\Psi^{n-1}_A(y))^{-1} \circ \big(I_{x_{n-1} y_{n-1}} + r_{n-1}\big) \circ \Psi^{n-1}_A(x) \\
&= (\Psi^{n-1}_A(y))^{-1} \circ I_{x_{n-1} y_{n-1}} \circ \Psi^{n-1}_A(x) + \\
&\quad + (\Psi^{n-1}_A(y))^{-1} \circ r_{n-1} \circ \Psi^{n-1}_A(x) \\
&= I_{xy} + \sum_{i=0}^{n-1} (\Psi^i_A(y))^{-1} \circ r_i \circ \Psi^i_A(x),
\end{aligned} \tag{4.2.6}$$

where we use recursively the argument in the first equality of (4.2.6) and $r_i = (\Psi_A(y_i))^{-1} \circ I_{x_{i+1} y_{i+1}} \circ \Psi_A(x_i) - I_{x_i y_i}$.

Since Ψ_A is fiber-bunched, there exists $\theta_2 < 1$ such that $\|\Psi_A(x)\| \cdot \|(\Psi_A(x))^{-1}\| \cdot \theta_1^{\tau(x) \cdot \beta} < \theta_2$ for all $x \in \Gamma$. For the function $\eta(x) = \theta_1^{\tau(x)}$ we denote

$$\eta_i(x) = \eta(x_0)\eta(x_1) \cdots \eta(x_{i-1}) = \theta_1^{\sum_{k=0}^{i-1} \tau(x_k)},$$

which is a multiplicative cocycle. Then it can be estimated that $d(P^n(x), P^n(y)) \leq d(x, y) \cdot \eta_n(y)$, for all $n \geq 1$ (see [17, Lemma 1.1]). We need the following auxiliary result.

Lemma 4.2.9. *[29, Lemma 4.3] If Ψ_A is fiber-bunched , then there exists $C_0 > 0$ such that $\|(\Psi^i_A)^{-1}(y)\| \cdot \|\Psi^i_A(x)\| \leq C_0 \theta_2^i \eta_i(y)^{-\beta}$, for all $x \in \Gamma$, $y \in W^s_\epsilon(x)$, and $i \geq 0$.*

Proof. Using (4.2.3), (4.2.4) and the fact of $\|\Psi_A(\cdot)\| \geq 1$, there is a uniform $C_2 > 0$ such that

$$\begin{aligned}
\frac{\|\Psi_A(x_k)\|}{\|\Psi_A(y_k)\|} &\leq \frac{\|\Psi_A(x_k) - I^{-1}_{x_{k+1} y_{k+1}} \circ \Psi_A(y_k) \circ I_{x_k y_k}\|}{\|\Psi_A(y_k)\|} + \frac{\|I^{-1}_{x_{k+1} y_{k+1}} \circ \Psi_A(y_k) \circ I_{x_k y_k}\|}{\|\Psi_A(y_k)\|} \\
&\leq C_1 (d(x_k, y_k))^\beta + \|I^{-1}_{x_{k+1} y_{k+1}}\| \cdot \|I_{x_k y_k}\| \\
&\leq 1 + C_2 (d(x_k, y_k))^\beta,
\end{aligned}$$

for all $k \geq 0$. We estimate

$$\begin{aligned}
\|(\Psi^i_A(y))^{-1}\| \cdot \|\Psi^i_A(x)\| &\leq \|(\Psi_A(y))^{-1}\| \cdot \|(\Psi_A(y_1))^{-1}\| \cdots \|(\Psi_A(y_{i-1}))^{-1}\| \cdot \\
&\quad \cdot \|\Psi_A(x_{i-1})\| \cdots \|\Psi_A(x_1)\| \cdot \|\Psi_A(x)\| \\
&= \prod_{k=0}^{i-1} \|\Psi_A(y_k)\| \cdot \|(\Psi_A(y_k))^{-1}\| \cdot \prod_{k=0}^{i-1} \frac{\|\Psi_A(x_k)\|}{\|\Psi_A(y_k)\|} \\
&\leq \prod_{k=0}^{i-1} \theta_2 \eta(y_k)^{-\beta} \cdot \prod_{k=0}^{i-1} (1 + C_2 (d(x_k, y_k))^\beta).
\end{aligned}$$

Since the distance between x_n and y_n decreases exponentially, the second product is uniformly bounded, and we obtain the existence of a uniform $C_0 > 0$ such that

$$\|(\Psi_A^i)^{-1}(y)\| \cdot \|\Psi_A^i(x)\| \leq C_0 \theta_2^i \eta_i(y)^{-\beta}.$$

$\square$

We now complete the proof of Proposition 4.2.8. Since Ψ_A is Hölder continuous (see (4.2.4)) we have

$$\begin{aligned}\|r_i\| &= \|(\Psi_A(y_i))^{-1} \circ I_{x_{i+1}y_{i+1}} \circ \Psi_A(x_i) - I_{x_iy_i}\| \\ &\leq \|(\Psi_A(y_i))^{-1} \circ I_{x_{i+1}y_{i+1}}\| \cdot \|\Psi_A(x_i) - I^{-1}_{x_{i+1}y_{i+1}} \circ \Psi_A(y_i) \circ I_{x_iy_i}\| \\ &\leq C_3(d(x_i, y_i))^\beta \leq C_3(C_4 d(x, y)\eta_i(y))^\beta.\end{aligned} \tag{4.2.7}$$

It follows from (4.2.7) and Lemma 4.2.9 that for all $i \geq 0$

$$\begin{aligned}\|(\Psi_A^i(y))^{-1} \circ r_i \circ \Psi_A^i(x)\| &\leq \|(\Psi_A^i(y))^{-1}\| \cdot \|\Psi_A^i(x)\| \cdot \|r_i\| \\ &\leq C_0 \theta_2^i \eta_i(y)^{-\beta} C_3 C_4^\beta d(x, y)^\beta \eta_i(y)^\beta \\ &= C_5 d(x, y)^\beta \theta^i.\end{aligned} \tag{4.2.8}$$

Using (4.2.6), (4.2.8) and $\sum_{i=0}^\infty \theta^i = \frac{1}{1-\theta} < \infty$, we conclude that there is a constant $C > 0$ (depending only on A and the identifications) such that

$$\begin{aligned}\|(\Psi_A^n(y))^{-1} \circ I_{x_ny_n} \circ \Psi_A^n(x) - I_{xy}\| &\leq \sum_{i=0}^{n-1} \|(\Psi_A^i(y))^{-1} \circ r_i \circ \Psi_A^i(x)\| \\ &\leq C d(x, y)^\beta.\end{aligned}$$

(b) It follows from the estimates in (4.2.6) that

$$\|(\Psi_A^{n+1}(y))^{-1} \circ I_{x_{n+1}y_{n+1}} \circ \Psi_A^{n+1}(x) - (\Psi_A^n(y))^{-1} \circ I_{x_ny_n} \circ \Psi_A^n(x)\| = \|(\Psi_A^n(y))^{-1} \circ r_n \circ \Psi_A^n(x)\|.$$

Therefore, it follows from (4.2.8) that $\{(\Psi_A^n(y))^{-1} \circ I_{x_ny_n} \circ \Psi_A^n(x)\}_n$ is a Cauchy sequence, and thus, since $SL(2, \mathbb{K})$ is complete, this sequence has a limit $H^s_{A,x,y} : \mathcal{E}_x \to \mathcal{E}_y$. Since the convergence is uniform in the set of the pairs (x, y) where $y \in W^s_\epsilon(x)$, the map $H^s_{A,x,y}$ is continuous at x and y. Clearly, the maps $H^s_{A,x,y}$ are linear and satisfy $H^s_{A,x,x} = Id$. It follows from (a) that $\|H^s_{A,x,y} - I_{xy}\| \leq C d(x, y)^\beta$. We also have

$$\begin{aligned}H^s_{A,x,y} &= \lim_{k\to\infty} (\Psi_A^n(y))^{-1} \circ (\Psi_A^{k-n}(P^n(y)))^{-1} \circ I_{P^k(x)P^k(y)} \circ \Psi_A^{k-n}(P^k(x)) \circ \Psi_A^n(x) \\ &= (\Psi_A^n(y))^{-1} \circ H^s_{A,P^n(x),P^n(y)} \circ \Psi_A^n(x),\end{aligned} \tag{4.2.9}$$

for all $n \geq 0$. To show $H^s_{A,y.z} \circ H^s_{A,x,y} = H^s_{A,x,z}$ we use (4.2.3) and Lemma 4.2.9 to obtain, as in (4.2.8), that

$$\|H^s_{A,x,z} - H^s_{A,y,z} \circ H^s_{A,x,y}\| \leq \|(\Psi^n_A(z))^{-1}\| \cdot \|(I_{x_n z_n} - I_{y_n z_n} \circ I_{x_n z_n})\| \cdot \|\Psi^n_A(x)\|$$

which tends to zero as $n \to \infty$.

(c) Suppose that H^1_A and H^2_A are two stable holonomies satisfying $\|H^i_{A,x,y} - I_{xy}\| \leq Cd(x,y)^\beta$, for $i = 1, 2$. Then using the equation (4.2.9) and the Lemma 4.2.9 we obtain

$$\begin{aligned} \|H^1_{A,x,y} - H^2_{A,x,y}\| &= \|(\Psi^n_A(y))^{-1} \circ (H^1_{A,P^n(x)P^n(y)} - H^2_{A,P^n(x)P^n(y)}) \circ \Psi^n_A(x)\| \\ &\leq C_0\theta_2^n \eta_n(y)^{-\beta} Cd(P^n(x), P^n(y))^\beta = C_6\theta_2^n \end{aligned}$$

which tends to zero as $n \to \infty$. Hence $H^1_A = H^2_A$. □

4.2.3 Reduction to a cocycle over a subshift of finite type

Let $R_1, ..., R_k$ be a Markov system for a flow $(X^t)_{t\in\mathbb{R}}$ on the hyperbolic set Λ, we consider the $k \times k$ matrix $\mathbf{R}$ with entries

$$r_{ij} = \begin{cases} 1 & \text{if } \mathrm{int}P(R_i) \cap \mathrm{int}R_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where P is the transfer map. We also consider the set $\Sigma_\mathbf{R} \subset \{1, ..., k\}^\mathbb{Z}$ given by

$$\Sigma_\mathbf{R} = \left\{(\cdots i_{-1} i_0 i_1 \cdots) : r_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z}\right\},$$

and the shift map $\sigma_\mathbf{R} : \Sigma_\mathbf{R} \to \Sigma_\mathbf{R}$. Recall that we define a coding map $\rho : \Sigma_\mathbf{R} \to \bigcup_{i=1}^k R_i$ by

$$\rho(\cdots i_0 \cdots) = \bigcap_{j\in\mathbb{Z}} \overline{(P|\Gamma)^{-j}(\mathrm{int}R_{i_j})}$$

and the diagram

$$\begin{array}{ccc} \Sigma_\mathbf{R} & \xrightarrow{\sigma_\mathbf{R}} & \Sigma_\mathbf{R} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \Gamma & \xrightarrow{P} & \Gamma \end{array} \tag{4.2.10}$$

commutes, that is,

$$\rho \circ \sigma_\mathbf{R} = P \circ \rho. \tag{4.2.11}$$

Defining

$$\widehat{\Psi}_A(\hat{p}) = \Psi_A(\rho(\hat{p})), \quad \text{with} \quad \hat{p} \in \Sigma_\mathbf{R},$$

we obtain a cocycle over $\sigma_\mathbf{R}$. The next lemma, proved by Backes, Poletti, Varandas and Lima in [2], shows that the product structure of μ_P can be lifted to a $\sigma_\mathbf{R}$-invariant measure in $\Sigma_\mathbf{R}$.

Lemma 4.2.10 ([2, Proposition 4.1]). *There is a $\sigma_{\mathbf{R}}$-invariant probability measure ν on $\Sigma_{\mathbf{R}}$, such that ν is $\sigma_{\mathbf{R}}$-ergodic, has local product structure and $\mu_P = \pi_* \nu$.*

Lemma 4.2.11. *The Lyapunov exponents of $\widehat{\Psi}_A$ coincide with the Lyapunov exponents of Ψ_A.*

Proof. Since μ_P is ergodic, by the Oseledets theorem, for μ_P-almost every $x \in \Gamma$ there is a $\Psi_A(x)$-invariant decomposition $T_x\Gamma = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^{k(x)}$ and there are Lyapunov exponents well defined by

$$\lambda_i(\Psi_A, P, \mu_P) = \lim_{n\to\infty} \frac{1}{n} \log \|\Psi_A^n(x) v_i\|$$

for every $v_i \in E_x^i \backslash \{0\}$ and $1 \le i \le k(x)$. Thus, given $\hat{p} \in \Sigma_{\mathbf{R}}$ regular for ν in the sense of Oseledets, we define $E_{\hat{p}}^i = E_{\pi(\hat{p})}^i$, for $1 \le i \le k(\pi(\hat{p}))$. Hence, given $w_i \in E_{\pi(\hat{p})}^i \backslash \{0\}$, we have

$$\begin{aligned}\lambda_i(\widehat{\Psi}_A, \sigma_{\mathbf{R}}, \nu) &= \lim_{n\to+\infty} \frac{1}{n} \log \|\widehat{\Psi}_A^n(\hat{p}) w_i\| \\ &= \lim_{n\to+\infty} \frac{1}{n} \log \|\Psi_A^n(\pi(\hat{p})) w_i\| = \lambda_i(\Psi_A, P, \mu_P).\end{aligned}$$

Therefore, the Lyapunov exponents of $\widehat{\Psi}_A$ coincide with the Lyapunov exponents of Ψ_A. $\square$

Proposition 4.2.12. *The cocycle $F_{\widehat{\Psi}_A} : \Sigma_{\mathbf{R}} \times \mathbb{K}^2 \to \Sigma_{\mathbf{R}} \times \mathbb{K}^2$, $(\hat{p}, v) \mapsto (\sigma_R(\hat{p}), \widehat{\Psi}_A(\hat{p})v)$, is continuous and admits stable and unstable holonomies .*

Proof. The continuity of $\widehat{\Psi}_A$ follows from the fact that Ψ_A and π are continuous. If $\hat{p}$ and $\hat{z}$ are in the same local stable manifold for $\sigma_{\mathbf{R}}$, then $\pi(\hat{p})$ and $\pi(\hat{z})$ are in the same local stable manifold for P. Thus, we can define stable holonomies for $\widehat{\Psi}_A$ as $\widehat{H}^s_{A,\hat{p},\hat{z}} := H^s_{A,\pi(\hat{p}),\pi(\hat{z})}$. Similarly, if $\hat{p}$ and $\hat{z}$ are in the same local unstable manifold for $\sigma_{\mathbf{R}}$, then $\pi(\hat{p})$ and $\pi(\hat{z})$ are in the same local unstable manifold for P. Thus, we can define unstable holonomies for $\widehat{\Psi}_A$ as $\widehat{H}^u_{A,\hat{p},\hat{z}} := H^u_{A,\pi(\hat{p}),\pi(\hat{z})}$. $\square$

As a consequence of Lemmas 4.2.7 and 4.2.11 and and Proposition 4.2.12, we obtain the following

Lemma 4.2.13. *The following three statements are equivalent:*

- *The cocycle $\widehat{\Psi}_A$ has simple spectrum for ν-almost every point in $\Sigma_{\mathbf{R}}$.*
- *The cocycle Ψ_A has simple spectrum for μ_P-almost every point in Γ.*
- *The cocycle Φ_A^t has simple spectrum for μ-almost every point in Λ.*

4.2.4 Density and openness of twisting and pinching cocycles

First we show that for any given infinitesimal generator and any small perturbation of the time-one map of its solution, there exists an infinitesimal generator close to the original one which realizes the perturbation map. Our main tool for that is the Lemma 4.2.14 which was proved by Bessa and Varandas in [9].

Given $S \in SL(2,\mathbb{K})$, we can see S as the time-one map of the linear flow solution of the linear variational equation $\dot{u}(t) = \mathbf{S}(t) \cdot u(t)$ with initial condition $u(0) = id$. In other words, $u(t) = \Phi^t_{\mathbf{S}}$ is solution of $\dot{u}(t) = \mathbf{S}(t) \cdot u(t)$ and $\Phi^1_{\mathbf{S}} = S$. By Gronwall's inequality, we have

$$\|\Phi^t_{\mathbf{S}}\|_{r,\nu} \leq \exp\left\{\int_0^t \|\mathbf{S}(s)\|_{r,\nu} ds\right\}, \quad \text{for all } t \geq 0.$$

Hence, we say that $S \in SL(2,\mathbb{K})$ is δ-$C^{r,\nu}$–close to identity if $\mathbf{S}$ is δ-$C^{r,\nu}$-small,that is, $\|\mathbf{S}\|_{r,\nu} < \delta$.

Lemma 4.2.14 ([9])**.** *Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator over a flow $(X^t)_t$ on M, $x \in M$ any nonperiodic point (or periodic with period 1) and $\epsilon > 0$. There exists $\delta = \delta(A,\epsilon) > 0$ such that if $S \in SL(2,\mathbb{K})$ is isotopic to the identity and δ-$C^{r,\nu}$-close to identity, then there exists $B \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ satisfying:*

(a) $\|B - A\|_{r,\nu} < \epsilon$ *and*

(b) $\Phi^1_B(x) = \Phi^1_A(x) \circ S$.

Proof. By the tubular flowbox theorem there exists a smooth change of coordinates so that there exists a local conjugation of X on a neighborhood of the segment of orbit $\{X^t(x) : t \in [0,1]\}$ to a constant vector field on $\mathbb{R}^d$, $d = \dim(M)$. With this assumption we consider $x = \vec{0}$ and $\{X^t(x) : t \in [0,1]\} = \{(t,0,\ldots,0) \in \mathbb{R}^d : t \in [0,1]\} \subset \frac{\partial}{\partial x_1}$, where $\frac{\partial}{\partial x_1}$ denotes the direction spanned by the direction $x_1 = (1,0,\ldots,0)$. Given $\rho > 0$, let $B(\vec{0},\rho) \subset \left(\frac{\partial}{\partial x_1}\right)^{\perp}$ denotes the ball centered in $\vec{0}$ of radius ρ contained in the hyperplane orthogonal to $\frac{\partial}{\partial x_1}$. The perturbation will be performed in the cylinder $\mathcal{C} = B(\vec{0},\rho)\times[0,1] = \{X^t(B(\vec{0},\rho)) : t \in [0,1]\}$. Using the fact that M is compact we can take

$$K := \max_{z\in M, t\in[0,1]} \{\|\Phi^t_A(z)\|_{r,\nu}, \|(\Phi^t_A(z))^{-1}\|_{r,\nu}, \|A\|_{r,\nu}\}. \tag{4.2.12}$$

Fix any $\epsilon > 0$ and choose $\delta := \frac{\epsilon}{6K^3}$. Consider the isotopy $S_t \in SL(2,\mathbb{K})$, $t \in [0,1]$, such that:

(1) $S_t = (1-t)id + tS$;

(2) S_t is the solution of the linear variational equation $\partial_t u(t) = \mathbf{S}(t)\cdot u(t)$ with infinitesimal generator $\mathbf{S} = (S_t)'(S_t)^{-1}$, where $(S_t)' = S - id$, satisfying the inequality

$$\|\mathbf{S}\|_{r,\nu} := \sup_{0\le j\le r}\sup_{t\in[0,1]} \|D^j\mathbf{S}(t)\| + \sup_{t\neq s}\frac{\|\mathbf{S}(t)-\mathbf{S}(s)\|}{|t-s|^\nu} < \delta.$$

Consider a C^∞ bump-function $\alpha : [0,\infty[\to [0,1]$, with $\alpha(s) = 0$ if $s \ge \rho$ and $\alpha = 1$ if $s \in [0,\rho/2]$. Given $z \in B(\vec{0},\rho)$, consider the linear isotopy $S_t(z) \in SL(2,\mathbb{K})$, $t \in [0,1]$, between $S_0(z) = id$ and $S_1(z) = \alpha(\|z\|^2)S$ obtained as solution of the equation $\partial_t u(t,z) = \mathbf{S}(t,z)\cdot u(t,z)$ with infinitesimal generator $\mathbf{S}$ satisfying

$$\|\mathbf{S}(t,z)\|_{r,\nu} := \sup_{0\le j\le r}\sup_{t\in[0,1]} \|D^j\mathbf{S}(z+(t,0,\dots,0))\| + \sup_{x\neq y}\frac{\|\mathbf{S}(x)-\mathbf{S}(y)\|}{d(x,y)^\nu} < \delta.$$

Then, if $\Upsilon_t(z) = \Phi_A^t(z)\alpha(\|z\|)S_t(z)$ and we consider time derivatives one notices that

$$\begin{aligned}
\Upsilon(z)' &= \Phi_A^t(z)'\alpha(\|z\|)S_t(z) + \Phi_A^t(z)(\alpha(\|z\|)S_t(z))' \\
&= A(X^t(z))\Phi_A^t(z)\alpha(\|z\|)S_t(z) + \Phi_A^t(z)(\alpha(\|z\|)S_t(z))' \\
&= A(X^t(z))\Upsilon_t(z) + \Phi_A^t(z)(\alpha(\|z\|)S_t(z))'(\Upsilon_t(z))^{-1}\Upsilon_t(z) \\
&= \left[A(X^t(z)) + \Phi_A^t(z)\alpha(\|z\|)S_t'(z)(\Phi_A^t(z)\alpha(\|z\|)S_t(z))^{-1}\right]\Upsilon_t(z) \\
&= \left[A(X^t(z)) + \Phi_A^t(z)\alpha(\|z\|)S_t'(z)(S_t(z))^{-1}(\alpha(\|z\|))^{-1}(\Phi_A^t(z))^{-1}\right]\Upsilon_t(z) \\
&= \left[A(X^t(z)) + \Phi_A^t(z)S_t'(z)(S_t(z))^{-1}(\Phi_A^t(z))^{-1}\right]\Upsilon_t(z) \\
&= \left[A(X^t(z)) + T(X^t(z))\right]\Upsilon_t(z),
\end{aligned}$$

where $T(X^t(z)) = \Phi_A^t(z)\mathbf{S}(t,z)(\Phi_A^t(z))^{-1}$, with $\mathbf{S}(t,z) = S_t'(z)(S_t(z))^{-1}$, in the flowbox coordinates $(z,t) \in \mathcal{C} = B(\vec{0},\rho)\times[0,1]$ and outside the flowbox cylinder $\mathcal{C}$ we let $T = [0]$. Consequently Υ_t is a solution of the equation $\partial_t u(t,z) = B(X^t(z))\cdot u(t,z)$ with initial condition equal to the identity, where $B(X^t(z)) = A(X^t(z)) + T(X^t(z))$ for all $t \in [0,1]$ and $z \in B(\vec{0},\rho)$.

We will prove condition (a) of the conclusions of the lemma, that is, that $\|B-A\|_{r,\nu} < \epsilon$ or, equivalently, that $\|T\|_{r,\nu} < \epsilon$. We will perform the computations for $r = 0$ with all the details. For $r \in \mathbb{N}$ we can estimate easily using the chain rule and Cauchy-Schwarz inequality. Whenever we consider points x, y in the tubular flowbox $\mathcal{C}$ (the support of the perturbation) we write them in the flowbox coordinates $x = (z,t)$, $y = (w,s)$, where $t, s \in [0,1]$ and $z, w \in B(\vec{0},\rho)$.

We shall estimate T in both coordinates and then the estimates on $\|T\|_{0,\nu} = \|T\|_\nu$ can be obtained on $B(\vec{0},\rho)\times[0,1]$ by means of a triangular inequality argument.

If z_t, w_t are inside the same laminar section in $\mathcal{C}$, that is, $z_t = (z,t)$ and $w_t = (w,t)$, then using (4.2.12), it follows that

$$\begin{aligned}
\|T(z_t)-T(w_t)\| &= \|\Phi_A^t(z)\mathbf{S}(t,z)(\Phi_A^t(z))^{-1}-\Phi_A^t(w)\mathbf{S}(t,w)(\Phi_A^t(w))^{-1}\| \\
&\le \|\Phi_A^t(z)\left[\mathbf{S}(t,z)-\mathbf{S}(t,w)\right](\Phi_A^t(z))^{-1}\|+ \\
&\quad +\|\left[\Phi_A^t(z)-\Phi_A^t(w)\right]\mathbf{S}(t,w)(\Phi_A^t(z))^{-1}\|+ \\
&\quad +\|\Phi_A^t(w)\mathbf{S}(t,w)\left[(\Phi_A^t(z))^{-1}-(\Phi_A^t(w))^{-1}\right]\| \\
&\le K^2\|\mathbf{S}(t,z)-\mathbf{S}(t,w)\|+K\|\Phi_A^t(z)-\Phi_A^t(w)\|\|\mathbf{S}(t,w)\| \\
&\quad +K\|\mathbf{S}(t,w)\|\|(\Phi_A^t(z))^{-1}-(\Phi_A^t(w))^{-1}\|,
\end{aligned}$$

and so

$$\sup_{z_t\neq w_t}\frac{\|T(z_t)-T(w_t)\|}{d(z_t,w_t)^\nu}\le K^2\delta+2Ke^\delta<\epsilon.$$

Analogously, for z_t, z_s inside the same orbit in $\mathcal{C}$, it follows

$$\begin{aligned}
\|T(z_t)-T(z_s)\| &= \|\Phi_A^t(z)\mathbf{S}(t,z)(\Phi_A^t(z))^{-1}-\Phi_A^s(z)\mathbf{S}(s,z)(\Phi_A^s(z))^{-1}\| \\
&\le \|\Phi_A^t(z)\left[\mathbf{S}(t,z)-\mathbf{S}(s,z)\right](\Phi_A^t(z))^{-1}\|+ \\
&\quad +\|\left[\Phi_A^t(z)-\Phi_A^s(z)\right]\mathbf{S}(s,z)(\Phi_A^t(z))^{-1}\|+ \\
&\quad +\|\Phi_A^s(z)\mathbf{S}(s,z)\left[(\Phi_A^t(z))^{-1}-(\Phi_A^s(z))^{-1}\right]\| \\
&\le K^2\|\mathbf{S}(t,z)-\mathbf{S}(s,z)\|+K\left\|\Phi_A^t(z)\left[id-\Phi_A^{s-t}\left(X^t(z)\right)\right]\right\|\|\mathbf{S}(s,z)\| \\
&\quad +K\|\mathbf{S}(s,z)\|\left\|(\Phi_A^t(z))^{-1}\left[id-(\Phi_A^{s-t}(X^t(z)))^{-1}\right]\right\|,
\end{aligned}$$

and so

$$\begin{aligned}
\sup_{z_t\neq z_s}\frac{\|T(z_t)-T(z_s)\|}{d(z_t,z_s)^\nu} &\le \sup_{t\neq s}\left[K^2\delta+K^2\delta\frac{\|id-\Phi_A^{s-t}(X^t(z))\|}{|t-s|^\nu}+\right. \\
&\quad \left.+K^2\delta\frac{\|id-(\Phi_A^{s-t}(X^t(z)))^{-1}\|}{|t-s|^\nu}\right] \\
&\le K^2\delta+\sup_{t\neq s}K^2\delta\left[\frac{\|id-\Phi_A^{s-t}(X^t(z))\|}{|t-s|^\nu}+\right. \\
&\quad \left.+\frac{\|id-(\Phi_A^{s-t}(X^t(z)))^{-1}\|}{|t-s|^\nu}\right] \\
&\le K^2\delta+\sup_{t\neq s}K^2\delta(2\|A\|) \\
&\le K^2\delta+2K^3\delta\le 3K^3\delta<\epsilon.
\end{aligned}$$

Notice that we consider $\nu=1$. This is enough to deduce condition (a) using a triangular inequality argument.

Finally, we will prove condition (b) of the conclusions of the lemma, that is, that we have the equality $\Phi_B^1(x)=\Phi_A^1(x)\circ S$. We are considering $x=\vec{0}$, so let us prove that

$\Phi^1_B(\vec{0}) = \Phi^1_A(\vec{0}) \circ S$. Just observe that $\Upsilon_t(z)$ is a solution of the linear differential equation

$$u'(t,z) = [A(X^t(z)) + T(X^t(z))] \cdot u(t,z) = B(X^t(z)) \cdot u(t,z). \tag{4.2.13}$$

But, given the initial condition $u(0,z) = z$, this solution is unique, say $\Phi^t_B(\vec{0})$. Since $\Upsilon_t(z) = \Phi^t_A(z)\alpha(\|z\|)S_t(z)$ and it also satisfies (4.2.13), we obtain that, for $z = \vec{0}$, $\Phi^t_B(\vec{0}) = \Phi^t_A(\vec{0})\alpha(\|\vec{0}\|)S_t(\vec{0})$. Thus, we obtain that, $\Phi^1_B(\vec{0}) = \Phi^1_A(\vec{0})\alpha(0)S_1(\vec{0}) = \Phi^1_A(\vec{0})S$, and the lemma is proved. □

Density and openness of twisting property

The next proposition shows that the set of infinitesimal generators for which the reduced cocycles are twisting, is an open subset of $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$.

Proposition 4.2.15. *Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator and Ψ_A be the corresponding reduced cocycle. Let $p \in \Gamma$ be a periodic point for the return map $P : \Gamma \to \Gamma$ and $z \in \Gamma$ be a homoclinic point for p. Suppose that Ψ_A satisfies the twisting property for p and z. There exists an open set $\mathcal{U} \subset C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ containing A, such that for all $B \in \mathcal{U}$, the reduced cocycle Ψ_B satisfies the twisting property for p and z.*

Proof. Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be such that its reduced cocycle Ψ_A satisfies the twisting property for the periodic point $p \in \Gamma$, with period $q(p) \geq 1$, and the homoclinic point $z \in \Gamma$ associated to p.

By Lemma 4.1.3, the map which associates $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ with the matrix $\Psi_A(p) = \Phi^{\tau(p)}_A(p)$ varies continuously with A. Thus the map $A \mapsto \Psi^{q(p)}_A(p)$, which associates the infinitesimal generator $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ to the matrix $\Psi^{q(p)}_A(p) \in SL(2,\mathbb{K})$, also vary continuously with A. Since the holonomies varies continuously with the infinitesimal generator, the map $A \mapsto \zeta_{A,p,z}$, which associates the infinitesimal generator $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ to the map $\zeta_{A,p,z} \in SL(2,\mathbb{K})$, also varies continuously with A.

Thus, for any $B \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ that is $C^{r,\nu}$-close to A, the invariant subspaces of $\Psi^{q(p)}_B(p)$ are close to invariant subspaces of $\Psi^{q(p)}_A(p)$. If the subspaces $E, F \subset \mathbb{K}^2$ are close, their images $\zeta_{A,p,z}(E)$ and $\zeta_{B,p,z}(F)$, under $\zeta_{A,p,z}$ and $\zeta_{B,p,z}$, respectively, are close.

Therefore, for each $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$, fix $E_A \subset \mathbb{K}^2$ and $F_A \subset \mathbb{K}^2$ invariant spaces under A, such that $\zeta_{A,p,z}(E_A) \cap F_A = \{0\}$. For any open neighborhood $\mathcal{V} \subset SL(2,\mathbb{K})$ of $\zeta_{A,p,z}$, there exists an open neighborhood $\mathcal{U} \subset C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ such that if $B \in \mathcal{U}$, then $\zeta_{B,p,z} \in \mathcal{V}$. Thus, taking $\mathcal{V}$ and $\mathcal{U}$ small enough, $E_B, F_B \subset \mathbb{K}^2$ invariant subspaces under $\Psi^{q(p)}_B(p)$, close to E_A and F_A, respectively, we have $\zeta_{B,p,z}(E_B) \cap F_B = \{0\}$. Since $\dim \mathbb{K}^2 = 2$, the number of choices of proper invariant subspaces is at most 2, it shows that B is twisting with respect to p and z. This completes the proof of proposition. □

Now we prove the set of cocycles satisfying the twisting property for any periodic point p and any homoclinic point $z$is dense in $C^{r,\nu}(M,\mathfrak{sl}(2,\mathbb{K}))$.

Proposition 4.2.16. *Let $A \in C^{r,\nu}(M,\mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator over a flow $(X^t)_t : M \to M$. For any neighborhood $\mathcal{V} \subset C^{r,\nu}(M,\mathfrak{sl}(2,\mathbb{K}))$ of A, any periodic point p and any homoclinic point z, there exists $B \in \mathcal{V}$ such that Ψ_B is twisting with respect to p and z.*

Proof. Let $A \in C^{r,\nu}(M,\mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator. Suppose that Ψ_A is not twisting. The condition of not satisfying the twisting property can be described as follows: for any periodic point p with a homoclinic point $z \in W^u_\epsilon(p,P)$, with $P^l(z) \in W^s_\epsilon(p,P)$, $l > 1$ multiple of $q(p)$, the period of p, there are subspaces E and F invariant under $\Psi_A^{q(p)}$ and satisfying $\dim E + \dim F = 2$, such that the transition map $\zeta_{A,p,z} = H^s_{A,P^l(z),p} \circ \Psi^l_A(z) \circ H^u_{A,p,z}$ satisfies $\zeta_{A,p,z}(E) = F$. Let $w_1 \in E$ be such that $\zeta_{A,p,z}(w_1) \in \zeta_{A,p,z}(E)$. Choose $1 < k < l$, we will perturb the cocycle Ψ_A in a neighborhood of the point $P^k(z)$. Let

$$\overline{w_1} = \left(\Psi^k_A(z) \circ H^u_{A,p,z}\right)(w_1) \in \mathbb{K}^2_{P^k(z)}.$$

Denote by R_θ the rotation of angle θ in $\mathbb{K}^2$. If $\theta_1 > 0$ is small enough, by Lemma 4.2.14, we can find $B \in C^{r,\nu}(M,\mathfrak{sl}(2,\mathbb{K}))$ that is $C^{r,\nu}$-close to A such that $\Phi^1_B(P^k(z)) = \Phi^1_A(P^k(z)) \circ R_{\theta_1}$. Thus

$$\begin{aligned}
\Psi_B(P^k(z)) &= \Phi_B^{\tau(P^k(z))}(P^k(z)) \\
&= \Phi_B^{\tau(P^k(z))-1}(X^1(P^k(z))) \circ \Phi^1_B(P^k(z)) \\
&= \Phi_A^{\tau(P^k(z))-1}(X^1(P^k(z))) \circ \Phi^1_A(P^k(z)) \circ R_{\theta_1} \\
&= \Phi_A^{\tau(P^k(z))}(P^k(z)) \circ R_{\theta_1} \\
&= \Psi_A(P^k(z)) \circ R_{\theta_1}.
\end{aligned}$$

So

$$\begin{aligned}
\Psi^l_B(z) &= \Psi_B(P^{l-1}(z)) \circ \ldots \circ \Psi_B(P^{k+1}(z)) \circ \Psi_B(P^k(z)) \circ \Psi_B(P^{k-1}(z)) \circ \ldots \circ \Psi_B(z) \\
&= \Psi_A^{l-k}(P^k(z)) \circ R_{\theta_1,\overline{v_1},\overline{w_1}} \circ \Psi^k_A(z).
\end{aligned}$$

Note that $H^s_{B,P^l(z),p} = H^s_{A,P^l(z),p}$ e $H^u_{B,p,z} = H^u_{A,p,z}$. In fact, since $1 < k < l$, the limits

$$H^s_{B,P^l(z),p} = \lim_{n\to+\infty} (\Psi^n_A(p))^{-1} \circ I_{P^n(P^l(z))P^n(p)} \circ \Psi^n_A(P^l(z))$$

and

$$H^u_{B,p,z} = \lim_{n\to-\infty} (\Psi^n_A(z))^{-1} \circ I_{P^n(p)P^n(z)} \circ \Psi^n_A(p)$$

do not depend on the expression $\Psi_B(P^k(z))$. Thus, the transition map $\zeta_{B,p,z}$ associated with infinitesimal generator B is given by

$$\begin{aligned}\zeta_{B,p,z} &= H^s_{A,P^l(z),p} \circ \Psi^l_B(z) \circ H^u_{A,p,z} \\ &= H^s_{A,P^l(z),p} \circ \Psi^{l-k}_A(P^k(z)) \circ R_{\theta_1} \circ \Psi^k_A(z) \circ H^u_{A,p,z}.\end{aligned}$$

Thus we have $\zeta_{B,p,z}(w_1) \notin F$. Since the number of choices of E and F is finite, we have that Ψ_B is twisting for p. □

Density and openness of the +pinching property

The next proposition will show that the set of infinitesimal generators for which the reduced cocycles are pinching with respect to some periodic point is an open set in $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$.

Proposition 4.2.17. *Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be such that its reduced cocycle Ψ_A satisfies the pinching property for a periodic point p. Then there is an open $A \in \mathcal{U} \subset C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ such that for all $B \in \mathcal{U}$, the reduced cocycle Ψ_B satisfies the pinching property for p.*

Proof. Let $p \in \Gamma$ be periodic point for the return map $P : \Gamma \to \Gamma$, with period $q(p)$. Again, the map $A \mapsto \Psi^{q(p)}_A(p)$, which associates the infinitesimal generator $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ with the matrix $\Psi^{q(p)}_A(p) \in SL(2,\mathbb{K})$, varies continuously with A.

We know from the Spectral Theory that the eigenvalues vary continuously with the matrix. So, if $\Psi^{q(p)}_A(p)$ has all eigenvalues with different norms, there is an open $\mathcal{V} \subset SL(2,\mathbb{K})$ containing $\Psi^{q(p)}_A(p)$ such that all matrices in $\mathcal{V}$ have eigenvalues with different norms. Therefore, the pre-image of $\mathcal{V}$ under the map $B \mapsto \Psi^{q(p)}_B(p)$ is an open $\mathcal{U} \subset C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ containing A such that if $B \in \mathcal{U}$, then $\Psi^{q(p)}_B(p)$ has all eigenvalues with different norms. This proves the proposition. □

The next lemma is inspired by [11] and shows that there is a dense set of fiber-bunched infinitesimal generators in $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ whose reduced cocycles are pinching.

Lemma 4.2.18. *Given a cocycle $\Psi \in C^{r,\nu}(\Gamma, SL(2,\mathbb{K}))$ and any $\epsilon > 0$, there are $\overline{\Psi} \in C^{r,\nu}(\Gamma, SL(2,\mathbb{K}))$ such that $\|\Psi - \overline{\Psi}\|_{r,\nu} < \epsilon$ and a periodic orbit $p \in M$, and $\overline{\Psi}^{q(p)}(p)$ has two eigenvalues with distinct norms.*

Proof. Let $\Psi \in C^{r,\nu}(\Gamma, SL(2,\mathbb{K}))$ be fixed. The lemma is clear in the case that $\mathbb{K} = \mathbb{C}$. In fact, suppose $\Psi \in C^{r,\nu}(\Gamma, SL(2,\mathbb{C}))$ and for a periodic point p with period $q(p)$ we have that $\Psi^{q(p)}(p)$ has two eigenvalues λ_1 and λ_2 with the same norm, then $\Psi^{q(p)}(p)$ is diagonalizable, that is, there exists a 2×2 invertible matrix Q and a 2×2 diagonal matrix D such that $\Psi^{q(p)}(p) = QDQ^{-1}$. We also have that $\lambda_1 \cdot \lambda_2 = \det \Psi^{q(p)}(p) = 1$. Define

$$\overline{\Psi} = Q \begin{pmatrix} 1+\epsilon_0 & 0 \\ 0 & \frac{1}{1+\epsilon_0} \end{pmatrix} Q^{-1} \cdot \Psi.$$

Then $\overline{\Psi}^{q(p)}(p)$ has two eigenvalues with distinct norms, $\det \overline{\Psi}^{q(p)}(p) = 1$ and, if ϵ_0 is small enough, $\overline{\Psi}$ is $C^{r,\nu}$ close to Ψ.

In the case that $\mathbb{K} = \mathbb{R}$, if there exists a periodic point p so that $\Psi^{q(p)}$ has two real eigenvalues then, up to a arbitrarily small perturbation we find a cocycle $\overline{\Psi}$ so that $\|\Psi - \overline{\Psi}\|_{r,\nu} < \epsilon$, and $\overline{\Psi}^{q(p)}$ has two distinct real eigenvalues. For that reason in what follows we are reduced to the case where the cocycle Ψ is so that $\Psi^{q(p)}$ has some complex eigenvalue for every periodic point p (here $q(p) \geq 1$ denotes the period of p).

For simplicity of the presentation, we suppose that p is a fixed point for P, and let z be a homoclinic point with respect to p. The general case follows along the same lines.

Suppose that $\Psi_A(p)$ has a pair of complex conjugate eigenvalues. Let

$$E^1_p \oplus E^2_p$$

be the splitting of $\mathbb{R}^2$ into eigenspaces of $\Psi_A(p)$.

Let Λ_p be the horseshoe generated by local stable and unstable manifolds of p crossing through z, that is, $\Lambda_p = \cap_{n\in\mathbb{Z}} P^n(U_0 \cup U_1)$ with U_0, U_1 disjoint neighborhoods of p and z, respectively. Up to a finite multiple of P we may assume that $P(U_0) \cap U_1 \neq \emptyset$. Hence, for each n there exists a periodic point x_n, of increasing period equal to $l+n$, such that the first n iterates of x_n belong to U_0 and the following l iterates belong to U_1 (see Figure 4.2.2). Those l iterates are precisely the ones equal to the orbit of z different from p. Defined in this way x_n, as n increases, the point x_n is as close as desired to p and the matrix $\Psi^{l+n}(x_n)$ inherits the dynamical behavior of $\Psi(p)$.

By continuity of the eigenvalues, every cocycle Ψ_0 in a C^0 neighborhood $\mathcal{U}$ of Ψ_A has a pair of complex eigenvalues over x_n for every large n (independent of Ψ_0).

The case when $\Psi_A^{l+n}(x_n)$ reverses the orientation of $E^1_{x_n} \oplus E^2_{x_n}$ is easy, as we shall see right after the statement of the next claim. For the time being, we suppose that $\Psi_A^{l+n}(x_n)$ preserves the orientation of $E^1_{x_n} \oplus E^2_{x_n}$. Hence, the same is true for every nearby cocycle Ψ_0. Then we denote $\rho(n, \Psi_0)$ the rotation number associated to $\Psi_0^{l+n}(x_n)$. Moreover, given a continuous arc $\mathcal{B} = \{\Xi_t\}$ of cocycles close to Ψ_A, we denote $\delta(n, \mathcal{B})$ the oscillation of $\rho(n, \Xi_t)$ over the whole parameterization interval. The main step in the proof of Lemma 4.2.18 is the following.

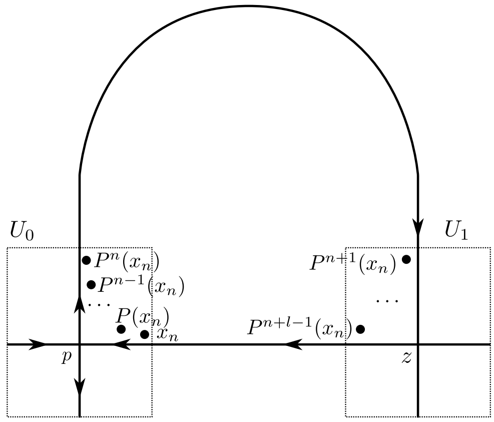

Figure 4.2.2: Periodic points x_n.

Claim 4.2.19. There exists a continuous arc $\mathcal{A} = \{\Xi_t : t \in [0,1]\}$ of C^ν cocycles in $\mathcal{U}$ with $\Xi_0 = \Psi_A$ and such that for every $t > 0$ there exists $n_t \geq 1$ so that

$$\delta(n, \{\Xi_t : t \in [0,1]\}) > 1,$$

for every $n \geq n_t$.

Let us explain how Lemma 4.2.18 follows from Claim 4.2.19, after which we shall prove the lemma.

Firstly, for every t and every large n the matrix $\Xi_t^{l+n}(x_n)$ has a pair of complex eigenvalues. Secondly, in the orientation preserving case we may use Claim 4.2.19 to conclude that there exists t arbitrarily close to zero and $n \geq 1$ for which the rotation number $\rho(n, \Xi_t)$ is an integer. This means that $\Xi_t^{l+n}(x_n)$ has some real eigenvalue. Observe that in the orientation reversing case this conclusion comes for free. So, in general, by an arbitrarily small perturbation close to x_n and preserving $E^1_{x_n,t} \oplus E^2_{x_n,t}$, the splitting of $\mathbb{R}^2$ into eigenspaces of $\Xi_t^{l+n}(x_n)$, we can obtain a cocycle Ξ' for which there are two real and distinct eigenvalues. Thus, we find a periodic point p_0 and a continuous cocycle Ψ_0, arbitrarily close to the initials p and Ψ_A such that all the eigenvalues of Ψ_0 over the orbit of p_0 are real. This concludes the proof of Lemma 4.2.18.

Finally, we prove Claim 4.2.19.

Proof (of Claim 4.2.19). We begin by fixing, once and for all, a basis of $\mathbb{R}^2$ coherent with the decomposition $E^1_p \oplus E^2_p$: each vector in the basis is in some E^i_p, and the matrix of $\Psi_A(p)$ is a rotation (of angle ρ_i), relative to this basis. We always consider the (constant) system of coordinates on the fibers $\{x\} \times \mathbb{R}^2$ defined by this basis. Given any θ, we define R_θ to be the linear map given by the rotation of angle θ along $\mathbb{R}^2$. In this system of coordinates, R_θ can be written as

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We choose

$$\Xi_t(x) = R_{t\epsilon} \cdot \Psi_A(x), \quad \text{for } t \in [0,1],$$

where $\epsilon > 0$ is fixed small enough so that all these cocycles be in $\mathcal{U}$. Reducing $\epsilon > 0$ if necessary, we may find $r > 0$ small enough so that every $E^i_{x_n,t}$ is a graph over E^i_p restricted to the r-neighborhood of p. Moreover, we write

$$\Psi_A^{l+n}(x) = \alpha_{t,n,n} \cdots \alpha_{t,n,1} \cdot \beta_{t,n}$$

where the $\alpha_{t,n,j}$ correspond to iterates inside the r-neighborhood of p, and $\beta_{t,n}$ encompasses the iterates outside that neighborhood. Since there are finitely many of the latter, $\beta_{t,n}$ converges uniformly to some β_t, as $n \to \infty$. Thus, in order to obtain the conclusion of the lemma, it suffices to show that the variation of the rotation number of the matrix $\alpha_{t,n,n} \cdots \alpha_{t,n,1}$ over every interval $[0,1]$ goes to infinity when $n \to \infty$. For this we observe that, by the definition of Ξ_t, the $\alpha_{t,n,i}$ are uniformly close to the rotation of angle $t\epsilon + \rho_\ell$ if the radius r is chosen small enough. Since the $\alpha_{t,n,i}$ preserve the orientation, all their contributions to the rotation number roughly add up, yielding the claim. □

□

Proposition 4.2.20. *Given a infinitesimal generator $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ and any $\epsilon > 0$, there are $B \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ such that $\|A - B\|_{r,\nu} < \epsilon$ and a periodic orbit $p \in M$, such that $\Psi_B^{q(p)}$ has two real and distinct eigenvalues.*

Proof. Let $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ be an infinitesimal generator. If there is a periodic point p such that $\Psi_A^{q(p)}$ has two real and distinct eigenvalues we are done. Otherwise, up to a small perturbation we may assume that $\Psi_A^{q(p)}$ has a complex eigenvalue γ for every periodic point p. Recall that complex conjugate $\bar{\gamma}$ is also an eigenvalue for $\Psi_A^{q(p)}$.

Let p be a periodic point of period $q(p) \geq 1$ for P. By Lemma 4.2.18 we can find a cocycle $\tilde{\Psi}$ that is $C^{r,\nu}$-close to Ψ_A and a periodic point x_n close to p such that $\tilde{\Psi}^{q(x_n)}(x_n)$ has two real and distinct eigenvalues. By Lemma 4.2.14 we can find $B \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ $C^{r,\nu}$-close to A such that $\Psi_B = \tilde{\Psi}$. This finish the proof of the Proposition. □

Denote by $Per(P)$ the set of periodic points for the map P. Now we prove that there is an open and dense set $\mathcal{O} \subset C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ such that for every $A \in \mathcal{O}$ we have that the reduced cocycle Ψ_A is both twisting and pinching for some $p \in Per(P)$ and some point z homoclinic with respect to p. For each $p \in Per(P)$ consider the following sets

$$T_p := \{A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K})) : \Psi_A \text{ is twisting for the point } p \in Per(P) \\ \text{and some homoclinic point z}\},$$

$$P_p := \{A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K})) : \Psi_A \text{ is pinching for the point } p \in Per(P)\},$$

$$TP_p := \{A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K})) : \Psi_A \text{ is twisting and pinching for the point } p \in Per(P) \\ \text{and some homoclinic point z}\},$$

and

$$TP := \cup_{p \in Per(P)} TP_p.$$

We will prove that TP is an open and dense subset of $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$.

By Proposition 4.2.15 and Proposition 4.2.17 the sets T_p and P_p are open for every fixed $p \in Per(P)$. Since $TP_p = T_p \cap P_p$, we also have TP_p is open for each $p \in Per(P)$, then TP is also open.

To show that TP is dense, take any $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$, $p \in Per(P)$ and a homoclinic point z for p. If Ψ_A is not pinching for p, by Proposition 4.2.20 we can find $B \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$, close to A, and periodic points x_n for P, so that $x_n \to p$ such that Ψ_B is pinching for x_n. If Ψ_B is not twisting to x_n, by Proposition 4.2.16 we can find $C \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ close to B, such that Ψ_C is twisting for x_n. Since P_{x_n} is open, we have that we can take $C \in TP_{x_n}$. Thus, taking approaches sufficiently small, $A \in C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$ is close to $C \in TP$, so TP is dense in $C^{r,\nu}(M, \mathfrak{sl}(2,\mathbb{K}))$.

This concludes the proof of Theorem A.

Chapter 5

Ergodic optimization for hyperbolic flows

The results in this chapter were obtained in collaboration with Paulo Varandas and Roberto Sant'Anna.

5.1 Hyperbolic flows

Our starting point is a result due Contreras for Ruelle expanding maps. Let Σ be a compact metric space and $T : \Sigma \to \Sigma$ be a Ruelle expanding map: there are numbers $k \in \mathbb{Z}^+$ and $0 < \lambda < 1$ such that for every point $x \in \Sigma$ there is a neighborhood U_x of x in Σ and continuous branches S_i, $i = 1, \ldots, \ell_x \leq k$ of the inverse of T such that

$$T^{-1}(U_x) = \bigcup_{i=1}^{\ell_x} S_i(U_x), \quad T \circ S_i = I_{U_x}$$

for all i, and

$$d\left(S_i(y), S_i(z)\right) \leq \lambda d(y, z)$$

for all $y, z \in U_x$. Assume without loss of generality $\operatorname{diam} \Sigma = 1$.

Theorem 5.1.1 (Contreras [19]). *If Σ is a compact metric space and $T : \Sigma \to \Sigma$ is a Ruelle expanding map then there is an open and dense set $\mathcal{O} \subset C^\alpha(\Sigma, \mathbb{R})$ such that for all $F \in \mathcal{O}$ there is a single F-maximizing measure and it is supported on a periodic orbit.*

Actually, in [19], Theorem 5.1.1 was proved for $\operatorname{Lip}(\Sigma, \mathbb{R})$, the space of Lipschitz observables, instead of $C^\alpha(\Sigma, \mathbb{R})$. But a Lipschitz function is a Hölder function with $\alpha = 1$ and the result remains true as we stated here up to a change of metric. In fact, a Hölder function $\phi \in C^\alpha(\Sigma, \mathbb{R})$ with respect to the metric $d(\cdot, \cdot)$ becomes a Lipschitz function if we just change the metric to $d_\alpha(\cdot, \cdot)$ defined by $d_\alpha(x, y) = d(x, y)^\alpha$.

For generic continuous observables Morris showed in [34, Corollary 1.3] the following.

Theorem 5.1.2. *Let $T : M \to M$ be a continuous transformation of a compact metric space satisfying Bowen's specification property. Then there is a dense G_δ set $Z \subset C^0(M, \mathbb{R})$ such that for every $f \in Z$, there is a single T-maximizing measure, such that it has support equal to M, has zero entropy and is not strongly mixing.*

5.1.1 Ergodic optimization for the shift map

We recall the following result by Bowen, whose proof will be included for reader's convenience. Recall that for $\mathbf{R}$ a $n \times n$ matrix of 0's and 1's, we denote

$$\Sigma_{\mathbf{R}} = \left\{ \underline{x} \in \Sigma_n : \mathbf{R}_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z} \right\}$$

and call $\sigma : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$, $\sigma\left(\{x_i\}_{i=-\infty}^{\infty}\right) = \{x_{i+1}\}_{i=-\infty}^{\infty}$, a subshift of finite type. Also recall that for $\phi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ continuous we define the variation of ϕ on k-cylinders by

$$\mathrm{var}_k \phi = \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : x_i = y_i \text{ for all } |i| \leq k\}$$

and denote $\mathscr{F}_{\mathbf{R}}$ the family of all continuous $\phi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ for which $\mathrm{var}_k \phi \leq bc^k$ (for all $k \geq 0$) for some positive constants b and $c \in (0,1)$.

Lemma 5.1.3 (Bowen [14, Lemma 1.6])**.** *If $\phi \in \mathscr{F}_{\mathbf{R}}$, then there exists a continuous function $u : \Sigma_{\mathbf{R}} \to \mathbb{R}$ such that $\psi := \phi + u \circ \sigma - u \in \mathscr{F}_{\mathbf{R}}$ and $\psi(\underline{x}) = \psi(\underline{y})$ whenever $x_i = y_i$ for all $i \geq 0$.*

Proof. For each $1 \leq t \leq n$ pick $\{a_{k,t}\}_{k=-\infty}^{\infty} \in \Sigma_{\mathbf{R}}$ with $a_{0,t} = t$. Define $\rho : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ by $\rho(\underline{x}) = \underline{x}^*$ where

$$x_k^* = \begin{cases} x_k & \text{for } k \geq 0 \\ a_{k,x_0} & \text{for } k \leq 0. \end{cases}$$

Let

$$u(\underline{x}) = \sum_{j=0}^{\infty} (\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\rho(\underline{x})))).$$

Since $\sigma^j(\underline{x})$ and $\sigma^j(\rho(\underline{x}))$ agree in places from $-j$ to $+\infty$,

$$|\phi(\sigma^j(\underline{x})) - \phi(\sigma^j \rho(\underline{x})))| \leq \mathrm{var}_j \phi \leq b\alpha^j.$$

As $\sum_{j=0}^{\infty} b\alpha^j < \infty$, u is well defined and continuous. If $x_i = y_i$ for all $|i| \leq n$, then, for $j \in [0, n]$,

$$|\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\underline{y}))| \leq \mathrm{var}_{n-j} \phi \leq b\alpha^{n-j}$$

and

$$|\phi(\sigma^j\rho((\underline{x}))) - \phi(\sigma^j\rho((\underline{y})))| \le b\alpha^{n-j}.$$

Hence

$$\begin{aligned}
|u(\underline{x}) - u(\underline{y})| &\le \sum_{j=0}^{[\frac{n}{2}]} |\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\underline{y})) + \phi(\sigma^j\rho((\underline{x}))) - \phi(\sigma^j\rho((\underline{y})))| + 2\sum_{j>[\frac{n}{2}]} \alpha^j \\
&\le 2b\left(\sum_{j=0}^{[\frac{n}{2}]} \alpha^{n-j} + \sum_{j>[\frac{n}{2}]} \alpha^j\right) \\
&\le \frac{4b\alpha^{[\frac{n}{2}]}}{1-\alpha}.
\end{aligned}$$

This shows that $u \in \mathscr{F}_{\mathbf{R}}$. Hence $\psi := \phi - u + u \circ \sigma \in \mathscr{F}_{\mathbf{R}}$. Furthermore

$$\begin{aligned}
\psi(\underline{x}) &= \phi(\underline{x}) + \sum_{j=-1}^{\infty} \left(\phi(\sigma^{j+1}\rho(\underline{x})) - \phi(\sigma^{j+1}(\underline{x}))\right) + \sum_{j=0}^{\infty} \left(\phi(\sigma^{j+1}(\underline{x})) - \phi(\sigma^j(\rho(\underline{x})))\right) \\
&= \phi(\rho(\underline{x})) + \sum_{j=0}^{\infty} \left(\phi(\sigma^{j+1}(\underline{x})) - \phi(\sigma^j(\rho(\underline{x})))\right).
\end{aligned}$$

The final expression depends only on $\{x_i\}_{i=0}^{\infty}$, as desired. □

Now we analyze the previous coboundary map as a function of the observable. Note that by Remark 3.1.1, up to a change of metric, we have that $\mathscr{F}_{\mathbf{R}} = C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$. So we have the following.

Lemma 5.1.4. *Let D^+ be defined as*

$$D^+ := \left\{\psi \in C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) : \psi(\underline{x}) = \psi(\underline{y}) \text{ whenever } x_i = y_i \text{ for all } i \ge 0\right\}.$$

Then the application $\Xi : C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) \to D^+$ given by $\Xi(\phi) = \phi + u \circ \sigma - u$, where $u = u_\phi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ is given by Lemma 5.1.3, is a submersion.

Proof. First we show that the transformation $\mathcal{U} : C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) \to C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ given by

$$\mathcal{U}(\phi)(\underline{x}) = \sum_{j=0}^{\infty} (\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\rho(\underline{x}))))$$

is linear on $\phi \in C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$. Then, for $\phi, \psi \in C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ and $\lambda \in \mathbb{R}$ we have that

$$\begin{aligned}
\mathcal{U}(\phi + \lambda\psi)(\underline{x}) &= \sum_{j=0}^{\infty} ((\phi + \lambda\psi)(\sigma^j(\underline{x})) - (\phi + \lambda\psi)(\sigma^j(\rho(\underline{x})))) \\
&= \sum_{j=0}^{\infty} ((\phi(\sigma^j(\underline{x})) + \lambda\psi(\sigma^j(\underline{x}))) - (\phi(\sigma^j(\rho(\underline{x}))) + \lambda\psi(\sigma^j(\rho(\underline{x}))))) \\
&= \sum_{j=0}^{\infty} ((\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\rho(\underline{x})))) + \lambda(\psi(\sigma^j(\underline{x})) - \psi(\sigma^j(\rho(\underline{x}))))).
\end{aligned}$$

Since
$$\mathcal{U}(\phi)(\underline{x}) = \sum_{j=0}^{\infty}(\phi(\sigma^j(\underline{x})) - \phi(\sigma^j(\rho(\underline{x}))))$$
and
$$\mathcal{U}(\psi)(\underline{x}) = \sum_{j=0}^{\infty}(\psi(\sigma^j(\underline{x})) - \psi(\sigma^j(\rho(\underline{x}))))$$
are convergent series, we have that
$$\mathcal{U}(\phi+\lambda\psi)(\underline{x}) = \mathcal{U}(\phi)(\underline{x}) + \lambda\mathcal{U}(\psi)(\underline{x}).$$

This proves that $\mathcal{U}$ is linear. Hence, $\Xi : C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R}) \to D^+$ is linear, since it is a sum of linear transformations. Note also that Ξ is surjective by construction of D^+. This implies that $\Xi : C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R}) \to D^+$ is submersion. $\square$

Remark 5.1.5. If $\phi \in C^\alpha(\Sigma^+_{\mathbf{R}},\mathbb{R})$ one can associate to ϕ an observable $\tilde{\phi} \in C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R})$ by
$$\tilde{\phi}\left(\{x_i\}_{i=-\infty}^{\infty}\right) = \phi\left(\{x_i\}_{i=0}^{\infty}\right)$$
where $\{x_i\}_{i=0}^{\infty} \in \Sigma^+_{\mathbf{R}}$ is the natural projection of $\{x_i\}_{i=-\infty}^{\infty} \in \Sigma_{\mathbf{R}}$. Note that $\tilde{\phi} : \Sigma_{\mathbf{R}} \to \mathbb{R}$ is constant along local stable leaves. Reciprocally, if $\tilde{\phi} \in C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R})$ satisfies $\tilde{\phi}(\underline{x}) = \tilde{\phi}(\underline{y})$ whenever $x_i = y_i$ for all $i \geq 0$, then one can associate to $\tilde{\phi}$ an observable in $\phi \in C^\alpha(\Sigma^+_{\mathbf{R}},\mathbb{R})$ by
$$\phi\left(\{x_i\}_{i=0}^{\infty}\right) = \tilde{\phi}\left(\{x_i\}_{i=-\infty}^{\infty}\right).$$
The observables in $C^\alpha(\Sigma^+_{\mathbf{R}},\mathbb{R})$ are thus identified with the subclass of $C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R})$ formed by observables that are constant on local stable leaves. More precisely, given identification
$$\Sigma^+_{\mathbf{R}} \quad\approx\quad \Sigma_{\mathbf{R}}/\sim,$$
where $\underline{x} \sim \underline{y}$ if $x_i = y_i$ for all $i \geq 0$ and $\underline{x},\underline{y} \in \Sigma_{\mathbf{R}}$, one can identify
$$C^\alpha(\Sigma^+_{\mathbf{R}},\mathbb{R}) \quad\approx\quad C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R})/\sim \;\approx\quad D^+,$$
where $D^+ := \big\{\psi \in C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R}) : \psi(\underline{x}) = \psi(\underline{y}) \text{ whenever } x_i = y_i \text{ for all } i \geq 0\big\}$.

The next result is a version of the main result in [19] for bilateral subshift of finite type.

Proposition 5.1.6. *There is an open and dense subset $\mathcal{R} \subset C^\alpha(\Sigma_{\mathbf{R}},\mathbb{R})$ such that for all $\phi \in \mathcal{R}$ there is a single ϕ-maximizing measure and it is supported on a periodic orbit of $\sigma : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$.*

Proof. By Remark 3.1.2, we can apply Theorem 5.1.1 to $\sigma : \Sigma_{\mathbf{R}}^{+} \to \Sigma_{\mathbf{R}}^{+}$ and obtain an open and dense set $\mathcal{O} \subset C^{\alpha}(\Sigma_{\mathbf{R}}^{+}, \mathbb{R})$ such that for all $\psi \in \mathcal{O}$ there is a single ψ-maximizing measure and it is supported on a periodic orbit. By Remark 5.1.5 we have that $\mathcal{O}$ is isomorphic to an open and dense set $\mathcal{O}^{+} \subset D^{+}$, such that for all $\psi \in \mathcal{O}^{+}$ there is a single ψ-maximizing measure and it is supported on a periodic orbit. In fact, for every μ σ-invariant measure in $\Sigma_{\mathbf{R}}^{+}$ there is a natural way to make μ into a measure on Σ_A.

Following [14, Section C], for $\phi \in C^0(\Sigma_{\mathbf{R}}, \mathbb{R})$ define $\phi^* \in C^0(\Sigma_{\mathbf{R}}^{+}, \mathbb{R})$ by

$$\phi^*\left(\{x_i\}_{i=0}^{\infty}\right) = \min\{\phi(\underline{y}) : \underline{y} \in \Sigma_{\mathbf{R}}, y_i = x_i \text{ for all } i \geq 0\}.$$

Notice that for $m, n \geq 0$ one has

$$\|(\phi \circ \sigma^n)^* \circ \sigma^m - (\phi \circ \sigma^{m+n})^*\| \leq \mathrm{var}_n \phi.$$

Hence

$$\begin{aligned} \left| \int (\phi \circ \sigma^n)^* d\mu - \int (\phi \circ \sigma^{n+m})^* d\mu \right| &= \left| \int (\phi \circ \sigma^n)^* \circ \sigma^m d\mu - \int (\phi \circ \sigma^{n+m})^* d\mu \right| \\ &\leq \mathrm{var}_n \phi \end{aligned}$$

which approaches 0, as $n \to \infty$, since ϕ is continuous. Hence

$$\int \phi d\tilde{\mu} = \lim_{n \to \infty} \int (\phi \circ \sigma^n)^* d\mu$$

exists by the Cauchy criterion. It is straightforward to check that $\tilde{\mu} \in C^0(\Sigma_{\mathbf{R}}, \mathbb{R})^*$. By the Riesz Representation Theorem we see that $\tilde{\mu}$ defines a probability measures on $\Sigma_{\mathbf{R}}$. Note that

$$\int \phi \circ \sigma d\tilde{\mu} = \lim_{n \to \infty} \int (\phi \circ \sigma^{n+1})^* d\mu = \int \phi d\tilde{\mu}.$$

proving that $\tilde{\mu}$ is σ-invariant. Also $\int \tilde{\psi} d\tilde{\mu} = \int \psi d\mu$ for $\phi \in C^0(\Sigma_{\mathbf{R}}^{+}, \mathbb{R})$ with $\tilde{\psi}$ as in Remark 5.1.5.

Note that if $\psi = \phi + u \circ \sigma - u$, then $M(\phi, \sigma) = M(\psi, \sigma)$ and the maximizing measures for ϕ and ψ are the same. Hence, by Lemma 5.1.4 the pre-image $\Xi^{-1}(\mathcal{O}^{+})$ is an open and dense subset of $C^{\alpha}(\Sigma_{\mathbf{R}}, \mathbb{R})$, and for every for every $\phi \in \Xi^{-1}(\mathcal{O}^{+})$ there exists a single ϕ-maximizing measure and it is supported on a periodic orbit. □

5.1.2 Proof of Theorem B

The next Lemma, similar to [3, Lemma 3.4], shows that the maximum value $M(\Phi, (X^t)_t)$ varies continuously with respect to the observable Φ.

Lemma 5.1.7. *Let (Φ_k) be a sequence of continuous observables converging to $\Phi : \Sigma^r \to \mathbb{R}$ in the C^0-topology. Let μ_k be any maximizing measure for Φ_k and μ be an accumulation point of the sequence $(\mu_k)_k$. Then, $\lim_{k\to\infty} M(\Phi_k, (X^t)_t) = M(\Phi, (X^t)_t)$ and μ is a Φ-maximizing measure.*

Proof. For any $\epsilon > 0$, and and for k sufficiently large,

$$\Phi(x,t) - \epsilon \leq \Phi_k(x,t) \leq \Phi(x,t) + \epsilon$$

for all $(x,t) \in \Sigma^r$.

This shows

$$M(\Phi, (X^t)_t) - \epsilon \leq M(\Phi_k, (X^t)_t) \leq M(\Phi, (X^t)_t) + \epsilon.$$

Furthermore, we have $M(\Phi_k, (X^t)_t) = \int \Phi_k d\mu_k$, $M(\Phi, (X^t)_t) = \int \Phi d\mu$ and (up to a subsequence),

$$\lim_{k\to\infty} \int \Phi_k d\mu_k = \int \Phi d\mu$$

because μ_k converges to μ in the weak* topology and Φ_k goes to Φ in the strong topology. $\square$

5.1.2.1 Reduction to base dynamics

In this subsection let $\sigma : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ be a two-sided subshift of finite type and let $(X^t)_t$ be the suspension flow associated to σ with a Hölder continuous height function $r : \Sigma_{\mathbf{R}} \to \mathbb{R}_+$ bounded away from zero. We also consider $\mu \in \mathcal{M}_1(\Sigma^r_{\mathbf{R}}, (X^t)_t)$ and $\bar{\mu} \in \mathcal{M}_1(\Sigma_{\mathbf{R}}, \sigma)$, such that μ is induced in $\Sigma^r_{\mathbf{R}}$ by $\bar{\mu}$. By Remark 3.2.7 we have that

$$\mu = \frac{\bar{\mu} \times Leb}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}}. \tag{5.1.1}$$

Lemma 5.1.8. *For each continuous function $\Phi : \Sigma^r_{\mathbf{R}} \to \mathbb{R}$, define $\varphi : \Sigma_{\mathbf{R}} \to \mathbb{R}$ by*

$$\varphi(x) = \int_0^{r(x)} \Phi(X^s(x))\, ds,$$

for every $x \in \Sigma_{\mathbf{R}}$. Then

$$\int_{\Sigma^r_{\mathbf{R}}} \Phi d\mu = \frac{\int_{\Sigma_{\mathbf{R}}} \varphi d\bar{\mu}}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}}. \tag{5.1.2}$$

Proof. By (5.1.1) we have

$$\begin{aligned}
\int_{\Sigma^r_{\mathbf{R}}} \Phi d\mu &= \int \Phi \circ \chi_{\Sigma^r_{\mathbf{R}}} d\mu \\
&= \frac{1}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}} \int_{\Sigma_{\mathbf{R}} \times \mathbb{R}} \Phi \circ \chi_{\Sigma^r_{\mathbf{R}}}(x,s)\, d\bar{\mu} \times Leb \\
&= \frac{1}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}} \int_{\Sigma_{\mathbf{R}}} \int_{\mathbb{R}} \Phi \circ \chi_{\Sigma^r_{\mathbf{R}}}(x,s)\, ds\, d\bar{\mu} \\
&= \frac{1}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}} \int_{\Sigma_{\mathbf{R}}} \int_0^{r(x)} \Phi(X^s(x))\, ds\, d\bar{\mu} \\
&= \frac{\int_{\Sigma_{\mathbf{R}}} \varphi d\bar{\mu}}{\int_{\Sigma_{\mathbf{R}}} r d\bar{\mu}}.
\end{aligned}$$

□

Lemma 5.1.9. *The map* $\mathfrak{F} : C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R}) \to C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ *given by*

$$\mathfrak{F}(\Phi) = \int_0^{r(x)} \Phi(X^s(x))\, ds$$

is a submersion.

Proof. $\mathfrak{F}$ is clearly linear in Φ. Therefore $D_\Phi \mathfrak{F}(H) = \mathfrak{F}(H)$ for $H \in C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R})$. To show that $D_\Phi \mathfrak{F}$ is surjective, we take any $\varphi \in C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ and present a $\Phi \in C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R})$ such that $\mathfrak{F}(\Phi) = \varphi$. It is sufficient to take $\Phi(x,t) = \dfrac{\varphi(x)}{r(x)}$, which is possible since $r(x) \neq 0$ for every $x \in \Sigma_{\mathbf{R}}$. In fact, since $\dfrac{\varphi(x)}{r(x)}$ does not depend on t,

$$\begin{aligned}
\mathfrak{F}(\Phi) &= \int_0^{r(x)} \Phi(X^s(x))\, ds \\
&= \int_0^{r(x)} \frac{\varphi(x)}{r(x)}\, ds \\
&= \frac{\varphi(x)}{r(x)} \cdot r(x) \\
&= \varphi(x).
\end{aligned}$$

Therefore $D_\Phi \mathfrak{F}$ is surjective and $\mathfrak{F}$ is a submersion. □

Next lemma plays an essential role in the proof of Theorem B. In its essence it provides a correspondence between maximizing measures for potentials on the Poincaré map and maximizing measures for suspension flows.

Lemma 5.1.10. *Let $(X^t)_t : M^r \to M^r$ be a suspension flow over a continuous map $f : M \to M$ on a compact metric space M with continuous height function $r : M \to \mathbb{R}$. Let $\Phi : M^r \to \mathbb{R}$ be continuous and $\varphi : M \to \mathbb{R}$ be given by*

$$\varphi(x) = \int_0^{r(x)} \Phi(X^s(x))\, ds.$$

Then the following are equivalent:

1. *μ is a maximizing measure for $(X^t)_t$ with respect to Φ*
2. *$\bar{\mu}$ is a maximizing measure for f with respect to $\tilde{\varphi} := \varphi - M(\Phi, (X^t)_t)r$. Moreover $M(\tilde{\varphi}, f) = 0$.*

Proof. First, note that by (5.1.2) we have

$$\begin{aligned} M(\Phi, (X^t)_t) &= \max\left\{ \int \Phi d\nu \,\middle|\, \nu \in \mathcal{M}_1(M^r, (X^t)_t) \right\} \\ &= \max\left\{ \frac{\int_M \varphi d\bar{\nu}}{\int_M r d\bar{\nu}} \,\middle|\, \bar{\nu} \in \mathcal{M}_1(M, f) \right\} \end{aligned}$$

and so

$$M(\Phi, (X^t)_t) \geq \frac{\int_M \varphi d\bar{\nu}}{\int_M r d\bar{\nu}}$$

for all $\bar{\nu} \in \mathcal{M}_1(M, f)$. So we have

$$\max_{\bar{\nu} \in \mathcal{M}_1(M,f)} \int_M \big(\varphi - M(\Phi, (X^t)_t)r\big)\, d\bar{\nu} \leq 0. \tag{5.1.3}$$

Therefore, if μ is a maximizing measure for $(X^t)_t$ with respect to Φ, from (5.1.2) we have that

$$\begin{aligned} \int_M \big(\varphi - M(\Phi, (X^t)_t)r\big)\, d\bar{\mu} &= \int_M \varphi d\bar{\mu} - M(\Phi, (X^t)_t) \int_M r d\bar{\mu} \\ &= \int_M \varphi d\bar{\mu} - \int_{M^r} \Phi d\mu \int_M r d\bar{\mu} \\ &= \int_M \varphi d\bar{\mu} - \frac{\int_M \varphi d\bar{\mu}}{\int_M r d\bar{\mu}} \int_M r d\bar{\mu} \\ &= 0. \end{aligned}$$

By (5.1.3), zero is the maximum possible value for $\int_M (\varphi - M(\Phi, (X^t)_t)r)\, d\bar{\mu}$. Thus $\bar{\mu}$ is a maximizing measure for $\varphi - M(\Phi, (X^t)_t)r$ with respect to f.

On the other hand, suppose that $\bar{\mu}$ is a maximizing measure for $\tilde{\varphi} := \varphi - M(\Phi, (X^t)_t)r$ with respect to f. We claim that $M(\tilde{\varphi}, f) = 0$. In fact, suppose by contradiction that

$$M(\tilde{\varphi}, f) = \max_{\bar{\nu} \in \mathcal{M}_1(M,f)} \int_M \big(\varphi - M(\Phi, (X^t)_t)r\big)\, d\bar{\nu} < 0.$$

In this case,

$$\frac{M(\tilde{\varphi}, f)}{\int_M r d\bar{\nu}} \leq \frac{M(\tilde{\varphi}, f)}{\|r\|_\infty} < 0$$

since, for any $\bar{\nu} \in \mathcal{M}_1(M, f)$, $\int_M r d\bar{\nu} \leq \|r\|_\infty$. Consequently

$$\begin{aligned}
\int_M \left(\varphi - M(\Phi, (X^t)_t) r\right) d\bar{\nu} &\leq M(\tilde{\varphi}, f) < 0 \\
\int_M \varphi d\bar{\nu} - M(\Phi, (X^t)_t) \int_M r d\bar{\nu} &\leq M(\tilde{\varphi}, f) < 0 \\
\frac{\int_M \varphi d\bar{\nu}}{\int_M r d\bar{\nu}} - \frac{M(\Phi, (X^t)_t) \int_M r d\bar{\nu}}{\int_M r d\bar{\nu}} &\leq \frac{M(\tilde{\varphi}, f)}{\int_M r d\bar{\nu}} < 0 \\
\int_{M^r} \Phi d\nu - M(\Phi, (X^t)_t) &\leq \frac{M(\tilde{\varphi}, f)}{\|r\|_\infty} < 0.
\end{aligned}$$

Therefore there is $a > 0$ such that $\int_{M^r} \Phi d\nu - M(\Phi, (X^t)_t) < -a$ for all $\nu \in \mathcal{M}_1(M^r, (X^t)_t)$ and taking the maximum over ν we have

$$\max_{\nu \in \mathcal{M}_1(M^r, (X^t)_t)} \int_{M^r} \Phi d\nu - M(\Phi, (X^t)_t) < -a < 0$$

leading to a contradiction.

It is straightforward from the condition $\int_M (\varphi - M(\Phi, (X^t)_t) r)\, d\bar{\mu} = 0$ and (5.1.2) that

$$M(\Phi, (X^t)_t) = \frac{\int_M \varphi d\bar{\mu}}{\int_M r d\bar{\mu}} = \int_{M^r} \Phi d\mu.$$

So μ is a maximizing measure for Φ with respect to $(X^t)_t$. □

Lemma 5.1.11. *Let $(X^t)_t$ be a suspension flow over $f : M \to M$ with α-Hölder continuous roof function $r : M \to \mathbb{R}$. If $\Phi : M^r \to \mathbb{R}$ is α-Hölder continuous (respectively continuous) in M^r. Then $\varphi : M \to \mathbb{R}$ given by*

$$\varphi(x) = \int_0^{r(x)} \Phi(X^s(x))\, ds$$

is Hölder continuous (respectively continuous).

Proof. Take $x, y \in M$ with $r(x) \geq r(y)$ (the case $r(x) \leq r(y)$ is analogous). Using that Φ and r are Hölder, we have

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &= \left| \int_0^{r(x)} \Phi(X^s(x)) ds - \int_0^{r(y)} \Phi(X^s(y)) ds \right| \\
&\leq \int_0^{r(y)} |\Phi(X^s(x)) - \Phi(X^s(y))|\, ds + \int_{r(y)}^{r(x)} \Phi(X^s(x)) ds \\
&\leq \sup r \cdot \sup_{s \in (0, r(y))} |\Phi(X^s(x)) - \Phi(X^s(y))| + \sup |\Phi| \cdot |r(x) - r(y)| \\
&\leq b \cdot \sup_{s \in (0, r(y))} d_{M^r}((x, s), (y, s))^\alpha + \sup |\Phi| \cdot L d_M(x, y)^\alpha \qquad (5.1.4)
\end{aligned}$$

for some positive constant b. It follows from Proposition 3.2.4, inequality (5.1.4) and the relation of d_{M^r} with the pseudo metric d_π expressed in (3.2.3) that

$$\begin{aligned}|\varphi(x)-\varphi(y)| &\leq \sup|\Phi|\cdot L d_M(x,y)^\alpha + bcd_\pi((x,s),(y,s))^\alpha \\ &\leq [\sup|\Phi|\cdot L + bc] d_M(x,y)^\alpha.\end{aligned}$$

This yields the desired result for Hölder continuous observables.

The case Φ continuous is immediate by composition of continuous functions. □

Proof. (of Theorem B) By Proposition 5.1.6, there exists an open and dense set $\mathcal{O} \subset C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ such that if $\varphi \in \mathcal{O}$ there is a unique φ-maximizing measure $\bar{\mu}$ and it is supported on a periodic orbit.

For $k \in \mathbb{R}$ we define the set

$$C_k := \{\psi \in C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) : M(\psi, \sigma) = k\}. \tag{5.1.5}$$

Note that $C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) = \bigcup_{k\in\mathbb{R}} C_k$. For $k = 0$, we define the map

$$\begin{aligned}\pi_0 : C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R}) &\longrightarrow C_0 \\ \varphi &\longmapsto \pi_0(\varphi) = \varphi - M(\varphi, \sigma).\end{aligned}$$

It is easy to see that if $\varphi \in \pi_0(\mathcal{O})$, then there is a single φ-maximizing measure $\bar{\mu}$ and it is supported on a periodic orbit. We claim the following

Claim 5.1.12. $\pi_0(\mathcal{O})$ is open and dense in C_0.

Proof. Take any $\varphi_1 \in \pi_0(\mathcal{O})$. We will show that φ_1 is an interior point of $\pi_0(\mathcal{O})$ in C_0. We have that there exists $\psi_1 \in \mathcal{O}$ such that $\varphi_1 = \psi_1 - M(\psi_1, \sigma)$. Denote $k_1 = M(\psi_1, \sigma)$ and consider the set C_{k_1}, as in (5.1.5). Since $\mathcal{O}$ is open in $C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$, $\mathcal{O} \cap C_{k_1}$ is open in C_{k_1}, so there is $\epsilon_1 > 0$ such that $B(\psi_1, \epsilon_1) \cap C_{k_1} \subset \mathcal{O} \cap C_{k_1}$, where $B(\psi_1, \epsilon_1)$ is the open ball in $C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$ with center in ψ_1 and radius ϵ_1. For any $\varphi_2 \in B(\varphi_1, \epsilon_1) \cap C_0$ define $\psi_2 := \varphi_2 + k_1$. Since $M(\varphi_2, \sigma) = 0$, we have that $M(\psi_2, \sigma) = k_1$, hence $\psi_2 \in C_{k_1}$ and

$$\|\psi_1 - \psi_2\| = \|\psi_1 - \varphi_2 - k_1\| = \|\varphi_1 - \varphi_2\| \leq \epsilon_1,$$

so $\psi_2 \in B(\psi_1, \epsilon_1)$. Therefore $\psi_2 \in C_{k_1}$ and $\varphi_2 \in \pi_0(\mathcal{O})$. Since φ_2 was taken arbitrarily, we have that $B(\varphi_1, \epsilon_1) \cap C_0 \subset \pi_0(\mathcal{O})$, which means that φ_1 is an interior point of $\pi_0(\mathcal{O})$ in C_0. Therefore $\pi_0(\mathcal{O})$ is an open subset of C_0.

In order to prove that $\pi_0(\mathcal{O})$ is dense, take any $\varphi_3 \in C_0 \backslash \pi_0(\mathcal{O})$ and show that φ_3 is a accumulation point for $\pi_0(\mathcal{O})$ in C_0. Since $\mathcal{O}$ is dense in $C^\alpha(\Sigma_{\mathbf{R}}, \mathbb{R})$, there is $\{\psi_n\}_n \subset \mathcal{O}$ such that $\psi_n \to \varphi_3$ when $n \to \infty$. Since $C^\alpha(M, \mathbb{R}) \ni \varphi \mapsto M(\varphi, \sigma)$ is continuous, we have that π_0 is also continuous, so $\pi_0(\psi_n) \to \pi_0(\varphi_3) = \varphi_3$ when $n \to \infty$. Therefore φ_3 is a accumulation point for $\pi_0(\mathcal{O})$, which means that $\pi_0(\mathcal{O})$ is dense in C_0.

This proves the claim. □

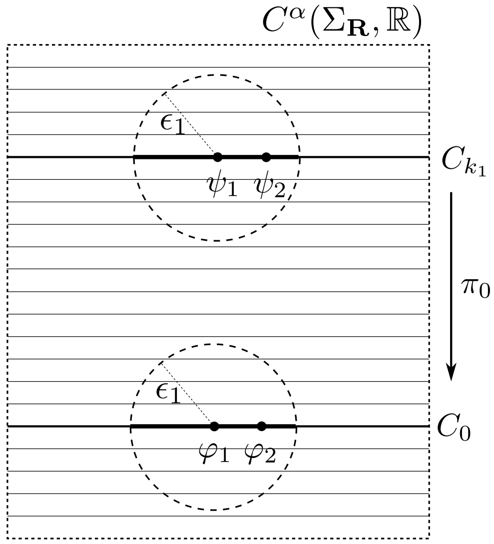

Figure 5.1.1: $\varphi_1 = \psi_1 - k_1$ and $\varphi_2 = \psi_2 - k_1$, where $k_1 = M(\psi_1, \sigma)$.

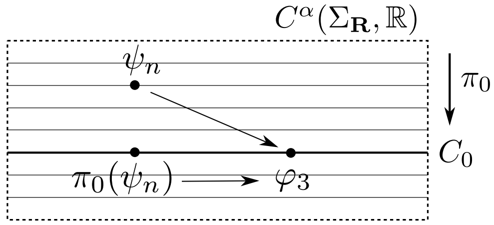

Figure 5.1.2: $\psi_n \to \varphi_3 \Rightarrow \pi_0(\psi_n) \to \varphi_3$.

We proceed with the proof of Theorem B. For every $\Phi \in C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R})$, defining $\Phi_0 = \Phi - M(\Phi, (X^t)_t)$ we have that $M(\Phi_0, (X^t)_t) = 0$. As before, we define the set

$$C^r_0 = \{\Phi \in C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R}) : M(\Phi, (X^t)_t) = 0\}.$$

The map $\mathfrak{F}(\Phi) = \int_0^{r(x)} \Phi(X^s(x))\, ds$ defined in Lemma 5.1.9 satisfies

$$\mathfrak{F}(C^r_0) = C_0.$$

Since $\mathfrak{F}$ is a submersion, by Lemma 5.1.9, we have that the pre-image $\mathfrak{F}^{-1}(\mathcal{O}_0)$ is an open and dense subset of C^r_0. Using Lemma 5.1.10 once more, every $\Phi \in \mathfrak{F}^{-1}(\mathcal{O}_0)$ has a unique maximizing measure, and it has the desired properties.

Now we claim the following

Claim 5.1.13. The set $\hat{\mathcal{O}} := \{\Phi \in C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R}) : \Phi - M(\Phi, (X^t)_t) \in \mathfrak{F}^{-1}(\mathcal{O}_0)\}$ is an open and dense subset of $C^\alpha(\Sigma^r_{\mathbf{R}}, \mathbb{R})$.

Proof. To show that $\hat{\mathcal{O}}$ is open, we take any $\Phi \in \hat{\mathcal{O}}$ and show that Φ is an interior point. Let $\Phi_0 = \Phi - M(\Phi, (X^t)_t) \in \mathfrak{F}^{-1}(\mathcal{O}_0)$. Since $\mathfrak{F}^{-1}(\mathcal{O}_0)$ is open in C^r_0, there is a $\epsilon > 0$

such that if $\Upsilon \in C_0^r$ and $\|\Phi_0 - \Upsilon\| < \epsilon$, then $\Upsilon \in \mathfrak{F}^{-1}(\mathcal{O}_0)$. On the other hand, since $\Phi \mapsto M(\Phi, (X^t)_t)$ is a continuous map, there is $\delta > 0$ such that if $\|\Phi - \Psi\| < \delta$, then $|M(\Phi, (X^t)_t) - M(\Psi, (X^t)_t)| < \dfrac{\epsilon}{2}$. Without loss of generality we can suppose that $\delta < \dfrac{\epsilon}{2}$. So taking $\Psi \in C^\alpha(\Sigma_{\mathbf{R}}^r, \mathbb{R})$ such that $\|\Phi - \Psi\| < \delta$ we have

$$\begin{aligned}\|\Phi_0 - \Psi_0\| &= \|\Phi - M(\Phi, (X^t)_t) - \Psi + M(\Psi, (X^t)_t)\| \\ &\leq \|\Phi - \Psi\| + \|M(\Phi, (X^t)_t) - M(\Psi, (X^t)_t)\| \\ &< \delta + \frac{\epsilon}{2} \\ &< \epsilon\end{aligned}$$

which means that $\Psi_0 \in C_0^r$. So $\Psi \in \hat{\mathcal{O}}$ and consequently $\hat{\mathcal{O}}$ is open.

In order to show that $\hat{\mathcal{O}}$ is dense, we take any $\Psi \in C^\alpha(\Sigma_{\mathbf{R}}^r, \mathbb{R}) \backslash \hat{\mathcal{O}}$ and show that Ψ is a accumulation point for $\hat{\mathcal{O}}$. Since $\mathfrak{F}^{-1}(\mathcal{O}_0)$ is dense in C_0^r, for any $\epsilon > 0$ there is $\Upsilon \in \mathfrak{F}^{-1}(\mathcal{O}_0)$ such that $\|\Psi_0 - \Upsilon\| < \epsilon$. Taking $\Phi = \Upsilon + M(\Psi, (X^t)_t)$, note that $M(\Phi, (X^t)_t) = M(\Psi, (X^t)_t)$. In fact, since $M(\Upsilon, (X^t)_t) = 0$ we have

$$\begin{aligned}M(\Phi, (X^t)_t) &= M(\Upsilon + M(\Psi, (X^t)_t), (X^t)_t) \\ &= M(\Upsilon, (X^t)_t) + M(\Psi, (X^t)_t) \\ &= M(\Psi, (X^t)_t).\end{aligned}$$

Moreover $\Phi \in \hat{\mathcal{O}}$, because $M(\Phi, (X^t)_t) = M(\Psi, (X^t)_t)$ and so

$$\begin{aligned}\Phi_0 &= \Phi - M(\Phi, (X^t)_t) \\ &= \Upsilon + M(\Psi, (X^t)_t) - M(\Psi, (X^t)_t) \\ &= \Upsilon \in \mathfrak{F}^{-1}(\mathcal{O}_0).\end{aligned}$$

We also have

$$\|\Psi - \Phi\| = \|\Psi - \Upsilon - M(\Psi, (X^t)_t)\| = \|\Psi_0 - \Upsilon\| < \epsilon.$$

Since ϵ was arbitrary we have that Ψ is a accumulation point for $\hat{\mathcal{O}}$. Therefore $\hat{\mathcal{O}}$ is a dense subset of $C^\alpha(\Sigma_{\mathbf{R}}^r, \mathbb{R})$. □

Note that every $\Phi \in \hat{\mathcal{O}}$ has a unique maximizing measure, as we can see by Lemma 5.1.10, and it is supported on a periodic orbit. □

5.1.3 Proof of Theorem C

Let $\Lambda \subset M$ be hyperbolic basic set for the flow $(X^t)_{t\in\mathbb{R}}$ embedding on a suspension flow over a subshift of finite type. This means that there is a subshift of finite type

$\sigma_{\mathbf{R}} : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$, a positive $r \in C^{\alpha}(\Sigma_{\mathbf{R}}, \mathbb{R})$ and a Hölder continuous bijection $\pi : \Sigma^r_{\mathbf{R}} \to \Lambda$ so that the diagram

$$\begin{tikzcd} \Sigma^r_{\mathbf{R}} \arrow[r, "Y^t"] \arrow[d, "\pi"'] & \Sigma^r_{\mathbf{R}} \arrow[d, "\pi"] \\ \Lambda \arrow[r, "X^t"] & \Lambda. \end{tikzcd} \tag{5.1.6}$$

commutes, where $\Sigma^r_{\mathbf{R}}$ is a quotient as in (3.2.1) and $(Y^t)_t : \Sigma^r_{\mathbf{R}} \to \Sigma^r_{\mathbf{R}}$ is the suspension flow over $\sigma_{\mathbf{R}}$ with height function r.

Since we suppose that $\pi : \Sigma^r_{\mathbf{R}} \to \Lambda$ is one-to-one, given an observable $\Phi \in C^{\alpha}(\Lambda, \mathbb{R})$ one can induce an observable $\Phi^* \in C^{\alpha}(\Sigma^r_{\mathbf{R}}, \mathbb{R})$ by doing $\Phi^* = \Phi \circ \pi$ and the map $\Theta : C^{\alpha}(\Lambda, \mathbb{R}) \to C^{\alpha}(\Sigma^r_{\mathbf{R}}, \mathbb{R})$ defined by $\Theta(\Phi) = \Phi \circ \pi$ is one-to-one.

By Theorem B there is an open and dense set $\mathcal{R}_r \subset C^{\alpha}(\Sigma^r_{\mathbf{R}}, \mathbb{R})$ of observables $\Phi : \Sigma^r_{\mathbf{R}} \to \mathbb{R}$ such that, for every $\Phi \in \mathcal{R}_r$, there is a single $(Y^t)_t$-maximizing measure with respect to Φ, and it is supported on a periodic orbit. Then $\Theta^{-1}(\mathcal{R}_r)$ is an open and dense set in $C^{\alpha}(\Lambda, \mathbb{R})$ such that, for every $\Phi \in \Theta^{-1}(\mathcal{R}_r)$, there is a single $(X^t)_t$-maximizing measure, with respect to Φ, and it is supported on a periodic orbit.

5.1.4 Proof of Theorem D

Morris proved in [34, Corollary 1.3] that if $f : M \to M$ satisfies Bowen's specification property, then there is a dense G_δ set (a countable intersection of open sets) $Z \subset C^0(M, \mathbb{R})$, the set of continuous observables in M, such that every $\varphi \in Z$ has a unique maximizing measure with full support. Let $(X^t)_t : M^r \to M^r$ be a suspension flow over f with the continuous height function $r : M \to \mathbb{R}$.

Our main tool to transfer results on discrete time maps for suspension flows is Lemma 5.1.10. It does not depend on the regularity of the observables, so we can utilize the same methods in the proof of Theorem B to transfer others results about uniqueness of maximizing measures for discrete time to suspension flows, even if those results are made for other classes of observables than Hölder or Lipschitz. Therefore we can, for instance, obtain Morris' result for suspension flows, which stated in Theorem 5.1.2.

As in the proof of Theorem B, we define

$$C_k = \{\psi \in C^0(M, \mathbb{R}) : M(\psi, f) = k\}$$

and conclude that there is a dense G_δ set $Z_0 \subset C_0$ such that every $\varphi \in Z_0$ has a unique maximizing measure with full support. Using again the map $\mathfrak{F}(\Phi) = \int_0^{r(x)} \Phi(X^s(x))\, ds$ defined in Lemma 5.1.9, since $\mathfrak{F}$ is a submersion, by Lemma 5.1.9, we have that the pre-image $\mathfrak{F}^{-1}(Z_0)$ is an open and dense subset of

$$C^r_0 = \{\Phi \in C^0(M^r, \mathbb{R}) : M(\Phi, (X^t)_t) = 0\}.$$

Using Lemma 5.1.10 once more, every $\Phi \in \mathfrak{F}^{-1}(Z_0)$ has a unique maximizing measure with full support.

Now we claim the following

Claim 5.1.14. The set $\hat{Z} := \{\Phi \in C^{\alpha}(\Sigma^r_{\mathbf{R}}, \mathbb{R}) : \Phi - M(\Phi, (X^t)_t) \in \mathfrak{F}^{-1}(\mathcal{O}_0)\}$ is an open and dense subset of $C^0(M^r, \mathbb{R})$.

The proof of this is identical of the proof of Claim 5.1.13. Since is clear that every $\Phi \in \hat{Z}$ has a unique maximizing measure with full support, this concludes the proof of Theorem D.

Chapter 6

Some open questions

In this section we collect some problems that arise or are related with the topics in this thesis.

6.1 Lyapunov spectra of linear cocycles over flows

First, we consider the case of linear cocycles over flows. Some two very natural questions are as follows:

Question 1: *Can Theorem A be extended to cocycles taking values on more general Lie groups, and for non-uniformly hyperbolic flows?*

A related question is whether these results can be extended to cocycles taking values on Banach or Hilbert spaces. The methods for proving the existence of a positive Lyapunov exponent and to prove simplicity of the Lyapunov spectrum are substantially different. Unfortunately, there is a minor step without a proof in [11], that if completed our results would hold for $SL(d, \mathbb{K})$ cocycles for any $d \geq 2$. We pose the following:

Question 2: *Let f be a Anosov diffeomorphism and $\mathcal{O}$ denote the space of Hölder continuous $SL(d, \mathbb{K})$ fiber-bunched linear cocycles A over f such that there exists a periodic point for f so that A has simple spectrum on p. If μ has local product structure, does an open and dense set of cocycles in $\mathcal{O}$ have simple Lyapunov spectrum with respect to μ?*

6.2 Ergodic optimization for flows

To the best of our knowledge, apart from the construction of sub-actions for Anosov flows [32], these are the first results concerning ergodic optimization for flows. Given the recent interest and development of ergodic optimization, there are many questions that can be addressed. First we consider less regular topologies. In [42, 41], Addas-Zanata and Tal proved that if M is a compact Riemannian manifold, and $Homeo(M)$ denotes the space of homeomorphisms in M then for every $\phi \in C^0(M,\mathbb{R})$ there exists a dense subset $\mathcal{D} \subset Homeo(M)$ of homeomorphisms so that every $f \in \mathcal{D}$ has a ϕ-maximizing measure supported on a periodic orbit, but that the there exists a Baire residual subset $\mathcal{R} \subset Homeo(M)$ so that no ϕ-maximizing measure is periodic. Let $\mathcal{F}^0(M)$ denote the space of continuous flows on M and $\mathfrak{X}^{0,1}(M)$ denote the space of Lipschitz continuous vector fields on M endowed with the C^0-topology.

Question 3: *Given $\phi \in C^0(M,\mathbb{R})$, does there exist a Baire residual subset $\mathcal{R} \subset \mathcal{F}^0(M)$ so that no ϕ-maximizing measure is periodic? Alternatively, the same question on $\mathfrak{X}^{0,1}(M)$.*

6.3 Hyperbolic and singular-hyperbolic flows

We now consider smooth dynamical systems with some hyperbolicity. A question that is often considered in ergodic optimization is to characterize the support of the maximizing measures (see e.g. [18, 23, 20] for both additive and non-additive sequence of observables). This raises the following:

Question 4: *Is there a subordination principle for hyperbolic flows?*

A positive answer to the previous question would lead to a better understanding of Aubry sets for flows and would require an extension of Atkinson's lemma for flows. Finally it is natural to look for extensions of Theorems C and D for the context of non-hyperbolic flows, as the Lorenz attractors. More precisely:

Question 5: *Let M be a 3-dimension compact boundless Riemannian manifold and Λ be a Lorenz-like attractor for a flow $(X^t)_t : M \to M$. Then*

1. Is there an open and dense set $R \subset C^\alpha(M,\mathbb{R})$ of α-Hölder observables such that, for every $\Phi \in R$, there is a unique $(X^t)_t$-maximizing measure, with respect to Φ, and it is supported on a periodic orbit?

2. Is there a C^0-residual subset $R \subset C^0(M,\mathbb{R})$ such that for every $\Phi \in R$ there is a unique $(X^t)_t$-maximizing measure, with respect to Φ, it has full support and zero entropy?

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