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On the rotation sets of homeomorphisms and flows on tori

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Salvador-Bahia
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Tese apresentada ao Colegiado do Programa de Pós-Graduação em Matemática UFBA/UFAL como requisito parcial para obtenção do título de Doutor em Matemática, aprovada em 17 de Dezembro de 2018.

Orientador: Prof. Dr. Paulo César Rodrigues Pinto Varandas

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Resumo

Estudamos os conjuntos de rotação para homeomorfismos homotópicos à identidade no toro $\mathbb{T}^d$, $d \geq 2$. No cenário conservativo, provamos que existe um subconjunto residual de Baire do conjunto $\text{Homeo}_{0,\lambda}(\mathbb{T}^2)$ de homeomorfismos conservativos homotópicos à identidade, de modo que o conjunto dos pontos com conjunto de rotação pontual selvagem é um Baire subconjunto residual em $\mathbb{T}^2$, e que ele carrega pressão topológica completa e dimensão média da métrica completa. Além disso, provamos que, para qualquer dimensão $d \geq 2$, o conjunto de rotação de homeomorfismos conservadores genéricos C^0 em $\mathbb{T}^d$ é convexo. Resultados relacionados são obtidos no caso de homeomorfismos dissipativos no toro. Os resultados anteriores baseiam-se na descrição da complexidade topológica do conjunto de pontos com comportamento histórico selvagem e na densidade de medidas periódicas para aplicações contínuos com a propriedade de colagem de órbita (em classes recorrentes de cadeia).

Palavras chaves: Conjunto de rotação; Homeomorfismo do toro, comportamento histórico, entropia total, dimensão media métrica; propriedade de colagem de órbita; Especificação.

Abstract

We study the rotation sets for homeomorphisms homotopic to the identity on the torus $\mathbb{T}^d$, $d \geq 2$. In the conservative setting, we prove that there exists a Baire residual subset of the set $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^2)$ of conservative homeomorphisms homotopic to the identity so that the set of points with wild pointwise rotation set is a Baire residual subset in $\mathbb{T}^2$, and that it carries full topological pressure and full metric mean dimension. Moreover, we prove that, for every $d \geq 2$, the rotation set of C^0-generic conservative homeomorphisms on $\mathbb{T}^d$ is convex. Related results are obtained in the case of dissipative homeomorphisms on tori. The previous results rely on the description of the topological complexity of the set of points with wild historic behavior and on the denseness of periodic measures for continuous maps with the gluing orbit property (on chain recurrent classes).

Keywords: Rotation sets; homeomorphisms on tori; historic behavior; topological entropy; metric mean dimension; gluing orbit property; specification.

Contents

Introduction

In this work we address and relate some fundamental concepts in topological dynamical systems, namely topological pressure (including topological entropy), metric mean dimension and generalized rotation sets for homeomorphisms on compact metric spaces. Topological entropy and metric mean dimensions are two measurements of the dynamical complexity, which are particularly important for continuous dynamical systems. While the first is a topological invariant, it is typically infinite for a C^0-Baire generic subset of homeomorphisms on surfaces [53]. On the other hand the second one, inspired by Gromov [21] and proposed by Lindenstrauss and Weiss, is a sort of dynamical analogue of the topological dimension, depends on the metric it is bounded above by the dimension of the ambient space [29]. In this way, the metric mean dimension may be used to distinguish the topological complexity of surface homeomorphisms with infinite topological entropy.

Our main motivation is to describe rotation sets for homeomorphisms homotopic to the identity on tori. The rotation number of a circle homeomorphism f, introduced by Poincaré [44], is defined by

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \ (\text{mod}1) \tag{0.0.1}$$

where $x \in \mathbb{S}^1$ and F is a lift of the circle homeomorphism to $\mathbb{R}$, a lift of a map f is a map $F : \mathbb{S}^1 \to \mathbb{S}^1$ that satisfies $f \circ \pi = \pi \circ F$. The rotation number is independent of F and x, and constitutes a very useful topological invariant (see e.g. [13]). The situation changes drastically in the case of one-dimensional endomorphisms and higher-dimensional homeomorphisms. This concept was first extended for continuous maps of degree one in the circle in which case the limit (0.0.1) does not necessarily exist, its accumulation points form a (possibly degenerate) interval and such a limit set defines a rotation interval which depends on the point x ([36]). A generalization of rotation theory to a higher dimensional setting was studied by Franks, Kucherenko, Kwapisz, Llibre, MacKay, Misiurewicz, Wolf and Ziemian among others (see [19, 20, 31, 33, 34, 35] and references therein) for homeomorphisms homotopic to the identity, where the notion of rotation sets extend the concept of rotation number for circle homeomorphisms. Although rotation sets are not a complete invariant, their shapes can be used to describe properties of the dynamical system, as we now illustrate. If f is a homeomorphism on the torus $\mathbb{T}^d$ ($d \geq 2$) homotopic to the identity, $\pi : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the natural projection and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a lift for f (cf. Appendix for more details), the *rotation set* of F is defined by

$$\rho(F) = \Big\{ v \in \mathbb{R}^d \;:\; \exists\, z_i \in \mathbb{R}^2, \exists\, n_i \nearrow \infty \;:\; \lim_{i\to\infty} \frac{F^{n_i}(z_i) - z_i}{n_i} = v \Big\}. \tag{0.0.2}$$

and the *rotation vector* of a point $z \in \mathbb{R}^d$ is (if the limit exists)

$$\rho(F, z) := \lim_{n\to\infty} \frac{F^n(z) - z}{n}. \tag{0.0.3}$$

Given $x \in \mathbb{T}^d$ we define $\rho(F, x)$ by (0.0.3) (note that the previous expression does not vary in $\pi^{-1}(x)$). The previous sets are compact and connected subsets of $\mathbb{R}^d$ (see e.g. Subsection 1.2 and [30, 34] for more details). In the 2-torus, the rotation set is convex (it may fail to be convex in higher dimensional torus, [35]) but there are compact convex sets of the plane that are not the rotation set of any torus homeomorphisms [27, 34]. Nevertheless, for every convex polygon $K \subset \mathbb{R}^2$ there exists a homeomorphism f on $\mathbb{T}^2$ homotopic to the identity so that $\rho(F) = K$ [33].

Building over the work of Koropecki and Guihéneuf [23], and Passeggi [39], we derive several consequences on the convergence of the rotation vectors in the case of homeomorphisms homotopic to the identity on the torus $\mathbb{T}^2$.

Our starting point is a result due to Passeggi [39] which asserts that there exists a C^0-open and dense subset $\mathcal{U}$ of the set of homeomorphisms of the torus $\mathbb{T}^2$ so that for every $f \in \mathcal{U}$ the rotation set $\rho(F)$ is a (eventually degenerate) polygon with rational vertices. The rotation set of an axiom A diffeomorphisms is a rational polygon. A. Guihéneuf and A. Koropecki [23] proved in the area-preserving setting that there exists a C^0-dense set of homeomorphisms whose rotation set is a polygon with nonempty interior. we are mainly interested in addressing rotation theory for continuous flows on tori. We will focus mainly in the three-dimensional setting, inspired by results for homeomorphisms on surfaces and what one can expect from suspension flows.

The specification property, introduced by Bowen [10], corresponds to a strong shadowing of pieces of orbits, it was very important to study the topological and ergodic features of the dynamical system. Other similar notions have been introduced to study specific problems where the specification property failed. In particular, the concept of gluing orbit property, introduced by Bomfim and Varandas in [7], proved to be an useful tool to replace the specification property e.g. in the study of multifractal formalism for non-uniformly hyperbolic flows. The flexibility of the gluing orbit property, comparing to specification, is the existence of jumps bounded above by a positive integer $m(\varepsilon)$ (see

Subsection 1.4.3). Just as an illustration it can be proved that the gluing orbit property hold for suspension flows over a map with gluing orbit property. It follows immediately that the gluing orbit property implies the specification property. We try to study the relationship between gluing orbit property and rotation set, expressed in Theorems C and D.

We will focus on the realization of convex sets as rotation sets (see Subsection 1.2 for the definition). More precisely, if f is a homeomorphism on $\mathbb{T}^d$, $g \geq 2$, and the map $F : \mathbb{R}^d \to \mathbb{R}^d$ is a lift:

1. given a compact and convex set $K \subset \rho(F)$, does there exist $x \in \mathbb{T}^d$ and $z \in \pi^{-1}(x) \in \mathbb{R}^d$ such that $\rho(F, z) = K$?

2. if the previous holds, what is the size of such set of points in $\mathbb{T}^d$?

3. how commonly (in f) is $\rho(F)$ convex?

Concerning the first question we note that if f is a homeomorphism isotopic to the identity on $\mathbb{T}^2$ and F is a lift then: (i) for every rational vector $v \in \rho(F)$ in the interior of $\rho(F)$ there exists a periodic point $x \in \mathbb{T}^d$ so that $\rho(F, \widetilde{x}) = v$ [19]; (ii) for any vector v in the interior of $\rho(F)$ there exists a non-empty compact set $\Lambda_v \subset \mathbb{T}^2$ so that $\rho(F, \widetilde{x}) = v$ for every $x \in \Lambda_v$ [34] (iii) for any compact connected C is the interior of the convex hull of vectors in $\rho(F)$ which represent periodic orbits of f there exists a point $x \in \mathbb{T}^2$ so that $\rho(F, \tilde{x}) = C$ [30].

It seems that much less is known as an answer to the second question. Building over [23, 39] we prove that C^0-generic conservative homeomorphisms homotopic to the identity on $\mathbb{T}^2$ are so that the set of points for which the rotation vector is not well defined (equivalently, the limit defined by (0.0.3) does not exist) form a Baire residual, full topological pressure and full metric mean dimension subset of $\mathbb{T}^2$. In the case of dissipative homeomorphisms homotopic to the identity we prove that the gluing orbit property is typical among the chain recurrent classes in the non-wandering set and we use this fact to prove that for "most" surface homeomorphisms of $\mathbb{T}^2$ homotopic to the identity, the set of points with non-trivial (here non-trivial is equivalently to say that is just a vector) pointwise rotation set is topologically large (Baire residual, full topological entropy and metric mean dimension) in a positive entropy chain recurrent class.

Finally, concerning the third question, we refer that Passeggi [39] proved that an open and dense subset set of homeomorphisms on $\mathbb{T}^2$ homotopic to the identity so that the rotation set is a rational polygon. Here we prove that C^0-generic conservative homeomorphisms homotopic to the identity on the torus have a convex rotation set, providing an answer to this question. We also obtain related results in the dissipative context.

The previous results fit in a more general framework, namely the description of the topological complexity of the set of points with historic behavior (also known as irregular, exceptional or non-typical points) from the topological viewpoint, and the density of periodic measures. Given a continuous map $f : X \to X$ on a compact metric space (X, d) and a continuous observable $\varphi : X \to \mathbb{R}^d$ $(d \geq 1)$, the set of points with historic behavior with respect to φ is

$$X_{\varphi,f} := \Big\{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \ \text{ does not exist}\Big\}.$$

The term historic behavior was coined after some dynamics where the phenomena of the persistence of points with this kind of behavior occurs [46, 50]. Birkhoff's ergodic theorem (applied to the coordinates of φ) ensures that $X_{\varphi,f}$ is negligible from the measure theoretic viewpoint, as it has zero measure with respect to any invariant probability measure. It was first proved by Pesin and [41], and by Barreira and Schmelling [4], that in the case of subshifts of finite type, conformal repellers and conformal horseshoes the sets $X_{\varphi,f}$ are either empty or carry full topological entropy, and full Hausdorff dimension [4, 41]. Several extensions of these results have been considered later on, building mainly over the concept of specification introduced by Bowen in the early seventies and the concept of shadowing (see e.g. [5, 11, 15, 25, 38, 51, 52] and references therein).

Here we obtain yet another mechanism to describe the topological complexity of the set of points with historic behavior, and to pave the way to multifractal analysis. In order to do so, we introduce the notion of relative metric mean dimension. Then, given a continuous map with the gluing orbit property (a concept introduced in [7] in the context of topological dynamical systems which bridges between uniform and non-uniform hyperbolicity and extends the concept of specification) we prove that any non-empty set of points with historic behavior has three levels of topological complexity: it is Baire generic, it has full topological pressure and it has full metric mean dimension (Theorems A and B). Moreover, we prove that the latter holds for typical pairs (f, φ) of homeomorphisms and continuous observables (Corollary A), building over the fact, of independent interest, that the gluing orbit property holds on the chain recurrent classes of C^0-generic homeomorphisms (Proposition 4.2.2).

This paper is organized as follows. Our main results are given in Chapter 2, where where we make an overview in the proof. In Chapter 1 we describe the setting, some preliminaries on the topological invariants and notions of complexity. In the Chapters 4 and 5 we prove the results on the rotation sets for homeomorphisms homotopic to identity, it is, Theorems D, C and E. Chapter 3 is devoted to the proofs for the results on the set of points with wild historic behavior for maps with the gluing orbit property, that are Theorem A and Theorem B.

Finally, in Section 7 we make some comments and discuss futures directions of research.

Chapter 1

Preliminaries

1.1 The space of homeomorphisms homotopic to identity

Let X be a compact metric space. Recall that we denote $\mathrm{Homeo}(X)$ as the space of homeomorphisms on X endowed with the C^0-topology given by the metric

$$d_{C^0}(f,g) = \max\left\{\sup_{x\in X}\{d(f(x),g(x))\},\ \sup_{x\in X}\{d(f^{-1}(x),g^{-1}(x))\}\right\}$$

for every $f,g \in \mathrm{Homeo}(X)$. Two homeomorphisms $f,g : X \to X$ are *homotopic* if there exists a continuous function $H : [0,1] \times X \to X$ (homotopy between f and g) such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for every $x \in X$. If H is a homotopy between f and g, then it defines a family of continuous functions $H_t : X \to X$ given by $H_t(x) = H(t,x)$. Two homeomorphisms $f,g : X \to X$ are *isotopic* if there exists a homotopy H between f and g such that for every $t \in [0,1]$ the map $H_t : X \to X$ is a homeomorphism. It follows from [18] that the previous concepts coincide for homeomorphisms on $\mathbb{R}^2$. More precisely:

Theorem 1.1.1. *[18, Theorem 6.4] If h is a homeomorphism of $\mathbb{R}^2$ onto itself, homotopic to the identity then h is isotopic to the identity.*

We denote $\mathrm{Homeo}_0(X) \subset \mathrm{Homeo}(X)$ the space of homeomorphisms on X homotopic to the identity and let $\mathrm{Homeo}_{0,\lambda}(X)$ be the subspace of $\mathrm{Homeo}_0(X)$ formed by the area-preserving homeomorphism (f is area-preserving if $\mathrm{Leb}(f^{-1}(A)) = \mathrm{Leb}(A)$ for all $A \subset X$ measurable). In other words, $\mathrm{Homeo}_{0,\lambda}(X) := \mathrm{Homeo}_0(X) \cap \mathrm{Homeo}_\lambda(X)$, where $\mathrm{Homeo}_\lambda(X)$ consisting of area-preserving homeomorphisms. Theorem 1.1.1 ensures the following

Proposition 1.1.2. *$Homeo_0(\mathbb{T}^2)$ is an open set in $Homeo(\mathbb{T}^2)$.*

Proof. Fix $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Let g C^0-near to f. Given $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ lifts of f and g, respectively. Take $H_t(x) = tF(x) + (1-t)G(x)$ the homotopy between F and G, with $t \in [0,\ 1]$. Note that

$$h_t(y) := \pi \circ H_t(x)$$

where $y = \pi(x)$, is a homotopy between f and g and, as f is homotopic to identity, we conclude that g is homotopic to identity. By Theorem 1.1.1, f and g are isotopic to the identity. $\square$

As consequence of the proposition above 1.1.2, $\mathrm{Homeo}_{0,\lambda}(\mathbb{T}^2)$ is C^0-open in $\mathrm{Homeo}_\lambda(\mathbb{T}^2)$.

1.2 Rotation sets for homeomorphisms in $\mathbb{T}^d$

In this subsection we recall briefly some notions and properties of rotation sets (see [34, 35] for more details and proofs).

Let $f : X \to X$ be a continuous map and $\varphi : X \to \mathbb{R}^d$ ($d \geq 1$) be a continuous function. The *rotation set* of φ, denoted by $\rho(\varphi)$, is the set of limits of convergent sequences $(\frac{1}{n_i}\sum_{i=0}^{n_i-1}\varphi(f^i(x_i)))_{i=1}^{\infty}$, where $n_i \to \infty$ and $x_i \in X$. Given $x \in X$, let $\mathcal{V}_\varphi(x)$ denote the accumulation points of the sequence $(\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)))_{n\geq 1}$, and let

$$\mathcal{V}_\varphi := \bigcup_{x\in X} \mathcal{V}_\varphi(x)$$

be the *pointwise rotation set* of φ. In the case that $\mathcal{V}_\varphi(x) = \{v\}$ we say that v is the *rotation vector* of x. Finally, given an f-invariant probability measure μ on X we say that $\int \varphi d\mu$ is the *rotation vector of* μ and denote it by $\mathcal{V}_\varphi(\mu)$.

In the special case that $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$, $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the natural projection, $F : \mathbb{R}^d \to \mathbb{R}^d$ a lift for f (cf. Appendix for more details), and the displacement function $\varphi_F : \mathbb{T}^d \to \mathbb{R}^d$ is defined by $\varphi_F(\pi(z)) = F(z) - z$, then

$$\frac{1}{n_i}\sum_{i=0}^{n_i-1}\varphi_F(f^i(\pi(z_i))) = \frac{1}{n_i}\sum_{i=0}^{n_i-1}(F^{i+1}(z_i) - F^i(z_i)) = \frac{F^{n_i}(z_i) - z_i}{n_i}$$

with $z_i \in \mathbb{R}^d$ and $n_i \geq 1$. Using that φ_F is constant on $\pi^{-1}(x)$ for every $x \in \mathbb{T}^d$ it induces a continuous observable in $\mathbb{T}^d$, which we still denote by φ_F with some abuse of notation. Hence, the *rotation set* of F, denoted by $\rho(F)$, defined in [35] as the limits of converging sequences

$$\Big(\frac{F^{n_i}(z_i) - z_i}{n_i}\Big)_{z_i\in\mathbb{R}^2,\, n_i\geq 1}.$$

In other words $v \in \rho(F)$, if and only if there are $x_i \in \mathbb{R}^d$ and integers n_i with $\lim_{i\to+\infty} n_i = \infty$ such that

$$\lim_{i\to+\infty} \frac{F^{n_i}(x_i) - x_i}{n_i} = v.$$

Given $x \in \mathbb{R}^2$, let $\rho(F, x) = \mathcal{V}_{\varphi_F}(x)$ and $\rho_p(F) = \mathcal{V}_{\varphi_F}$ denote the *pointwise rotation set* of x and the *pointwise rotation set* of F as defined before with respect to the observable φ_F, which fit in the previous context. It is, given $x \in \mathbb{T}^d$, we define pointwise rotation set of x, as the limits of converging sequences

$$\Big(\frac{F^{n_i}(\widetilde{x}) - \widetilde{x}}{n_i}\Big)_{n_i\geq 1},$$

where $\widetilde{x} \in \pi^{-1}(x)$.

and pointwise rotation set of F by $\rho_p(F) = \bigcup_{x\in\mathbb{R}^d} \rho(F, x)$.

The rotation set induced by the ergodic probability measures is $\rho_{erg}(F) := \{\int \varphi_F \, d\mu : \mu \in \mathcal{M}_e(f)\}$, where $\mathcal{M}_e(f)$ denote the set of f-invariant and ergodic probability measures (analogous for $\rho_{inv}(F)$ using the space $\mathcal{M}_{inv}(f)$ of f-invariant probability measures).

We recall that

$$\rho_{erg}(F) \subseteq \rho_p(F) \subseteq \rho(F) \subseteq \rho_{inv}(F) \tag{1.2.1}$$

and that $\rho_{inv}(F)$ is convex. Moreover, if $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ then

$$\rho(F) = \mathrm{Conv}\ \rho(F) = \mathrm{Conv}(\rho_p(F)) = \mathrm{Conv}\ (\rho_{erg}(F)) = \rho_{inv}(F)$$

where $\mathrm{Conv}(K)$ denotes the convex hull of K, ie, $\mathrm{Conv}(K) = \bigcap\{C : C \text{ is convex and } K \subset C\}$, (see [34]).

Remark 1.2.1. Given $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and two lifts F and G of f, we have that $F - G \in \mathbb{Z}^d$. In addition, $\rho(F) = \rho(G) \mod \mathbb{Z}^2$.

For homeomorphism homotopic to identity on torus $\mathbb{T}^2$ we have the following results:

Theorem 1.2.2. *[35] If $f \in Homeo_0(\mathbb{T}^2)$ and $v \in \rho(F) \cap \mathbb{Q}^2$ is an extremal point, then there exists $x \in \mathbb{R}^2$ such that $\rho(F, x) = v$; In particular, there exists an ergodic measure f-invariant μ such that $\int \varphi d\mu = v$;*

Theorem 1.2.3. *[19] If $f \in Homeo_0(\mathbb{T}^2)$ and $u \in int(\rho(F)) \cap \mathbb{Q}^2$, then there exists a periodic point $x \in \mathbb{T}^2$ of f such that $\rho(F, \widetilde{x}) = u$.*

Theorem 1.2.4. *[34] If $f \in Homeo_0(\mathbb{T}^2)$ and $u \in int(\rho(F)) \cap \mathbb{Q}^2$, then there exists a non-empty closed f-invariant $K \subset \mathbb{T}^2$ such that $\rho(F, x) = u$ for every $x \in \pi^{-1}(K)$. Moreover, there exists an ergodic measure f-invariant μ such that $\int \varphi_F d\mu = u$;*

The following useful results are due the Libre and Mackay [30].

Theorem 1.2.5. *[30, Theorem 1] If $f \in Homeo_0(\mathbb{T}^2)$ and F is a lift of f then the following hold: (i) if $\rho(F)$ has nonempty interior, then f has positive topological entropy; and (ii) if $\Delta \subset \rho(F)$ is a polygon whose vertices are given by the rotation vectors of (finitely many) periodic points of f, then for any compact connected $D \subset \Delta$ there is $x \in \mathbb{T}^2$ and $\tilde{x} \in \pi^{-1}(x)$ so that $\rho(F, \tilde{x}) = D$.*

The rotation set can be calculated on non-wandering. More precisely:

Proposition 1.2.6. *If $f \in Homeo_0(\mathbb{T}^2)$, then $\rho(F) = Conv(\rho(F|\pi^{-1}(\Omega(f))))$, where $\Omega(f)$ is the non-wandering set of f.*

Proof. Clearly $\rho(F) \supseteq \mathrm{Conv}(\rho(F|\pi^{-1}(\Omega(f))))$. Let v an extremal point of $\rho(F)$, then by item (i) of Theorem 1.2.2, there is an ergodic measure f-invariant μ such that $\int \varphi d\mu = v$, consequently $v \in \rho_{erg}(F) \subset \rho_p(F)$. Thus, we obtain $x \in \mathbb{R}^2$ such that $\rho_p(F, x) = v$, where $x \in \pi^{-1}(\Omega(f))$. Since

$$\rho_p(F|\pi^{-1}(\Omega(f))) \subset \rho(F|\pi^{-1}(\Omega(f))) \subset \mathrm{Conv}(\rho(F|\pi^{-1}(\Omega(f))))$$

and $\rho(F)$ is convex we get that $\rho(F) \subset \mathrm{Conv}(\rho(F|\pi^{-1}(\Omega(f))))$. $\square$

1.3 Rotation sets for continuous flows in $\mathbb{T}^d$

Let us now recall the definition of rotation vectors and rotation set for flows proposed by Franks and Misiurewicz in [20].

Let M be an n-dimensional Riemaniann closed manifold with $n \geq 2$. Let $\mathfrak{X}^0(M)$ the set of continuous vector fields $X : M \to TM$. Let $X \in \mathfrak{X}^0(\mathbb{T}^d)$, $(X_t)_t$ a flow generated by X and $(Y_t)_t$ lift of $(X_t)_t$ (cf. Appendix for more details about lift). The *rotation set* of a continuous flows $(Y_t)_t$, denoted by $\rho((Y_t)_t)$, is the set of limits of convergent sequences of

$$\left\{ \frac{Y_{t_i}(x_i) - x_i}{t_i} \right\}_{i\,=\,1}^{\infty},$$

in other words $v \in \rho((Y_t)_t)$ if and only if there are x_i and t_i with $\lim_{i \to +\infty} t_i = \infty$ such that

$$\lim_{i \to +\infty} \frac{Y_{t_i}(x_i) - x_i}{t_i} = v.$$

The *rotation vector* of $x \in \mathbb{R}^d$ is defined by

$$\rho((Y_t)_t, x) = \lim_{t \to +\infty} \frac{Y_t(x) - x}{t}, \tag{1.3.1}$$

if the limit exists. The *pointwise rotation set* of the continuous flows $(Y_t)_t$ is the set

$$\rho((Y_t)_t) = \bigcup_{x\in\mathbb{R}^d} \rho((Y_t)_t, x).$$

Remark 1.3.1. Two comments are in order. The rotation set can also be defined via time-1 maps of the flows (see Proposition 1.3.2 for more details). Moreover, if the flows $(Y_t)_t$ on $\mathbb{R}^3$ are generated by a vector field Y, the rotation vector (1.3.1) can be the computed by time average of then vector field along the orbit:

$$\rho((Y_t)_t, v) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t Y(Y_s(v))ds. \tag{1.3.2}$$

Proposition 1.3.2. *If $(X_t)_t$ is a flow generated by X and $(Y_t)_t$ is a lift of $(X_t)_t$, then $\rho((Y_t)_t) = \rho(Y_1)$.*

Proof. Let $v \in \rho((Y_t)_t)$, then there exist $t_i \in \mathbb{R}$ with $t_i \nearrow +\infty$ and $x_i \in \mathbb{R}^d$ such that

$$v = \lim_{i\to+\infty} \frac{Y_{t_i}(x_i) - x_i}{t_i},$$

Using Euclidean algorithm, one can write $t_i = n_i \cdot 1 + r_i$, with $0 \leq r_i < 1$, to get

$$\begin{aligned} v &= \lim_{i\to+\infty} \frac{Y_{t_i}(x_i) - x_i}{t_i} \\ &= \lim_{i\to+\infty} \frac{Y_{_i\cdot 1+r_i}(x_i) - x_i}{n_i\cdot 1 + r_i} \\ &= \lim_{i\to+\infty} \frac{Y_{n_i}(Y_{r_i}(x_i)) - x_i}{n_i + r_i}. \end{aligned}$$

By definition of r_i we can rewrite the limit above as follows

$$v = \lim_{i\to+\infty} \frac{Y_{n_i}(Y_{r_i}(x)) - Y_{r_i}(x)}{n_i}.$$

Taking $z_i = Y_{r_i}(x)$ and $F = Y_1$,

$$v = \lim_{i\to+\infty} \frac{F^{n_i}(z_i) - z_i}{n_i}$$

This implies that $v \in \rho(Y_1)$.

The other inclusion is immediate. Therefore, $\rho((Y_t)_t) = \rho(Y_1)$. □

In comparison with the case of homeomorphisms we highlight that: the rotation set of a flows on $\mathbb{T}^2$ is a segment or a point [20]. Also, by Proposition 1.3.2 we have that if $(Y_t)_t$ is a continuous flows on the torus $\mathbb{T}^d$ with $d \geq 2$, then the rotation set $\rho((Y_t)_t)$ is compact and connected.

1.4 Shadowing, irregular set, specification and gluing orbit properties

The concept of reconstruction of orbits in topological dynamics has gained substantial importance for its wide range of applications in ergodic theory. Among these properties it is worth mentioning the shadowing, specification and the gluing orbit properties. Throughout, let $f : X \to X$ be a continuous map on a compact metric space X.

First we recall the definition of the shadowing property. Given $\delta > 0$, we say that $(x_k)_k$ is a *δ-pseudo-orbit* for f if $d(f(x_k), x_{k+1}) < \delta$ for every $k \in \mathbb{Z}$. If there exists $N > 0$ so that $x_k = x_{k+N}$ for all $k \in \mathbb{Z}$ we say that $(x_k)_k$ is a *periodic δ-pseudo-orbit.* Given $x, y \in X$, we say that $x \sim y$ if for any $\delta > 0$ there exists a δ-pseudo-orbit $(x_i)_{n_i}$ for $i = 1, ..., k$ such that $x_1 = x$ and $f^{n_k}(x_k) = y$. It is easy to check that $\sim$ is an equivalence relation. Each of the equivalence classes of $\sim$ is called a *chain recurrence class.* Chain recurrence classes are disjoint, compact and invariant subsets of X. The set of all chain recurrent points encloses the topological complexity of the dynamics and every non-wandering point is also chain recurrent.

Definition 1.4.1. We say that f satisfies the (periodic) *shadowing property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any (periodic) δ-pseudo-orbit $(x_k)_k$ there exists $y \in X$ (periodic of f) satisfying $d(f^k(y), x_k) < \varepsilon$ for all $k \in \mathbb{Z}$.

Let $f : X \to X$ be a continuous map on compact metric space (X, d). For $x \in X$ and $n \in \mathbb{N}$, let $f^n(x)$ denotes the $n-$th iterate of x under f. That is, $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$. Let $\varphi : X \to \mathbb{R}^d$ be a continuous function. The Birkhoff average of φ, denoted by $S_n(\varphi, x)$, is defined by

$$S_n(\varphi, x) = \frac{1}{n} \sum_{i\,=\,0}^{n-1} \varphi(f^i(x)). \tag{1.4.1}$$

Define $C_\varphi(v) = \{x \in X : \lim_{n\to\infty} S_n(\varphi, x) = v\}$, as the set of points for which the Birkhoff averages converge to v, note that $X = \bigcup_{v\in\rho(F)} C_\varphi(v) \cup X_{\varphi,f}$ for F lift of f. Consider also the set $\mathcal{L}_\varphi = \{v \in \mathbb{R}^d : X_\varphi(v) \neq \emptyset\}$.

There are points $x \in X$ such that the limit $\lim_{n\to\infty} S_n(\varphi, x)$ may not exist. The set of those points for which the above limit does not exist is called the *irregular set* for φ and it is denoted by $X_{\varphi,f}$. That is,

$$X_{\varphi,f} = \left\{x \,\in\, X : \lim_{n\,\to\,\infty} S_n(\varphi, x) \ \text{ does not exists}\right\}. \tag{1.4.2}$$

Note that by Birkhoff ergodic theorem, the irregular set has zero measure with respect to any invariant measure.

The specification property, introduced by Bowen [10], roughly means that an arbitrary number of pieces of orbits can be "glued together" to obtain a real orbit that shadows the previous ones with a prefixed number of iterates in between. Moreover, it configures itself as an indicator of chaotic behavior (e.g. it implies the dynamics has positive topological entropy).

Definition 1.4.2. We say that f satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon) \geq 1$ so that for any points $x_1, x_2, \ldots, x_k \in X$ and for any positive integers $n_1, \ldots, n_k$ and $0 \leq p_1, \ldots, p_{k-1}$ with $p_i \geq m(\varepsilon)$ there exists a point $y \in X$ such that $d(f^j(y), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d(f^{j+n_1+p_1+\ \cdots\ +n_{i-1}+p_{i-1}}(y), f^j(x_i)) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Finally, the gluing orbit property, introduced in [7], bridges between completely non-hyperbolic dynamics (equicontinuous and minimal dynamics [8, 49]) and uniformly hyperbolic dynamics (see e.g. [7]). Both of these properties imply on a rich structure on the dynamics and the space of invariant measures (see e.g. [14, 8]).

Definition 1.4.3. We say that f satisfies the *gluing orbit property* if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon) \geq 1$ so that for any points $x_1, x_2, \ldots, x_k \in X$ and any positive integers $n_1, \ldots, n_k$ there are $0 \leq p_1, \ldots, p_{k-1} \leq m(\varepsilon)$ and a point $y \in X$ so that $d(f^j(y), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(y), f^j(x_i)) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. If, in addition, $y \in X$ can be chosen periodic with period $\sum_{i=1}^{k}(n_i + p_i)$ for some $0 \leq p_k \leq m(\varepsilon)$ then we say that f satisfies the *periodic gluing orbit property*.

We say that f satisfies *2-gluing orbit property* if f satisfies the gluing orbit property for $k = 2$ in the definition above.

It is not hard to check that irrational rotations satisfies the gluing orbit property [8], but fail to satisfy the shadowing or specification properties. Partially hyperbolic examples exhibiting the same kind of behavior have been constructed in [9]. Also notice that the gluing orbit property is clearly a topological invariant.

Remark 1.4.4. It is clear from the definitions that the specification property implies the gluing orbit property, which implies transitivity. For continuous dynamics on the interval gluing orbit property and transitivity are equivalent, cf. dissertation [1]. It will be useful to consider the (periodic) gluing orbit property on compact invariant subsets Γ, in which case

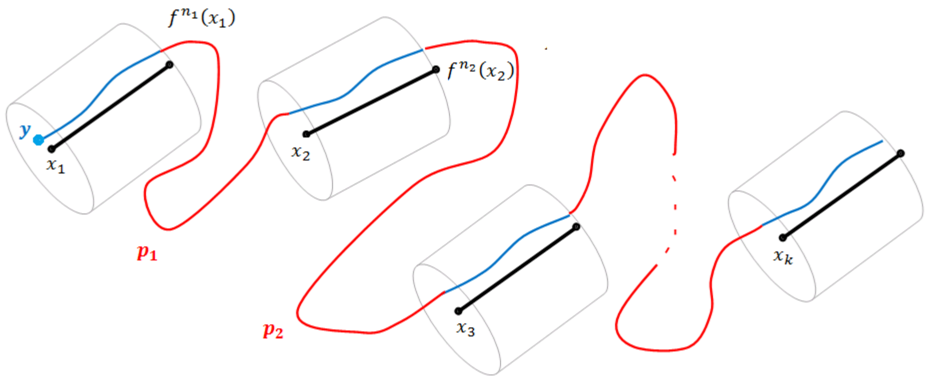

Figure 1.4.1: An illustration of the gluing orbit property.

we demand only Definition 1.4.3 to hold for every small ε but we require the shadowing point z to belong to Γ.

Proposition 1.4.5. *Let $f : X \to X$ a continuous function with shadowing property. f satisfies 2-gluing orbit property if and only if f has the gluing orbit property.*

Proof. Suppose that f satisfies 2-gluing orbit property, we will show that f satisfies gluing orbit property. Let $\varepsilon > 0$, let $x_1, x_2, \dots, x_k \in X$ and positive integers $n_1, n_2, \dots, n_k$. Let ε, $m(\delta)$ where δ is of the shadowing property associated to $\frac{\epsilon}{2}$, x_1, x_2, n_1 and n_2 by 2-gluing orbit property there exists $y_{1,2}$ so that

$$d_{n_1}(y_{1,2}, x_1) \leq \frac{\varepsilon}{2} \quad \text{and } d_{n_2}(f^{n_1+p_1}(y_{1,2}), x_2) \leq \frac{\varepsilon}{2}.$$

Analogous for x_2, x_3, n_2 and n_3 there exists $y_{2,3}$ so that

$$d_{n_2}(y_{2,3}, x_2) \leq \frac{\varepsilon}{2} \quad \text{and } d_{n_3}(f^{n_2+p_2}(y_{2,3}), x_3) \leq \frac{\varepsilon}{2}.$$

In this way until integer k.

Note that $d_{n_2}(f^{n_1+p_1}(y_{1,2}), y_{2,3}) < \epsilon$. We can consider the following δ-pseudo orbit (δ of the shadowing property):

$$\begin{gathered} y_{1,2}, f(y_{1,2}), \dots, f^{n_1}(y_{1,2}), f^{n_1+1}(y_{1,2}), \dots, f^{n_1+p_1-1}(y_{1,2}), \\ y_{2,3}, f(y_{2,3}), \dots, f^{n_2}(y_{2,3}), f^{n_2+1}(y_{2,3}), \dots, f^{n_2+p_2-1}(y_{2,3}), \\ \dots, \\ y_{k-1,k}, f(y_{k-1,k}), \dots, f^{n_{k-1}}(y_{k-1,k}), f^{n_k+1}(y_{k-1,k}), \dots, f^{n_{k-1}+p_2}(y_{k-1,k}), \\ f^{n_{k-1}+p_2+1}(y_{k-1,k}), \dots, f^{n_{k-1}+p_2+n_k-1}(y_{k-1,k}). \end{gathered}$$

By property of the shadowing there exist a point $y \in X$ satisfying $d(f^k(y), x_k) < \varepsilon$. Take $p_1, \dots, p_k$ and y as above we have the desired. $\square$

1.5 Reparametrization gluing orbit properties for continuous flow

In this Section we define a notion of gluing orbit property for continuous flows and reparametrized gluing orbit property for continuous flows, wich is weaker than specification. The notion of reparametrized gluing orbit property was introduced in [6]. First we define gluing orbit property for continuous flow.

Definition 1.5.1. Let $X \in \mathfrak{X}^0(M)$, $(X_t)_t$ a continuous flow generated by X we say that $(X_t)_t$ satisfies gluing orbit property if for any $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{R}_+$ such that for any points $x_1, x_2, \dots, x_k$ in M and times $t_1, \dots, t_k \geq 0$ there are $p_1, \dots, p_{k-1} \leq K(\epsilon)$ and a point $y \in M$ so that

$$d(X_t(y), X_t(x_1)) < \epsilon, \quad \forall\ t \in [0, t_1]$$

and

$$d(X_{t+\sum_{j=0}^{i-1} t_j + p_j}(y), X_t(x_1)) < \epsilon, \quad \forall\ t \in [0, t_i]$$

for every $2 \leq i \leq k$.

Denote by *Rep* the set of all increasing homeomorphisms $\tau : \mathbb{R} \to \mathbb{R}$ satisfying $\tau(0) = 0$. τ is called reparametrization. Fixing $\varepsilon > 0$, we define the set

$$Rep(\varepsilon) = \{\tau \in Rep : \mid \frac{\tau(s) - \tau(t)}{s - t} - 1 \mid < \varepsilon, \ \ s, t \in \mathbb{R}\}$$

of the reparametrizations ε-close to the identity. It is, the reparametrization τ belongs to $Rep(\varepsilon)$ whenever the slopes formed by any two points in its graph belong to the interval $(1 - \varepsilon, 1 + \varepsilon)$.

Now we define the reparametrized gluing orbit property for flow.

Let $X \in \mathfrak{X}(M)$, $(X_t)_t$ a continuous flow generated by X. We say that $(X_t)_t$ satisfies reparametrized gluing orbit property if for any $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{R}_+$ such that for any points $x_1, \dots, x_k$ in M and times $t_1, \dots, t_k \geq 0$, there are $p_1, \dots, p_{k-1} \leq K(\epsilon)$, a reparametrization $\tau \in Rep(\epsilon)$ and a point $y \in M$ so that

$$d(X_{\tau(t)}(y), X_t(x_1)) < \epsilon, \quad \forall\ t \in [0, t_1]$$

and

$$d(X_{\tau(t+\sum_{j=0}^{i-1} t_j + p_j)}(y), X_t(x_1)) < \epsilon, \quad \forall\ t \in [0, t_i]$$

for every $2 \leq i \leq k$.

This notion is an extension of the original notion of gluing introduced by Bomfim and Varandas in [7].

Note that if $\tau \in Rep(\varepsilon)$ and p_1 is as above on the definition of gluing orbit property, then $\tau(t + p_1) - \tau(t) \leq (1 + \epsilon)p_1 \leq (1 + \epsilon)K(\varepsilon)$.

1.6 Pressure, entropy and mean dimensions

In this section we recall two important measurements of topological complexity, namely the concepts of topological entropy and metric mean dimension, and introduce a relative notion of the later. Our interest in the second notion is that, while a dense set of homeomorphisms on a compact Riemannian manifold have positive and finite topological entropy (by denseness of C^1-diffeomorphisms) it is known that typical homeomorphisms may have infinite topological entropy. In opposition, metric mean dimension is always bounded by the dimension of the compact manifold and can be seen as a smoothened measurement of topological complexity as we now detail.

Topological pressure

Let (X, d) be a compact metric space and $\psi \in C^0(X, \mathbb{R})$. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we say that $E \subset X$ is (n, ε)-separated if for every $x \neq y \in E$ it holds that $d_n(x, y) > \varepsilon$, where $d_n(x, y) = \max\{d(f^j(x), f^j(y)); j = 0, \ldots, n-1\}$ is the Bowen's distance. Bowen's dynamical balls are the sets $B_n(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}$. The *topological pressure* of f with respect to ψ is defined by

$$P_{top}(f, \psi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_E \sum_{x \in E} e^{S_n \psi(x)} \right),$$

where $S_n \psi(x) = \sum_{j=0}^{n-1} \psi(f^j(x))$ and the supremum is taken over every (n, ε)-separated sets E contained in X. In the case that $\psi \equiv 0$, if $s(n, \varepsilon)$ denotes the maximal cardinality of a (n, ε)-separated subset of X, then the *topological entropy* is defined by

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon).$$

The previous notion does not depend on the metric d and is a topological invariant. Moreover, by the classical variational principle for the pressure, it holds that $P_{top}(f, \psi) = \sup\{h_\mu(f) + \int \psi d\mu : \mu \in \mathcal{M}(f)\}$. However, the topological entropy of C^0-generic homeomorphisms on a closed manifold of dimension at least two is infinite [53] (the same holds for the topological pressure as a consequence of the variational principle), in which case neither the topological preassure nor topological entropy can distinguish such dynamics.

Topological and metric mean dimension

Gromov [21] proposed an invariant for dynamical systems called *mean dimension*, that was further studied by Lindenstrauss and Weiss [29]. The upper and lower *metric mean* *dimension*, which may depend on the metric, are defined in [28, 29] by

$$\overline{\mathrm{mdim}}(f) = \lim_{\varepsilon\to 0} \frac{\overline{\lim}_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon)}{-\log\varepsilon}$$

and

$$\underline{\mathrm{mdim}}(f) = \lim_{\varepsilon\to 0} \frac{\underline{\lim}_{n\to\infty}\frac{1}{n}\log s(n,\varepsilon)}{-\log\varepsilon},$$

respectively. If the supremum and infimum limits agree, we denote the common value by $\mathrm{mdim}(f)$. Observe that the latter quantitities are only meaningful whenever f has infinite topological entropy. In the case that the metric space satisfies a tame growth of covering numbers, the metric mean dimension satisfies a variational principle involving a concept of measure theoretical mean dimension (cf. [28]).

Relative metric mean dimension

Since we aim to describe the topological complexity of (not necessarily compact) f-invariant subsets we now introduce a concept of relative metric mean dimension using a Carathéodory structure. Let $Z \subset X$ be a f-invariant Borel set. Given $s \in \mathbb{R}$ and $\psi \in C^0(X,\mathbb{R})$ define

$$Q(Z,\psi,s,\Gamma) = \sum_{B_{n_i}(x_i,\varepsilon)\in\Gamma} e^{-s\,n_i + S_{n_i}\psi(B_{n_i}(x_i,\varepsilon))} \text{ and } M(Z,\psi,s,\varepsilon,N) = \inf_{\Gamma}\{Q(Z,\psi,s,\Gamma)\},$$

where $S_{n_i}\psi(B_{n_i}(x_i,\varepsilon)) := \sup_{x\in B_{n_i}(x_i,\varepsilon)} \sum_{k=0}^{n_i-1}\psi(f^k(x)))$ and the infimum is taken over all countable collections $\Gamma = \{B_{n_i}(x_i,\varepsilon)\}_i$ that cover Z and so that $n_i \geq N$. Since the function $M(Z,\psi,s,\varepsilon,N)$ is non-decreasing in N the limit $m(Z,\psi,s,\varepsilon) = \lim_{N\to\infty} M(Z,\psi,s,\varepsilon,N)$ does exist. Then let

$$P_Z(f,\psi,\varepsilon) = \inf\{s \in \mathbb{R} \colon m(Z,\psi,s,\varepsilon) = 0\} = \sup\{s \in \mathbb{R} \colon m(Z,\psi,s,\varepsilon) = \infty\}.$$

The existence of $P_Z(f,\psi,\varepsilon)$ follows by the Carathéodory structure [40]. The (relative) *topological pressure* of f on Z with respect to ψ is defined by

$$P_Z(f,\psi) = \lim_{\varepsilon\to 0} P_Z(f,\psi,\varepsilon).$$

We set $h_Z(f,\varepsilon) = P_Z(f,0,\varepsilon)$ for every $\varepsilon > 0$ and define the *relative entropy* of f on Z by $h_Z(f) = P_Z(f,0)$ (which corresponds to the potential $\psi \equiv 0$).

The upper and lower *relative metric mean dimension* of Z are given by

$$\overline{\mathrm{mdim}}_Z(f) = \overline{\lim}_{\varepsilon\to 0}\frac{h_Z(f,\varepsilon)}{-\log\varepsilon} \quad\text{and}\quad \underline{\mathrm{mdim}}_Z(f) = \underline{\lim}_{\varepsilon\to 0}\frac{h_Z(f,\varepsilon)}{-\log\varepsilon}$$

respectively. If the previous limits coincide, we represent simply by $\mathrm{mdim}_Z(f)$ and refer to this as the relative metric mean dimension of Z.

Definition 1.6.1. We say that the f-invariant subset $Z \subset X$ has full topological entropy if $h_Z(f) = h_{\text{top}}(f)$. We say that the f-invariant subset $Z \subset X$ has full metric mean dimension if $\underline{mdim}_Z(f) = \underline{mdim}(f)$ and $\overline{mdim}_Z(f) = \overline{mdim}(f)$.

Remark 1.6.2. If $f : X \to X$ is a continuous map on a compact metric space and $\psi \in C^0(X, \mathbb{R})$ then $P_{top}(f, \psi) = P_X(f, \psi)$. Moreover, if the limits exist and coincide then $\text{mdim}_X(f) = \text{mdim }(f)$. This follows from the fact that $h_X(f, \varepsilon) = h_{top}(f, \varepsilon)$ for any $\varepsilon > 0$, which can be read from the proof of [41, Proposition 4] (actually in [41] the authors use the definition of entropy using coverings and prove that $h_X(f, \mathcal{U}) = h_{top}(f, \mathcal{U})$ for every open cover $\mathcal{U}$).

Remark 1.6.3. The notion of Hausdorff dimension also involves a Carathéodory structure, associated to the function $Q(Z, s, \Gamma) = \sum_{B_{n_i}(x_i,\varepsilon)\in\Gamma} \text{diam}(B_{n_i}(x_i, \varepsilon))^s$ (see [40, Section 6]). Inspired by [4] we expect that for continuous and transitive maps on the interval (these satisfy the gluing orbit property, [1]) the set of points with historic behavior is either empty or has Hausdorff dimension equal to one. We neither claim nor prove this fact here.

We use the following generalization of Katok's formula for pressure:

Proposition 1.6.4. *[52, Proposition 2.5] Let (X, d) be a compact metric space, f be a continuous map on X and μ be an f-invariant, ergodic probability. Given $\varepsilon > 0$, $\gamma \in (0, 1)$ and $\psi \in C^0(X, \mathbb{R})$ set $N^\mu(\psi, \gamma, \varepsilon, n) = \inf_E \sum_{x\in E} \exp\big\{ \sum_{i=0}^{n-1} \psi(f^i(x))\big\}$, where the infimum is taken over all sets E that (n, ε)-span a set Z with $\mu(Z) \geq 1 - \gamma$. Then*

$$h_\mu(f) + \int \psi d\mu = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N^\mu(\psi, \gamma, \varepsilon, n).$$

Remark 1.6.5. Given $\varepsilon > 0$ and $\varphi \in C^0(X, \mathbb{R}^d)$, the variation in balls of radius ε is

$$\text{var}(\varphi, \varepsilon) = \sup\{| \varphi(x) - \varphi(y) |: d(x, y) < \varepsilon\}.$$

Since X is compact then $\text{var}(\varphi, \varepsilon) \to 0$ as $\varepsilon \to 0$. As $\varphi : X \to \mathbb{R}^d$ is continuous (hence uniformly continuous) and f is continuous then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(y))\| < \varepsilon$ whenever $d_n(x, y) < \delta$.

Chapter 2

Statement of the main results

In this chapter we expose our results. Before, recall that we denote by $\text{Homeo}_0(X)$ the space of homeomorphisms on X homotopic to the identity and $\text{Homeo}_{0,\lambda}(X)$ the space of homeomorphisms on X homotopic to the identity that area-preserving.

2.0.1 Points with historic behavior for maps with gluing orbit property

The results in this section, despite their own interest, will be key technical ingredients in the characterization of rotation sets for homeomorphisms on tori. These applications motivate to describe the set of points with historic behavior for observables taking values on $\mathbb{R}^d$, $d \geq 1$, and dynamical systems with the gluing orbit property (see Subsection 1.4 for the definition).

Let X denote a compact metric space, $f : X \to X$ be a continuous map, $d \geq 1$ be an integer and $\varphi : X \to \mathbb{R}^d$ be a continuous observable. Given $x \in X$, let us denote by $\mathcal{V}_\varphi(x)$ the (connected) set obtained as accumulation points of $(\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)))_{n\geq 1}$. In the higher dimensional setting context, $(d > 1)$ the set $\mathcal{V}_\varphi = \bigcup_{x\in X}\mathcal{V}_\varphi(x) \subset \mathbb{R}^d$ of all vectors obtained as pointwise limits of Birkhoff averages does not need connected or convex.

A point $x \in X$ has *historic behavior* for φ (also known as *exceptional*, *irregular* or *non-typical* behavior) if the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))$ does not exist. Moreover, we say that $x \in X$ has *wild historic behavior* if $\mathcal{V}_\varphi(x)$ and $\mathcal{V}_\varphi(x) = \mathcal{V}_\varphi$ and we denote $X_{\varphi,f}^{wild} := \{x \in X : \mathcal{V}_\varphi(x) = \mathcal{V}_\varphi\}$. In rough terms, a point has wild historic behavior if the Birkhoff averages have the largest oscillation in $\mathcal{V}_\varphi$. We say that $B \subset X$ is *Baire residual* if it contains a countable intersection of open and dense subsets of X. Our first result asserts that, if non-empty, the set of points with wild historic behavior is large from the category point of view.

Theorem A. *Let X be a compact metric space, let $f : X \to X$ be a continuous map with the gluing orbit property and let $\varphi : X \to \mathbb{R}^d$ be continuous. Then:*

1. *either there is $v \in \mathbb{R}^d$ so that $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = v$ for all $x \in X$,*
2. *or the set of points $x \in X$ so that the sequence $(\frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)))_{n\geq 1}$ accumulates in a non-trivial connected subset of $\mathbb{R}^d$ is Baire residual on X.*

Moreover, if $X_{\varphi,f} \neq \emptyset$ then $\mathcal{V}_\varphi$ is connected and the set of points with wild historic behavior is a Baire residual subset of X.

The next result establishes that the set of points with historic behavior has also large complexity, now measured in terms of topological entropy and metric mean dimension. We refer the reader to Subsection 1.6 for the notions of full topological pressure and full metric mean dimension.

Theorem B. *Let $f : X \to X$ be a continuous map with the gluing orbit property on compact metric space X and let $\varphi : X \to \mathbb{R}^d$ be a continuous observable. Assume that $X_{\varphi,f} \neq \emptyset$. Then $X_{\varphi,f}$ carries full topological pressure and full metric mean dimension.*

Under the previous assumptions, the set of points with historic behavior for φ is empty if and only if there exists $v \in \mathbb{R}^d$ so that $\int \varphi \, d\mu = v$ for every f-invariant probability measure (cf. Lemma 3.2.1). We also prove the following:

Corollary A. *Let X be a compact Riemannian manifold. There exists a C^0-Baire residual subset $\mathfrak{R} \subset Homeo(X) \times C^0(X, \mathbb{R}^d)$ so that either $h_{\text{top}}(f) = 0$ or $X_{\varphi,f}$ has full topological entropy for any pair $(f, \varphi) \in \mathfrak{R}$.*

2.0.2 Pointwise rotation sets of homeomorphisms on the torus $\mathbb{T}^2$

In this section we address the questions concerning the pointwise rotation sets of torus homeomorphisms homotopic to the identity (check Section 1.2 for more details about rotation set). We note that the pointwise rotation set may fail to be connected and all (see e.g. [30, Example 1]). Our first results ensure that there is a large set of points with non-trivial pointwise rotation set of x. First we consider in the case of volume preserving homeomorphisms.

Theorem C. *There exists a Baire residual subset $\mathfrak{R}_1 \subset Homeo_{0,\lambda}(\mathbb{T}^2)$ so that, for every $f \in \mathfrak{R}_1$ and every lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f:*

1. *the pointwise rotation set $\rho_p(F)$ is connected;*

2. *the set of points $x \in \mathbb{T}^2$ such that $\rho(F, \widetilde{x})$ is non-trivial (ie, $\rho(F, x) \neq \{v\}$ for some v) which coincide with $\rho_p(F)$ is a Baire residual subset of $\mathbb{T}^2$, it carries full topological pressure and full metric mean dimension in $\mathbb{T}^2$.*

Now we describe the counterpart of Theorem C on the space $\mathrm{Homeo}_0(\mathbb{T}^2)$ of homeomorphisms homotopic to the identity. Consider the set

$$\mathcal{A} = \left\{ f \in \mathrm{Homeo}_0(\mathbb{T}^2) : \mathrm{int}(\rho(F)) \neq \emptyset \right\}.$$

All homeomorphisms in $\mathcal{A}$ have positive topological entropy [30]. Let $\Omega(f)$ denote as usual the non-wandering set of f and consider the finite decomposition of $\Omega(f)$ in its chain recurrent classes. We prove the following:

Theorem D. *There exists a Baire residual subset $\mathfrak{R}_2 \subset \mathcal{A}$ so that, for every $f \in \mathfrak{R}_2$ there exists a positive entropy chain recurrent class $\Gamma \subset \Omega(f)$ such that the set of points $x \in \Gamma$ for which $\rho(F, \widetilde{x})$ is non-trivial is a Baire residual subset of Γ that carries full topological entropy and full metric mean dimension in Γ.*

We observe that the Theorem C is due to specification, but in case dissipative, the Theorem D, specification is not sufficient, thus we need of the concept of gluing orbit property.

2.0.3 On the rotation set of homeomorphisms on the torus $\mathbb{T}^d$ ($d \geq 2$)

The shape of the different rotation sets for an homeomorphism f homotopic to identity on the torus $\mathbb{T}^d$ have drawn the attention since these have been introduced (see Subsection 1.2 for definitions). Focusing first on connectedness, the rotation set $\rho(F)$ (and each pointwise rotation set of x, denoted by $\rho(F, \widetilde{x})$, where $\widetilde{x} \in \pi^{-1}(x)$) is a compact and connected set in $\mathbb{R}^d$ [30, 35]. However, the pointwise rotation set $\rho_p(F)$ may fail to be connected even when $d = 2$ [30]. As for convexity, $\rho(F)$ is convex when $d = 2$, but there are higher dimensional examples where it fails to be convex [35]. Our next result ensures that rotation sets of torus homeomorphisms are typically convex.

Theorem E. *For every $d \geq 2$:*

1. *there exists a Baire residual subset $\mathfrak{R}_3 \subset Homeo_{0,\lambda}(\mathbb{T}^d)$ so that $\rho(F)$ is convex, for every lift F of a homeomorphism $f \in \mathfrak{R}_3$; and*

2. *there exists a Baire residual subset $\mathfrak{R}_4 \subset Homeo_0(\mathbb{T}^d)$ so that $\rho(F\mid_{\pi^{-1}(\Gamma)})$ is convex, for every chain recurrent class $\Gamma \subset \Omega(f)$ and every lift F of $f \in \mathfrak{R}_4$.*

While the rotation set is always connected, in the case of dissipative homeomorphisms $\mathrm{Homeo}_0(\mathbb{T}^d)$ (e.g. Morse-Smale diffeomorphisms on the torus) the pointwise rotation set need not to be connected. If the pointwise rotation set is connected then one can hope that the "local" convexity statement in item (2) can be used to prove the convexity of the rotation set.

2.0.4 Points with historic behavior for a flows with reparametrized gluing orbit property

Our next results describe the set of points with historic behavior for continuous flows with the reparametrized gluing orbit property. Before some definitions.

Let M be an n-dimensional Riemaniann closed manifold with $n \geq 2$. Let $L > 0$, denote $\mathfrak{X}^0(M)$ the set of continuous vector fields $X : M \to TM$ and $\mathfrak{X}^{0,1}_L(M)$ the set of Lipschitz continuous vector fields $X : M \to TM$ with Lipschitz constant $\leq L$. We endow $\mathfrak{X}^0(M)$ and $\mathfrak{X}^{0,1}_L(M)$ with the C^0-topology, ie, given $X, Y \in \mathfrak{X}^{0,1}_L(M)$, X is ϵ-close Y if $\max_{x\in M} \|X(x) - X(y)\| < \epsilon$. We denote by $(X_t)_t$ the flow associated to $X \in \mathfrak{X}^{0,1}_L(M)$.

Let M a compact metric space, $X \in \mathfrak{X}^0(M)$, $(X_t)_t$ a flow generated by X and $\varphi : M \to \mathbb{R}^d$ continuous. Define the *irregular set* for $(X_t)_t$ by

$$I_\varphi = \Big\{ x \in M : \lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi \circ X_s(x)\, ds \text{ does not exist } \Big\}. \tag{2.0.1}$$

Recall that $A \subset M$ is $(X_t)_t$-invariant if $X_t(A) = A$, for all $t \in \mathbb{R}$ and that a finite measure μ on M is $(X_t)_t$-invariant if $\mu(X_t(A) = A$, for every measurable $A \subset M$ and $t \in \mathbb{R}$. Also note that I_φ has zero measure with respect to any Φ-invariant measure.

A point $x \in M$ has *historic behavior* for φ (on the flow) if the limit $\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi \circ X_s(x) ds$ does not exist.

For $d \geq 2$, define

$$\mathcal{L}_\varphi = \{\vec{v} \in \mathbb{R}^d : A_\varphi(\vec{v}) \neq \emptyset\}$$

where

$$A_\varphi(\vec{v}) := \{x \in M : \lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi \circ X_s(x)\, ds = \vec{v}\}.$$

We prove, for continuous flow with reparametrized gluing orbit property (check Subsection 1.5 for definition) that:

Theorem F. *Let $Y \in \mathfrak{X}^0(M)$. If $(Y_t)_t$ is a flow generated by Y with the reparametrized gluing orbit property on Δ compact and invariant and $\varphi : M \to \mathbb{R}^d$ is continuous, then* *the irregular set of φ either trivial or Baire residual subset on Δ. In other word, if the irregular set of φ is non-trivial, then the set of points with historic behavior is a Baire residual subset of Δ.*

As consequence we have the following:

Corollary B. *Let $X \in \mathfrak{X}^0(M)$. If $(X_t)_t$ is a flow generated by X with the reparametrized gluing orbit property on Γ where Γ is a chain recurrent class on $\Omega((X_t)_t)$ and $\varphi : M \to \mathbb{R}^d$ is continuous, then the irregular set of φ either trivial or Baire residual subset on Γ.*

In [6] is proved that: If M be a compact manifold there exists a C^0-residual subset of vector fields X in $\mathfrak{X}^0(M)$ so that every such vector field X has reparametrized gluing orbit property in chain recurrent class $\Gamma \subset \Omega((X_t)_t)$. This ensure the following.

Corollary C. *Let M be a compact manifold. There exists a C^0-residual subset of vector fields X in $\mathfrak{X}^0(\mathbb{T}^2)$ so that if $(X_t)_t$ is a generated flows by X, $\varphi : M \to \mathbb{R}^d$ continuous and $\Gamma \subset \Omega((X_t)_t)$ is a chain recurrent class, then the irregular set I_φ is either trivial or Baire residual subset of Γ.*

2.0.5 Points with historic behavior for a suspension flow and rotation set

Let M be a measurable space and $f : M \to M$ a measurable map and a measurable roof function $r : \Sigma \longrightarrow [0, \infty)$. We can define the *suspension flows* $(X_t)_t$ over f by

$$X_t(x, s) = (x, s + t),$$

whenever the expression is well defined, acting on

$$M_r = \{(x, t) \in X \times \mathbb{R}^+ : 0 \leq t \leq r(x)\}/\sim$$

where $\sim$ is the equivalence relation given by $(x, r(x)) \sim (f(x), 0)$ for all $x \in \Sigma$. Note that the coordinates of $(X_t)_t$ coincides with the flow along the vertical direction. More precisely,

$$X_t(x, s) = \Big(f^k(x), s + t - \sum_{j=0}^{k-1} r(f^j(x))\Big),$$

where $k = k(x, s, t)$ is determined by $\sum_{j=0}^{k-1} r(f^j(x)) \leq s + t < \sum_{j=0}^{k} r(f^j(x))$.

Our next results describe the set of points with historic behavior for suspension flows over a map with gluing orbit property. First, consider the following number:

$$C_\xi = \sup_{n\geq 1} \sup_{y\in B(x,n,\xi)} \|S_nR(x) - S_nR(y)\| < \infty, \quad \text{satisfy } \lim_{\xi\to 0} C_\xi = 0, \tag{2.0.2}$$

The condition under C_ξ is a bounded distortion property for the roof function. It is not hard to check it holds e.g. for Hölder continuous observables and uniformly expanding dynamics. So, we have:

Theorem G. *Let M be an n-dimensional Riemaniann closed manifold, $f \in$ Homeo(M) with gluing orbit property and $\varphi : M_r \to \mathbb{R}^d$ continuous. Suppose that $(X_t)_t$ is a suspension flow over f satisfying condition (2.0.2). Then, the irregular set of φ either trivial or Baire residual subset on M. In other word, if I_φ is non-trivial, then the set of points with historic behavior is a Baire residual.*

Now, consider the suspension flows when $f \in \text{Homeo}_0(\mathbb{T}^2)$. Since that f admits a lift F we can then define a suspension flow $(Y_t)_t$ over F so that $(Y_t)_t$ is a lift of $(X_t)_t$. The roof function in $(Y_t)_t$ is $R : \mathbb{R}^2 \longrightarrow (0, \infty)$, defined by $R(v) = r(\pi(v))$. In this context, we define the suspension flow to $(Y_t)_t$ acting on

$$(x, t) \in \widetilde{M_r} := \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ : 0 \leq t \leq R(x)\}/\sim,$$

where $R : \mathbb{R}^2 \longrightarrow (0, \infty)$ is the roof function and $\sim$ is the equivalence relation given by $(x, R(x)) \sim (F(x), 0)$.

Passeggi in [39] proved that there exists an open and dense set $\mathcal{D} \subset \text{Homeo}_0(\mathbb{T}^2)$ such that for every $f \in \mathcal{D}$ the rotation set $\rho(F)$ is a polygon with finite rational vertices. So, we have the following:

Theorem H. *Let $f \in$ Homeo$(\mathbb{T}^2)$. Suppose that $f \in \mathcal{D}$, that $(X_t)_t$ is a suspension flow over f and the roof function r is coboundary, then the rotation set $\rho((Y_t)_t)$ is a polygon rational, (where $(Y_t)_t$ is a lift of $(X_t)_t$). Moreover, if r is cohomologous to a rational number, then the polygon has rational vertices.*

2.0.6 Overview of the proof

Theorems A and B provide three distinct measurements of the topological complexity of the set of points with historic behavior. Their proofs use the construction of points with non-convergent Birkhoff averages by exploring the oscillatory behavior in the Birkhoff averages of points that shadow pieces of orbits that are typical for invariant measures with different space averages. The existence of such points is granted by the gluing orbit property.

If, on the one hand, the proof of Theorems A and B are inspired by [4, 25, 52], the arguments in the proof of Theorem B is much more challenging and presents novelties on how to construct a 'large amount' of points whose finite pieces of orbits up to time n have a controlled behavior and that are separated by the dynamics. This is crucial to estimate topological pressure and metric mean dimension. While the construction of points with non-convergent behavior can be obtained as a consequence of the gluing orbit property, it is natural to inquire on the control on the number of such distinct orbits (measured in terms of (n, ε)-separability). We overcome this issue by selecting of a large amount of orbits that are glued the same (bounded) time. Since this bound depends on ε, so does the estimates on the number of (n, ε)-separated points with controlled recurrence. This requires shadowing times to be chosen large in order to compensate the latter. In [15] the authors obtain similar flavored results using shadowing. Although both occur properties hold C^0-generically there are several examples that satisfy the gluing orbit property and fail to satisfy shadowing, which justifies our approach.

The first ingredient in the proof of Theorems C and D relies on the fact that each chain recurrent class of C^0-generic homeomorphisms satisfy the gluing orbit property. This will ensure that any connected subset of the rotation set can be realized by the rotation set along the orbit of a point relies on any vectors. Such a reconstruction of rotation vectors as the orbit of a single point is formalized in Theorems A and B. In comparison with the former, extra difficulties arise from the fact that the dynamics and the observables are not decoupled and the fact that, in the case of dissipative homeomorphisms, the chain recurrent classe(s) that concentrate topological pressure vary as the potential changes. One could ask whether the Baire generic conclusion of Theorem D could extend to a generic set of points in the whole chain-recurrent set (or the non-wandering set). For instance, it is easy to construct an Axiom A diffeomorphism f on $\mathbb{S}^2$ so that $\Omega(f) = \{p_1\} \cup \Lambda \cup \{p_2\}$, where p_1 is a repelling fixed point, p_2 is an attracting fixed point and Λ is an horseshoe. Recall that f is an Axiom A diffeomorphism if the set of periodic points is dense in the non-wandering set $\Omega(f)$ and $\Omega(f)$ is hyperbolic (we refer, for instance, to [47] for the construction of such examples). The existence of a filtration for homeomorphisms C^0-close to f imply that the Baire generic subset in the statement of Theorem D can only be contained in a neighborhood of the basic piece Λ for all C^0-close homeomorphisms. Moreover, the assertion concerning positive entropy seems optimal. Indeed, it may occur that there exists a unique chain recurrent class of largest positive topological entropy and whose (restricted) rotation set have empty interior or even reduce to a point, in the case of pseudo-rotations. Related constructions include [17, 45].

Theorems E relies on the fact that under the specification, or the gluing orbit property, the space of periodic measures is dense in the space of all invariant measures. Under any of these assumptions, the generalized rotation set coincides with the rotation set obtained by means of invariant measures, thus it is convex.

The Theorem F and G are an attempt to generalize from Theorem A, the strategy to proof of the F is also similar. Finally, the Theorem H is a relationship between suspension flow and rotation set.

Chapter 3

The set of points with historic behavior

The main goal of this chapter is to prove Theorems A and B, which claim that the set of points with historic behavior for continuous maps with the gluing orbit property is topologically large. Actually, this is established by means of three different measurements of topological complexity: Baire genericity, full topological entropy and full metric mean dimension. The arguments involved in the proofs of Theorems A and B are substantially different and their proofs occupy Sections 3.1 and 3.2, respectively.

3.1 Baire genericity of historic behavior and proof of the Theorem A

This section is devoted to the proof of Theorem A, whose strategy is strongly inspired by [2, 25]. The differences lie on the fact that, due to the higher dimensional features of observables, we need to restrict to connected subsets in the set of all accumulation vectors, and that we have transition time functions instead of a determined time to shadow pieces of orbits (see Remark 3.1.3 below).

Let $f : X \to X$ be a continuous map with the gluing orbit property on a compact metric space X and let $\varphi : X \to \mathbb{R}^d$ be a continuous function so that $\mathcal{V}_\varphi$ is non-trivial. Let $\Delta \subset \mathcal{V}_\varphi$ a non-trivial connected set, define

$$X_\Delta := \Big\{ x \in X : \Delta \subset \mathcal{V}_\varphi(x) \Big\}. \tag{3.1.1}$$

Note that for all $\Delta \subset \mathcal{V}_\varphi$ hold that $X_\Delta \subset X_{\varphi,f}$. Theorem A will be a consequence of the following:

Proposition 3.1.1. *Let X be a compact metric space, $f \in Homeo(X)$ satisfy the gluing orbit property, $\varphi : X \to \mathbb{R}^d$ be continuous such that $\mathcal{V}_\varphi$ is non-trivial. If $\Delta \subset \mathcal{V}_\varphi$ is a non-trivial connected set, then X_Δ is Baire residual in X.*

Remark 3.1.2. The set $\mathcal{V}_\varphi$ corresponds to the pointwise rotation set of φ, which needs not be connected in general. Since Baire residual subsets are preserved by finite intersection, a simple argument by contradiction ensures that under the assumptions of Proposition 3.1.1 the set $\mathcal{V}_\varphi$ is connected. In particular, since $\mathcal{V}_\varphi(x) \subset \mathcal{V}_\varphi$, the latter ensures that $X^{wild}_{\varphi,f} := \{x \in X \colon \mathcal{V}_\varphi(x) = \mathcal{V}_\varphi\}$. is Baire residual.

[Proof of the Proposition 3.1.1]: The remaining of the section is devoted to the proof of Proposition 3.1.1. Let $D \subset X$ be countable and dense. Let $\varepsilon > 0$ be arbitrary and fixed and let $m(\varepsilon)$ be given by the gluing orbit property (cf. Section 1.4). As $\mathcal{V}_\varphi$ is non-trivial, then $\mathcal{V}_\varphi$ is not a singleton. Let $\Delta \subset \mathcal{V}_\varphi$ be a non-trivial connected set and, for any $k \geq 1$ let $(v_{k,i})_{1\leq i\leq a_k}$ be a $\frac{1}{k}$-dense set of vectors in Δ so that

$$\|v_{k,i+1} - v_{k,i}\| < \frac{1}{k} \text{ for } 1 \leq i \leq a_k \quad \text{and} \quad \|v_{k,a_k} - v_{k+1,1}\| < \frac{1}{k}. \tag{3.1.2}$$

For $w \in \mathcal{V}_\varphi$, $\delta > 0$, $n \in \mathbb{N}$ let

$$P(w, \delta, n) = \Big\{x \in \ X : \|\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) - w\| < \delta\Big\}.$$

Given $k \geq 1$ and $1 \leq i \leq a_k$, let $\{\delta_{k,i}\}_{k\geq 1, 1\leq i\leq a_k}$ be a sequence of positive real and $\{n_{k,i}\}_{k\geq 1, 1\leq i\leq a_k}$ be a sequence of integers tending to zero and infinity, respectively, so that

$$\delta_{1,1} > \delta_{1,2} > \cdots > \delta_{1,a_1} > \delta_{2,1} > \delta_{2,2} > \cdots > \delta_{2,a_2} > \dots,$$
$$n_{1,1} < n_{1,2} < \cdots < n_{1,a_1} < n_{2,1} < n_{2,2} < \cdots < n_{2,a_2} < \dots$$

that $n_{k,i} \gg m_k$, meaning here $\lim_{k\to\infty} \frac{m_k}{n_{k,i}} = 0$, for all $1 \leq i \leq a_k$, where $m_k := m(\varepsilon/2^{b_k})$, with $b_0 = 0$ and $b_k = \sum_{1\leq i\leq k} a_i$, and $P(v_{k,i}, \delta_{k,i}, n_{k,i}) \neq \emptyset$, for all $k \geq 1$ and $1 \leq i \leq a_k$. Note that $b_k \to \infty$ as $k \to \infty$.

Given $q \in D$, $k \geq 1$ and $1 \leq i \leq a_k$, let $W_{k,i}(q)$ be a maximal $(n_{k,i}, 8\varepsilon)$-separated subset of $P(v_{k,i}, \delta_{k,i}, n_{k,i})$. We index the elements of $W_{k,i}(q)$ by $x^{k,i}_j$, for $1 \leq j \leq \#W_{k,i}(q)$.

Choose also a strictly increasing sequence of integers $\{N^q_{k,i}\}_{k\geq 1, 1\leq i\leq a_k}$ so that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{n_{k,i+1}+m_k}{N^q_{k,i}} = 0, \quad \text{for every } 1\leq i\leq a_k-1 \\
&\lim_{k\to\infty} \frac{n_{k+1,1}+m_{k+1}}{N^q_{k,a_k}} = 0, \quad \text{for } k\geq 1 \\
&\lim_{k\to\infty} \frac{N^q_{1,1}(n_{1,1}+m_1)+\cdots+N^q_{k,i}(n_{k,i}+m_k)}{N^q_{k,i+1}} = 0 \quad \text{for every }, \ 1\leq i\leq a_k-1, \quad \text{and} \\
&\lim_{k\to\infty} \frac{N^q_{1,1}(n_{1,1}+m_1)+\cdots+N^q_{k,a_k}(n_{k,a_k}+m_k)}{N^q_{k+1,i}} = 0 \quad \text{for every }, \ k\geq 1
\end{aligned} \tag{3.1.3}$$

We shall omit the dependence of $W_{k,i}(q)$ and $N^q_{k,i}$ on q when no confusion is possible. The idea is to construct points that shadow finite pieces of orbits associated with the vectors $v_{k,i}$ repeatedly.

We need the following auxiliary construction. The gluing orbit property ensures that for every $\underline{x_{k,i}} := (x^{k,i}_1, \dots, x^{k,i}_{N_{k,i}}) \in (W_{k,i})^{N_{k,i}}$ there exists a point $y = y(\underline{x_{k,i}}) \in X$ and transition time functions

$$p^j_{k,i} : W^{N_{k,i}}_{k,i} \times \mathbb{R}_+ \to \mathbb{N}, \qquad j = 1, 2, \dots, N_{k,i}-1$$

bounded above by $m(\frac{\varepsilon}{2^{b_{k-1}+i}}) \leq m_k$ so that

$$d_{n_{k,i}}(f^{\mathrm{e}_j}(y), x^{k,i}_j) < \frac{\varepsilon}{2^{b_{k-1}+i}}, \quad \text{for every } \ j = 1, 2, \dots, N_{k,i}-1, \tag{3.1.4}$$

where

$$\mathrm{e}_j = \begin{cases} 0 & \text{if } j = 1 \\ (j-1)\, n_{k,i} + \sum_{r=1}^{j-1} p^r_{k,i} & \text{if } j = 2, \dots, N_{k,i}. \end{cases} \tag{3.1.5}$$

Remark 3.1.3. For k and j as above we have that $p^j_{k,i} = p^j_{k,i}(x^{k,i}_1, x^{k,i}_2, \dots, x^{k,i}_{N_{k,i}}, \varepsilon)$ is a function that describes the time lag that the orbit of $y = y(\underline{x_{k,i}})$ takes to jump from a $\frac{\varepsilon}{2^k}$-neighborhood of $f^{n_{k,i}}(x^{k,i}_j)$ to a $\frac{\varepsilon}{2^k}$-neighborhood of $x^{k,i}_{j+1}$, and it is bounded above by m_k. In contrast with the case when f has the specification property, the previous functions need not be constant and, consequently, the collection of points of the form $y = y(\underline{x_{k,i}})$ need not be 3ε-separated by a suitable iterate of the dynamics. For that reason, not only an argument to select a 'large set' of distinguishable orbits would require to compare points with the same transition times, which strongly differs from [2, 25].

We order the family $\{W_{k,i}\}_{k\geq 1, 1\leq i\leq a_k}$ lexicographically: $W_{k,i} \prec W_{s,j}$ if and only if $k \leq s$ and $i \leq j$ whenever $k = s$. We proceed to make a recursive construction of points in a neighborhood of q that shadow points $N_{k,i}$ in the family $W_{k,i}$ successively with bounded time lags in between. More precisely, we construct a family $\{L_{k,i}(q)\}_{k\geq 0, 1\leq i\leq a_k}$ of sets (guaranteed by the gluing orbit property) contained in a neighborhood of q and a family of positive integers $\{l_{k,i}\}_{k\geq 0, 1\leq i\leq a_k}$ (also depending on q) corresponding to the time during the shadowing process. Set:

- $L_{0,i}(q) = \{q\}$ and $l_{0,i} = N_{0,i}n_{0,i} = 0$;
- $L_{1,1}(q) = \{z = z(q, y(\underline{x_{1,1}})) \in X : \underline{x_{1,1}} \in W_{1,1}^{N_{1,1}}\}$ and $l_{1,1} = p_{1,1}^0 + t_{1,1}$ with $t_{1,1} = N_{1,1}n_{1,1} + \sum_{r=1}^{N_{1,1}-1} p_{1,1}^r$, where $z = z(q, y(\underline{x_{1,1}}))$ satisfies $d(z,q) < \frac{\varepsilon}{2}$ and $d_{t_{1,1}}(f^{p_{1,1}^0}(z), y(\underline{x_{1,1}})) < \frac{\varepsilon}{2}$, and $y(\underline{x_{1,1}})$ is defined by (3.1.4) and $0 \leq p_{1,1}^0 \leq m(\frac{\varepsilon}{2^2})$ is given by the gluing orbit property;
- if $i = 1$
  $L_{k,1}(q) = \{z = z(z_0, y(\underline{x_{k,1}})) \in X\colon \underline{x_{k,1}} \in W_{k,1}^{N_{k,1}}$ and $z_0 \in L_{k-1,a_{k-1}}\}$, and $l_{k,1} = l_{k-1,a_{k-1}} + p_{k,1}^0 + t_{k,1}$, with $t_{k,1} = N_{k,1}n_{k,1} + \sum_{r=1}^{N_{k,1}-1} p_{k,1}^r$, where the shadowing point z satisfies
  $$d_{l_{k-1,a_{k-1}}}(z, z_0) < \frac{\varepsilon}{2^{b_{k-1}+1}} \quad \text{and} \quad d_{t_{k,1}}(f^{l_{k-1,a_{k-1}}+p_{k,1}^0}(z), y(\underline{x_{k,1}})) < \frac{\varepsilon}{2^{b_{k-1}+1}}.$$
- if $i \neq 1$
  $L_{k,i}(q) = \{z = z(z_0, y(\underline{x_{k,i}})) \in X\colon \underline{x_{k,i}} \in W_{k,i}^{N_{k,i}}$ and $z_0 \in L_{k,i-1}\}$, and $l_{k,i} = l_{k,i-1} + p_{k,i}^0 + t_{k,i}$, with $t_{k,i} = N_{k,i}n_{k,i} + \sum_{r=1}^{N_{k,i}-1} p_{k,i}^r$, where the shadowing point z satisfies
  $$d_{l_{k,i-1}}(z, z_0) < \frac{\varepsilon}{2^{b_{k-1}+i}} \text{ and } d_{t_{k,i}}(f^{l_{k,i-1}+p_{k,i}^0}(z), y(\underline{x_{k,i}})) < \frac{\varepsilon}{2^{b_{k-1}+i}}.$$

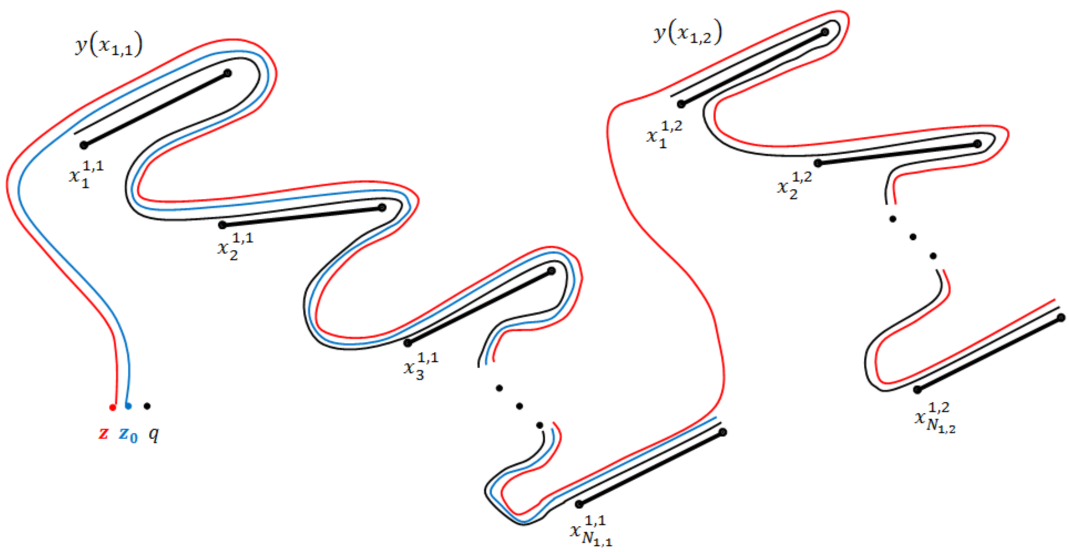

Figure 3.1.1: Construction of $L_{1,1}(q)$ and $L_{1,2}(q)$.

The previous points $y = y(\underline{x_{k,i}})$ are defined as in (3.1.4). By construction, for every $k \geq 1$ and $1 \leq i \leq a_k - 1$,

$$l_{k,i} = \sum_{r=1}^{k} \sum_{s=1}^{a_r} N_{r,s} n_{r,s} + \sum_{r=1}^{k} \sum_{s=1}^{a_r-1} \sum_{t=0}^{N_{r,s}-1} p_{r,s}^{t}. \tag{3.1.6}$$

Remark 3.1.4. Note that $l_{k,i}$ and $t_{k,i}$ are functions (as these depend on $p_{k,i}^{j}$) and, by definition of $N_{k,i}$ cf. (3.1.3), one has that $\frac{\|l_{k,i}\|}{N_{k,i+1}} \leq \frac{\sum_{r=1}^{k}\sum_{s=1}^{a_r} N_{r,s}(n_{r,s}+m_r)}{N_{k,i+1}}$ tends to zero as $k \to \infty$.

For every $k \geq 0$, $1 \leq i \leq a_k$, $q \in D$ and $\varepsilon > 0$ define

$$R_k(q, \varepsilon, i) = \bigcup_{z \in L_{k,i}(q)} \widetilde{B}_{l_{k,i}}\Big(z, \frac{\varepsilon}{2^{b_{k-1}+i}}\Big) \quad \text{and} \quad R(q,\varepsilon) = \bigcap_{k=0}^{\infty} \bigcap_{i=1}^{a_k} R_k(q, \varepsilon, i),$$

where $\widetilde{B}_{l_{k,i}}(x, \delta)$ is the set of points $y \in X$ so that $d(f^{\alpha}(x), f^{\alpha}(y)) < \delta$ for all iterates $0 \leq \alpha \leq l_{k,i-1} - 1$ and $d(f^{\beta}(x), f^{\beta}(y)) \leq \delta$ for every $l_{k,i-1} \leq \beta \leq l_{k,i} - 1$.

Consider also the sets

$$\widetilde{\mathcal{R}} = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} R(q, \frac{1}{j}) = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{k=0}^{\infty} \bigcap_{i=1}^{a_k} \bigcup_{z \in L_{k,i}(q)} \widetilde{B}_{l_{k,i}}\Big(z, \frac{1}{j 2^{b_{k-1}+i}}\Big), \tag{3.1.7}$$

and finally

$$\mathcal{R} = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{i=1}^{a_k} \bigcup_{z \in L_{k,i}(q)} \widetilde{B}_{l_{k,i}}\Big(z, \frac{1}{j 2^{b_{k-1}+i}}\Big). \tag{3.1.8}$$

It is clear from the construction that $R(q, \varepsilon) \subset B(q, \varepsilon)$ for every $q \in D$ and $\varepsilon > 0$, and that $\tilde{\mathcal{R}} \subset \mathcal{R}$. The following lemma, identical to Propositions 2.2 and 2.3 in [25], ensures that $\mathcal{R}$ is a Baire generic subset of X.

Lemma 3.1.5. *$\mathcal{R}$ is a G_δ-set and it is dense in X.*

Proof. First we prove denseness. Since $\widetilde{\mathcal{R}} \subset \mathcal{R}$, it is enough to show that $\widetilde{\mathcal{R}} \cap B(x, r) \neq \emptyset$ for every $x \in X$ and $r > 0$. In fact, given $x \in X$ and $r > 0$, there exists $j \in \mathbb{N}$ and $q \in D$ such that $d(x, q) < 1/j < r/2$. Given $y \in R(q, \frac{1}{j})$ it holds that $d(q, y) < \frac{1}{j}$ because $R(q, \frac{1}{j}) \subset B(q, \frac{1}{j})$. Therefore, $d(x, y) \leq d(x, q) + d(q, y) < 2/j < r$. This ensures that $\widetilde{\mathcal{R}} \cap B(x, r) \neq \emptyset$.

Now we prove that $\mathcal{R}$ is a G_δ-set. Fix $j \in \mathbb{N}$ and $q \in D$. For any $k \geq 1$ and $1 \leq i \leq a_k$, consider the open set

$$G_k(q, \varepsilon, i) := \bigcup_{z \in L_{k,i}(q)} B_{l_{k,i}}\Big(z, \frac{\varepsilon}{2^{b_{k-1}+i}}\Big)$$

and note that $G_k(q,\varepsilon,i) \subset R_k(q,\varepsilon,i)$ for any $k \geq 1$ and $1 \leq i \leq a_k$. We claim that $R_k(q,\varepsilon,i+1) \subset G_k(q,\varepsilon,i)$ and $R_{k+1}(q,\varepsilon,1) \subset G_k(q,\varepsilon,a_k)$, for any $k \geq 1$ and $1 \leq i \leq a_k - 1$. The claim implies that

$$\bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{i=1}^{a_k} R_k(q,\varepsilon,i) = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{i=1}^{a_k} G_k(q,\varepsilon,i),$$

and guarantees that $\mathcal{R}$ is a G_δ-set.

Now we proceed to prove the claim. We prove that $R_k(q,\varepsilon,i+1) \subset G_k(q,\varepsilon,i)$ for any $k \geq 1$ and $1 \leq i \leq a_k - 1$ (the proof of the $R_{k+1}(q,\varepsilon,i) \subset G_k(q,\varepsilon,a_k)$ is analogous). Given $y \in R_k(q,\varepsilon,i+1)$, there exists $z \in L_{k,i+1}(q)$ such that $y \in \widetilde{B}_{l_{k,i+1}}(z, \frac{1}{j2^{b_{k-1}+(i+1)}})$. By definition of $L_{k,i+1}(q)$, there exists z_0 such that $d_{l_{k,i}}(z,z_0) < \frac{1}{j2^{b_{k-1}+(i+1)}}$. Therefore,

$$d_{l_{k,i}}(y,z_0) \leq d_{l_{k,i}}(y,z) + d_{l_{k,i}}(z,z_0) < \frac{1}{j2^{b_{k-1}+(i+1)}} + \frac{1}{j2^{b_{k-1}+(i+1)}} = \frac{1}{j2^{b_{k-1}+i}}$$

and consequently $y \in G_k(q,\varepsilon,i)$. This proves the claim and completes the proof of the lemma. □

We must show that $\mathcal{R} \subset X_\Delta$, that is $\Delta \subseteq \mathcal{V}_\varphi(x)$ for every $x \in \mathcal{R}$. The proof follows some ideas from [25, Proposition 2.1]. We provide a sketch of the argument for completeness. Given $x \in \mathcal{R}$ fixed, for any $k > 1$, there exist integers $j \in \mathbb{N}$, $q \in D$ and $z \in L_{k,i+1}(q)$ such that

$$d_{l_{k,i+1}}(z,x) < \frac{1}{j2^{b_{k-1}}} \quad \text{for every } 1 \leq i \leq a_k. \tag{3.1.9}$$

We prove that $\Delta \subseteq \mathcal{V}_\varphi(x)$. If $v \in \Delta$, then for any $k \geq 1$ there exists $1 \leq i_k \leq a_k$ such that $v \in B(v_{k,i_k}, \frac{1}{k})$. We need the following:

Lemma 3.1.6. *Take $k \geq 1$ and $1 \leq i_k \leq a_k$. If*

$$R^q_{k,i} := \max_{z \in L_{k,i}(q)} \Big\| \sum_{r=0}^{l_{k,i}-1} \varphi(f^r(z)) - l_{k,i}\, v_{k,i} \Big\|,$$

then $\frac{R^q_{k,i}}{l_{k,i}} \to 0, \quad$ *as* $k \to \infty$.

Proof. Let k and i be as above, let $\underline{x_{k,i}} \in (W_{k,i})^{N_{k,i}}$ and $y = y(\underline{x_{k,i}})$. Recall that $\| \sum_{i=0}^{n-1} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(y)) \| \leq n \operatorname{var}(\varphi, c)$ if $d_n(x,y) < c$. Then, using $d_{n_{k,i}}(x^{k,i}_t, f^{e_t}(y)) < \frac{1}{j2^{b_{k-1}}}$ where e_t is defined in (3.1.5) with $t \in \{1,\dots,N_{k,i}\}$ we conclude that

$$\Big\| \sum_{r=0}^{n_{k,i}-1} \varphi(f^r(x^{k,i}_t)) - \sum_{r=0}^{n_{k,i}-1} \varphi(f^{e_t+r}(y)) \Big\| \leq n_{k,i} \operatorname{var}(\varphi, \frac{1}{j2^{b_{k-1}}}).$$

Since $x_t^{k,i} \in W_{k,i}$, we have that

$$\Big\| \sum_{r=0}^{n_{k,i}-1} \varphi(f^{e_t+r}(y)) - n_{k,i}\, v_{k,i} \Big\| \leq n_{k,i}(\mathrm{var}(\varphi, \frac{1}{j2^{b_{k-1}}}) + \delta_{k,i}). \tag{3.1.10}$$

We decompose the time interval $[0, t_{k,i} - 1]$ as follows:

$$\bigcup_{t\,=\,1}^{N_{k,i}} [e_t, e_t + n_{k,i} - 1] \cup \bigcup_{t\,=\,1}^{N_{k,i}-1} [e_t + n_{k,i}, e_t + n_{k,i} + p_{k,i}^t - 1].$$

On the intervals $[e_t, e_t + n_{k,i} - 1]$ we will use the estimate (3.1.10), while in the time intervals $[e_t + n_{k,i}, e_t + n_{k,i} + p_{k,i}^t - 1]$ we use

$$\Big\| \sum_{r=0}^{p_{k,i}^t-1} \varphi(f^{e_t+n_{k,i}+r}(y)) - p_{k,i}^t\, v_{k,i} \Big\| \leq m_k(\|\varphi\|_\infty + \|v_{k,i}\|) \leq 2\, m_k \|\varphi\|_\infty.$$

Therefore,

$$\Big\| \sum_{r=0}^{t_{k,i}-1} \varphi(f^r(y)) - t_{k,i}\, v_{k,i} \Big\| \leq N_{k,i}\, n_{k,i}(\mathrm{var}(\varphi, \frac{1}{j2^{b_{k-1}}}) + \delta_{k,i}) + 2(N_{k,i} - 1) m_k \|\varphi\|_\infty \tag{3.1.11}$$

On the other hand, by definition of $L_{k,i}(q)$ that for every $z \in L_{k,i}(q)$ there exist $z_0 \in L_{k,i-1}(q)$ and $y = y(\underline{x_{k,i}}) \in X$ such that

$$d_{l_{k,i-1}}(x, z) < \frac{1}{j2^{b_{k-1}}}, \qquad d_{t_{k,i}}(y, f^{l_{k,i-1}+p_{k,i}^0}(z)) < \frac{1}{j2^{b_{k-1}}}. \tag{3.1.12}$$

By triangular inequality,

$$\begin{aligned}
\Big\| \sum_{r=0}^{l_{k,i}-1} \varphi(f^r(z)) - l_{k,i}\, v_{k,i} \Big\| \;&\leq\; \Big\| \sum_{r=0}^{l_{k,i-1}-1} \varphi(f^r(z)) - l_{k,i-1}\, v_{k,i} \Big\| \\
&+\; \Big\| \sum_{r=p_{k,i}^0}^{l_{k,i-1}+p_{k,i}^0-1} \varphi(f^r(z)) - p_{k,i}^0\, v_{k,i} \Big\| \\
&+\; \Big\| \sum_{r=l_{k,i-1}+p_{k,i}^0}^{t_{k,i}-1} \varphi(f^r(z)) - t_{k,i}\, v_{k,i} \Big\|,
\end{aligned}$$

where the first and second terms are bounded by $2l_{k,i-1}\|\varphi\|_\infty$ and $2m_k\|\varphi\|_\infty$, respectively.

Inequalities (3.1.11)-(3.1.12) imply

$$\begin{aligned}
&\Big\| \sum_{r=t_{k,i-1}+p_{k,i}^0}^{t_{k,i}-1} \varphi(f^r(z)) - t_{k,i}\, v_{k,i} \Big\| \\
\leq\;& \Big\| \sum_{r=0}^{t_{k,i}-1} \varphi(f^{l_{k,i-1}+p_{k,i}^0+r}(z)) - \sum_{r=0}^{t_{k,i}-1} \varphi(f^r(y)) \Big\| + \Big\| \sum_{r=0}^{t_{k,i}-1} \varphi(f^r(y)) - t_{k,i}\, v_{k,i} \Big\| \\
\leq\;& t_{k,i}\, \mathrm{var}(\varphi, \frac{1}{j2^{b_{k-1}}}) + N_{k,i} n_{k,i}\, \mathrm{var}(\varphi, \frac{1}{j2^{b_{k-1}}} + \delta_{k,i}) + 2(N_{k,i} - 1) m_k \|\varphi\|_\infty
\end{aligned}$$

and, consequently,

$$R^q_{k,i} \le 2(l_{k,i-1} + N_{k,i}m_k)\|\varphi\|_\infty + (t_{k,i} + N_{k,i}n_{k,i})\operatorname{var}(\varphi, \frac{1}{j2^{b_{k-1}}}) + N_{k,i}n_{k,i}\delta_{k,i}.$$

By definition of $N_{k,i}$ in (3.1.3) we obtain that $R^q_{k,i}/l_{k,i} \to 0$ as $k \to \infty$, which proves the lemma. $\square$

Given $z = z(z_0, y(\underline{x_{k,i}})) \in L_{k,i+1}(q)$ satisfying (3.1.9) with $z_0 \in L_{k,i}(q)$, by triangular inequality we have $d_{l_{k,i}}(z_0, x) < \frac{1}{j2^{b_{k-1}-1}}$. Thus,

$$\begin{aligned} & \Big\| \sum_{r=0}^{l_{k,i_k}-1} \varphi(f^r(x)) - l_{k,i_k} v_{k,i_k} \Big\| \qquad (3.1.13) \\ \le\ & \Big\| \sum_{r=0}^{l_{k,i_k}-1} \varphi(f^r(x)) - \sum_{r=0}^{l_{k,i_k}-1} \varphi(f^r(z_0)) \Big\| + \Big\| \sum_{r=0}^{l_{k,i_k}-1} \varphi(f^r(z_0)) - l_{k,i_k} v_{k,i_k} \Big\| \\ \le\ & l_{k,i_k} \operatorname{var}(\varphi, \frac{1}{j2^{b_{k-1}-1}}) + R^q_{k,i_k}. \end{aligned}$$

Lemma 3.1.6 and the uniform continuity of φ ensures that

$$\Big\|\frac{1}{l_{k,i}} \sum_{r=0}^{l_{k,i}-1} \varphi(f^r(x)) - v\Big\| \le \Big\|\frac{1}{l_{k,i}} \sum_{r=0}^{l_{k,i}-1} \varphi(f^r(x)) - v_{k,i_k}\Big\| + \|v_{k,i_k} - v\| \to 0 \qquad (3.1.14)$$

as $k \to \infty$ and, consequently, $v \in \mathcal{V}_\varphi(x)$. This proves that $\Delta \subseteq \mathcal{V}_\varphi(x)$.

Altogether we conclude that X_Δ is a Baire residual subset of X, and finish the proof of Proposition 3.1.1 and Theorem A.

3.2 Full topological pressure and metric mean dimension

In this section we prove Theorem B. Assume that f is a continuous map with the gluing orbit property on a compact metric space X, $\varphi : X \to \mathbb{R}^d$ is continuous and $X_{\varphi,f} \neq \emptyset$. We will prove that $\underline{\mathrm{mdim}}_{X_{\varphi,f}}(f) = \underline{\mathrm{mdim}}(f)$, that $\overline{\mathrm{mdim}}_{X_{\varphi,f}}(f) = \overline{\mathrm{mdim}}(f)$ and $h_{top}(f) = h_{X_{\varphi,f}}(f)$.

Actually we will prove that $h_{X_{\varphi,f}}(f,\varepsilon) = h_{\mathrm{top}}(f,\varepsilon)$ every $\varepsilon > 0$, which is a sufficient condition. Fix $\varepsilon > 0$ and let $m(\varepsilon)$ be given by the gluing orbit property.

3.2.1 Measures with large entropy and distinct rotation vectors

The proof explores the construction of an exponentially large (with exponential rate close to topological entropy) number of points that oscillate between distinct vectors in $\mathbb{R}^d$.

We use some auxiliary results. We say that a observable $\varphi : X \to \mathbb{R}^d$ is *cohomologous to a vector* if there exists $v \in \mathbb{R}^d$ and a continuous function $\chi : X \to \mathbb{R}^d$ so that $\varphi = v + \chi - \chi \circ f$, and denote by Cob the set of all such observables and by $\overline{Cob}$ its closure in the C^0-topology.

Lemma 3.2.1. *Assume that f has the gluing orbit property. The following are equivalent:*

(i) $X_{\varphi,f} \neq \emptyset$,

(ii) $\inf_{\mu \in \mathcal{M}_e(f)} \int \varphi \, d\mu < \sup_{\mu \in \mathcal{M}_e(f)} \int \varphi \, d\mu$, *and*

(iii) there exist periodic points p_1, p_2 of period k_1, k_2 respectively such that

$$\frac{1}{k_1} \sum_{j=0}^{k_1-1} \varphi(f^j(p_1)) \neq \frac{1}{k_2} \sum_{j=0}^{k_2-1} \varphi(f^j(p_2)).$$

(iv) $\varphi \notin \overline{Cob}$.

(v) $\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j$ *does not converge uniformly to a constant.*

Proof. Although this is similar to [52, Lemma 1.9] we include it for completeness.

(iii) $\Rightarrow$ *(iv)*: If $\varphi \in \overline{Cob}$, then there is $\{\varphi_k\}$ in Cob such that $\varphi = \lim_{k\to\infty} \varphi_k$. In particular there exists $v_k \in \mathbb{R}^d$ and χ_k so that

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi_k(f^j(x)) = \frac{\chi_k \circ f^n(x)}{n} - \frac{\chi_k(x)}{n} + v_k \tag{3.2.1}$$

for every $x \in X$. By Birkhoff's ergodic theorem and dominated convergence theorem $\mathcal{M}_e(f) \ni \mu \mapsto \int \varphi d\mu$ is constant, which contradicts (iii).

(iv) $\Rightarrow$ *(v)*: If $\varphi \notin \overline{Cob}$, then the sequence $\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j$ is not uniformly convergent to a vector v. Indeed, otherwise the sequence of continuous function $(h_n)_n$ given by $h_n = \frac{1}{n} \sum_{i=0}^{n-1} (n-i) \varphi \circ f^{i-1}$ satisfy the cohomological equation

$$h_n(x) - h_n(f(x)) = \varphi(x) - \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)), \quad \forall \, x \in X$$

and so $\varphi(x) = \lim_{n\to\infty} [h_n(x) - h_n(f(x)) + \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))] \in \overline{Cob}$, leading to a contradiction.

(v) $\Rightarrow$ *(ii)*:

Let μ be an f-invariant probability measure and suppose that $\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$ does not converge uniformly to $\int \varphi d\mu$. There exists $\varepsilon > 0$ so that for every $k \geq 1$ there are $n_k \geq k$ and $x_k \in X$ for which $\| \frac{1}{n_k} \sum_{j=0}^{n_k-1} \varphi(f^j(x_k)) - \int \varphi d\mu \| \geq \varepsilon$. Consider $\nu_k :=$

$\frac{1}{n_k}\sum_{j=0}^{n_k-1}\delta_{f^j(x_k)}$ and let ν be a weak* accumulation point of the sequence $(\nu_k)_k$. Note that ν is f-invariant. Choose k such that $\|\frac{1}{n_k}\sum_{j=0}^{n_k-1}\varphi(f^j(x_k)) - \int\varphi d\nu\| \leq \varepsilon/2$, so

$$\begin{aligned}\|\int\varphi d\mu - \int\varphi d\nu\| \;&\geq\; \|\int\varphi d\mu - \frac{1}{n_k}\sum_{j=0}^{n_k-1}\varphi(f^j(x_k))\| \\ &-\; \|\frac{1}{n_k}\sum_{j=0}^{n_k-1}\varphi(f^j(x_k)) - \int\varphi d\nu\| \geq \varepsilon/2.\end{aligned}$$

The conclusion follows from the ergodic decomposition theorem.

$(ii) \Rightarrow (i)$: The construction in the proof of Theorem A ensures that if there exist f-invariant measures μ_1, μ_2 so that $\int\varphi\, d\mu_1 \neq \int\varphi\, d\mu_2$, then $X_{\varphi,f} \neq \emptyset$.

$(i) \Rightarrow (iii)$: If the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))$ does not exist for some $x \in X$, then the empirical measures $(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)})_{n\geq 1}$ accumulate on f-invariant probability measures μ_1, μ_2 so that $\int\varphi\, d\mu_1 \neq \int\varphi\, d\mu_2$. Now, the result follows as a simple consequence of the weak* convergence and the fact that periodic measures are dense in the space of f-invariant probability measures ([48, p102-104] proves the latter for maps with specification, but it is straightforward to see that this holds for maps with the gluing orbit property, to see 5.0.4). $\square$

Lemma 3.2.2. *Given $\psi \in C^0(X,\mathbb{R})$ and $\gamma > 0$ there are $\mu_1, \mu_2 \in \mathcal{M}_1(f)$ so that μ_1 is ergodic, $\int\varphi d\mu_1 \neq \int\varphi d\mu_2$ and $h_{\mu_i}(f) + \int\psi d\mu_i > P_{top}(f,\psi) - \gamma$, for $i = 1, 2$.*

Proof. By the variational principle there exists an ergodic $\mu_1 \in \mathcal{M}_1(f)$ so that $h_{\mu_1}(f) + \int\psi d\mu_1 > P_{top}(f,\psi) - \gamma$. As $X_{\varphi,f} \neq \emptyset$ there is $\nu \in \mathcal{M}_1(f)$ satisfying $\int\varphi d\mu_1 \neq \int\varphi d\nu$ (recall Lemma 3.2.1). Consider the family of measures

$$\mu_2^t = t\mu_1 + (1-t)\nu, \quad t \in (0,1) \tag{3.2.2}$$

and observe that, by convexity, $h_{\mu_2^t}(f) + \int\psi d\mu_2^t > P_{top}(f,\psi) - \gamma$, provided that t is sufficiently close to one. the probability measure $\mu_2 := \mu_2^t$ satisfies the requirements of the lemma. $\square$

Although the previously defined measures μ_1, μ_2 depend on the potential $\psi \in C^0(X,\mathbb{R})$ we shall omit its dependence for notational simplicity.

3.2.2 Exponential growth of points with averages close to $\int\varphi\, d\mu_i$, $i = 1, 2$

Take the probability measures μ_1, ν and μ_2 given by Lemma 3.2.2, consider the sequence $\{\zeta_k\}_k$ of real numbers

$$\zeta_k = \max\Big\{\frac{\|\int\varphi d\mu_1 - \int\varphi d\nu\|}{2^k}, \operatorname{var}(\varphi, \frac{\varepsilon}{2^k})\Big\}, \tag{3.2.3}$$

which tend to zero as $k \to \infty$, and take $m_k = m(\frac{\varepsilon}{2^k})$. Given $\gamma \in (0,1)$, by Birkhoff's ergodic theorem one can choose $n_k^1 \gg m_k$ so that

$$m_k/n_k^1 \to 0 \quad \text{as } k \to \infty \tag{3.2.4}$$

such that $\mu_1(Y_{k,1}) \geq 1-\gamma$, where

$$Y_{k,1} = \Big\{ x \in X : \Big\| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\mu_1 \Big\| < \zeta_k \;\text{ for every } n \geq n_k^1 \Big\}. \tag{3.2.5}$$

The following lemma will be instrumental.

Lemma 3.2.3. *There exists $\varepsilon_0 > 0$ so that for any $0 < \varepsilon < \varepsilon_0$, there is a collection $\{S_k^1\}_k$ so that every S_k^1 is a $(n_k^1, 6\varepsilon)$ separated subset of $Y_{k,1}$ and $M_k^1 := \sum_{x \in S_k^1} \exp(\sum_{i=0}^{n_k^1-1} \psi(f^i(x)))$ satisfies $M_k^1 \geq \exp\,(n_k^1(P_{top}(f,\psi,\varepsilon) - 4\gamma))$.*

Proof. The proof is a standard consequence of Proposition 1.6.4. □

For any $k \geq 1$, we now construct large sets of points S_k^2 with averages close to $\int \varphi \, d\mu_2$ at large instants $n_k^2 \geq 1$ (to be defined below). First, as ν is ergodic, there exists $\widetilde{x}_k \in X$ and $\tilde{n}_k \geq 1$ so that $\big\| \frac{1}{\widetilde{n}_k} \sum_{j=0}^{\widetilde{n}_k - 1} \varphi(f^j(\widetilde{x}_k)) - \int \varphi d\nu \big\| < \zeta_k$. Let t of the proof of Lemma 3.2.2, there are $r_k, s_k \geq 1$ be integers satisfying

$$\frac{r_k n_k^1}{s_k \widetilde{n}_k} \to \frac{t}{1-t} \qquad \text{as} \quad k \to \infty. \tag{3.2.6}$$

For any fixed $k \geq 1$, any string $(x_1^k, \dots, x_{r_k}^k) \in (S_k^1)^{r_k}$ and s_k copies of the point $\widetilde{x}_k$, by the gluing orbit property there exists $y = y(x_1^k, \dots, x_{r_k}^k) \in X$ satisfying

$$d_{n_k^1}(f^{a_i}(y), x_i^k) < \varepsilon, \qquad \text{and} \qquad d_{\widetilde{n}_k}(f^{b_j}(y), \widetilde{x}_k) < \varepsilon$$

for every $i = 1, 2, \dots, r_k$ and $j = 1, 2, \dots, s_k$, where

$$a_i = \begin{cases} 0 & \text{if } i = 1 \\ (i-1)n_k^1 + \sum_{r=1}^{i-1} p_{k,r} & \text{if } i = 2, \dots, r_k \end{cases}$$

and

$$b_j = \begin{cases} a_{r_k} + p_{k,r_k} & \text{if } j = 1 \\ (j-1)\widetilde{n}_k + \sum_{r=0}^{j-1} p_{k,r_k+r} + a_{r_k} & \text{if } j = 2, \dots, s_k \end{cases}$$

where $0 \leq p_{k,r} \leq m(\varepsilon)$ are the transition time functions defined similarly as in the proof of Theorem A. We define the auxiliary set $\widehat{S_k^2}$ as the set of points y obtained by the previous process.

Remark 3.2.4. For every point $x \in \widehat{S_k^2}$ we associate the size

$$n_k^2(\cdot) := r_k n_k^1 + s_k \widetilde{n}_k + \sum_{r=1}^{r_k+s_k-1} p_{k,r}(\cdot),$$

of the finite piece of orbit, which is a function of $(x_1^k, \ldots, x_{r_k}^k, \widetilde{x}_k, s_k)$. In strong contrast with the case when f satisfies the specification property, at this moment we can not claim that the cardinality of $\widehat{S_k^2}$ is large. Indeed, since n_k^2 varies with the elements in $\widehat{S_k^2}$ then the $(n_k^1, 4\varepsilon)$-separability of the points in S_k^1 is not sufficient to ensure the shadowing point map $(S_k^1)^{r_k} \times \{\widetilde{x}_k\}^{s_k} \to X$ to be injective. This issue is solved by Lemma 3.2.5.

Now, for any $\underline{j} = (j_1, j_2, \ldots, j_{r_k+s_k-1}) \in \mathbb{Z}_+^{r_k+s_k-1}$ so that $0 \le j_i \le m(\varepsilon)+1$ define the set $S_k^2(\underline{j}) := \{x \in \widehat{S_k^2} \ : \ p_{k,1} = j_1, \ p_{k,2} = j_2, \ldots, p_{k,r_k+s_k-1} = j_{r_k+s_k-1}\}$. The size of the finite orbit of all points in $S_k^2(\underline{j})$ is constant and, by some abuse of notation, we will denote it by

$$n_k^2(\underline{j}) := r_k n_k^1 + s_k \widetilde{n}_k + \sum_{r=1}^{r_k+s_k-1} j_r. \tag{3.2.7}$$

It is not hard to check that (3.2.6) implies

$$\frac{r_k n_k^1}{r_k n_k^1 + s_k \widetilde{n}_k} \to t \quad \text{and, consequently,} \quad \frac{r_k n_k^1}{n_k^2(\underline{j})} \to t \tag{3.2.8}$$

as $k \to \infty$. Moreover,

$$\frac{r_k + s_k}{n_k^2} \le \frac{1}{n_k^1} + \frac{1}{\widetilde{n}_k} \to 0 \quad \text{as } k \to \infty. \tag{3.2.9}$$

The next lemma says that one can choose a large set S_k^2 of points whose n_k^2-time average is close to the one determined by μ_2. More precisely:

Lemma 3.2.5. *For every large $k \ge 1$ there exists $\underline{j_k} = (j_1^k, \ldots, j_{r_k+s_k-1}^k)$ so that if $S_k^2 := S_k^2(\underline{j_k})$ and $n_k^2 = n_k^2(\underline{j_k})$, then the following hold:*

1. *S_k^2 is $(n_k^2, 4\varepsilon)$-separated,*
2. *if $M_k^2 := \sum_{x \in S_k^2} \exp\{\sum_{i=0}^{n_k^2-1} \psi(f^i(x))\}$, then*
$$M_k^2 \ge \exp\left(n_k^2[tP_{top}(f, \psi, \varepsilon) - \mathrm{var}(\psi, \varepsilon) - 6\gamma]\right),$$
3. *there exists a sequence $(a_k)_{k\ge1}$ converging to zero so that*
$$\Big\|\frac{1}{n_k^2} \sum_{j=0}^{n_k^2-1} \varphi(f^j(y)) - \int \varphi \, d\mu_2\Big\| \le \mathrm{var}(\varphi, \varepsilon) + a_k \|\varphi\|_\infty,$$
for every $k \ge 1$ and every $y \in S_k^2$.

Proof. In order to prove item (1), let $\underline{j} = (j_1, j_2, \dots, j_{r_k+s_k-1}) \in \mathbb{Z}_+^{r_k+s_k-1}$ be arbitrary so that $0 \le j_i \le m(\varepsilon) + 1$. Let $y_1 \neq y_2 \in S_k^2(\underline{j})$ shadow the orbits of points in the strings $(x_1^k, \dots, x_{r_k}^k) \neq (z_1^k, \dots, z_{r_k}^k) \in (S_k^1)^{r_k}$ and also s_k times the finite piece of orbit of $\widetilde{x}_k$, respectively. There exists $1 \le i \le r_k$ such that $x_i \neq z_i$ and, using that S_k^1 is $(n_k^1, 6\varepsilon)$-separated,

$$d_{n_k^2(\underline{j})}(y_1, y_2) \ge d_{n_k^1}(x_i^k, z_i^k) - d_{n_k^1}(y_1, x_i^k) - d_{n_k^1}(z_i^k, y_2) \ge 4\varepsilon.$$

Therefore, $S_k^2(\underline{j})$ is $(n_k^2(\underline{j}), 4\varepsilon)$-separated for every $\underline{j}$. This implies (1).

Now we prove (3). Take $j_0^k = 0$ and write the Birkhoff sum $\sum_{j=0}^{n_k^2-1} \varphi(f^j(x))$ by

$$\begin{aligned}\sum_{j=0}^{n_k^2-1} \varphi(f^j(x)) &= \sum_{l=1}^{r_k} \sum_{j=0}^{n_k^1-1} \varphi(f^{j+(l-1)n_k^1+\sum_{t\le l-1} j_t^k}(x)) \\ &+ \sum_{l=1}^{s_k} \sum_{j=0}^{\widetilde{n}_k-1} \varphi(f^{j+(l-1)\widetilde{n}_k+r_k n_k^1+\sum_{t\le r_k+l-1} j_t^k}(x)) \\ &+ \sum_{i=1}^{r_k+s_k-1} \sum_{j=0}^{j_i^k-1} \varphi(f^{j+\chi_i+\sum_{t\le i-1} j_t^k}(x)),\end{aligned} \tag{3.2.10}$$

where

$$\chi_i = \begin{cases} i n_k^1 & \text{if } 1 \le i \le r_k \\ r_k n_k^1 + (i - r_k)\widetilde{n}_k & \text{if } r_k < i \le r_k + s_k - 1. \end{cases}$$

The third expression in the right hand-side of (3.2.10) satisfies

$$\Big\| \sum_{i=1}^{r_k+s_k-1} \sum_{j=0}^{j_i-1} \varphi(f^{j+\chi_i+\sum_{t\le i-1} j_t}(x)) \Big\| \le (r_k + s_k - 1)(m(\varepsilon) + 1)\|\varphi\|_\infty.$$

Using (3.2.9) one can estimate the Birkhoff sums in terms of the periods of shadowing and the remainder terms as follows:

$$\begin{aligned}\Big\| \sum_{j=0}^{n_k^2-1} \varphi(f^j(x)) - n_k^2 \int \varphi d\mu_2 \Big\| &\le (r_k n_k^1 + s_k \widetilde{n}_k) \operatorname{var}(\varphi, \varepsilon) \\ &+ \Big\| \sum_{l=1}^{r_k} \Big(\sum_{j=0}^{n_k^1-1} \varphi(f^{j+(l-1)n_k^1+\sum_{t\le l-1} j_t}(x_l^k)) - n_k^1 \int \varphi d\mu_1 \Big) \Big\| \\ &+ \Big\| \sum_{l=1}^{s_k} \Big(\sum_{j=0}^{\widetilde{n}_k-1} \varphi(f^{j+r_k n_k^1+\sum_{t\le l-1} j_{r_k+t}+(l-1)\widetilde{n}_k}(\widetilde{x}_k)) - \widetilde{n}_k \int \varphi d\nu \Big) \Big\| \\ &+ \Big\| n_k^2 \int \varphi d\mu_2 - r_k n_k^1 \int \varphi d\mu_1 - s_k \widetilde{n}_k \int \varphi d\nu \Big\| \\ &+ (r_k + s_k - 1)(m(\varepsilon) + 1)\|\varphi\|_\infty \\ &\le n_k^2 (\operatorname{var}(\varphi, \varepsilon) + \zeta_k) + |t n_k^2 - r_k n_k^1| \|\varphi\|_\infty + |(1-t) n_k^2 - s_k \widetilde{n}_k| \|\varphi\|_\infty \\ &+ (r_k + s_k - 1)(m(\varepsilon) + 1)\|\varphi\|_\infty.\end{aligned}$$

Dividing all terms in the previous estimate by n_k^2 and using (3.2.8) - (3.2.9) we conclude that item (3) holds.

We are now left to prove item (2). First, computations similar to (3.2.10) for the potential $\psi \in C^0(X,\mathbb{R})$ yield

$$\begin{aligned}\sum_{x\in\widehat{S_k^2}} \exp\{\sum_{i=0}^{n_k^2-1}\psi(f^i(x))\} &\geq \Big[\sum_{z\in S_k^1}\exp\{\sum_{i=0}^{n_k^1-1}\psi(f^i(z))\}\Big]^{r_k} \\ &\quad\times e^{-n_k^2\left[\operatorname{var}(\psi,\varepsilon)+\frac{s_k\tilde{n}_k}{n_k^2}|\psi|_\infty+\frac{r_k+s_k-1}{n_k^2}|\psi|_\infty\right]} \\ &\geq \exp\left(n_k^2[tP_{top}(f,\psi,\varepsilon)-\operatorname{var}(\psi,\varepsilon)-5\gamma]\right) \qquad (3.2.11)\end{aligned}$$

for every large $k \geq 1$. Here we used equations (3.2.9), (3.2.8) and Lemma 3.2.3. Recall the definition of $\widehat{S_k^2}$ and consider the shadowing point map

$$\begin{array}{rccc}\mathcal{S}: & \{0,1,\dots,m(\varepsilon)\}^{r_k+s_k-1}\times(S_k^1)^{r_k}\times\{\tilde{x}_k\}^{s_k} & \to & \widehat{S_k^2}\subset X \\ & (\underline{j},\underline{x},(\tilde{x}_k,\dots,\tilde{x}_k)) & \mapsto & y(\underline{j},\underline{x},\tilde{x}_k,s_k).\end{array}$$

Observe that

$$\widehat{S_k^2} = \bigsqcup_{\underline{j}} S_k^2(\underline{j}) = \bigsqcup_{\underline{j}} \operatorname{Image}(\mathcal{S}(\underline{j},\cdot))$$

where the union is over all possible $\underline{j}\in\{0,1,\dots,m(\varepsilon)\}^{r_k}$. Now, equations (3.2.9) and (3.2.11), the separability condition proved in item (1) and the pigeonhole principle ensure that there exists a string $\underline{j_k}=(j_1^k,\dots,j_{r_k+s_k-1}^k)$ such that

$$\begin{aligned}\sum_{x\in S_k^2(\underline{j}_k)}\exp\{\sum_{i=0}^{n_k^2-1}\psi(f^i(x))\} &\geq \frac{1}{(m(\varepsilon)+1)^{r_k+s_k-1}}\sum_{x\in\widehat{S_k^2}}\exp\{\sum_{i=0}^{n_k^2-1}\psi(f^i(x))\} \\ &\geq \exp\left(n_k^2[tP_{top}(f,\psi,\varepsilon)-\operatorname{var}(\psi,\varepsilon)-6\gamma]\right)\end{aligned}$$

for every large $k\geq 1$. The set $S_k^2 = S_k^2(\underline{j}_k)$ satisfies the requirements of item (2). This proves the lemma. □

3.2.3 Construction of sets of points with oscillatory behavior

Consider the sequences $\{S_k\}_k$ and $\{n_k\}_k$ given by

$$S_k = \begin{cases} S_k^1, & \text{if } k \text{ is odd} \\ S_k^2, & \text{if } k \text{ is even,}\end{cases} \quad\text{and}\quad n_k = \begin{cases} n_k^1, & \text{if } k \text{ is odd} \\ n_k^2, & \text{if } k \text{ is even.}\end{cases}$$

Lemmas 3.2.3 and 3.2.5 ensure that

$$M_k := \sum_{x\in S_k}\exp\{\sum_{i=0}^{n_k-1}\psi(f^i(x))\} \geq \exp\left(n_k[tP_{top}(f,\psi,\varepsilon)-\operatorname{var}(\psi,\varepsilon)-6\gamma]\right) \qquad (3.2.12)$$

for every large $k \geq 1$. Since we will construct sets of points that interpolate between those in the sets S_k within a $\frac{\varepsilon}{2^k}$-distance (in the Bowen metric) we need the transition times $m_k = m(\frac{\varepsilon}{2^k})$ to be negligible in comparison with the total size of the orbits. For that, choose a strictly increasing sequence of integers $\{N_k\}_{k\geq 0}$ so that $N_0 = 1$,

$$\lim_{k\to\infty} \frac{n_{k+1}+m_k}{N_k} = 0, \qquad \text{and}$$
$$\lim_{k\to\infty} \frac{1+N_1(n_1+m_k)+\cdots+N_k(n_k+m_k)}{N_{k+1}} = 0. \tag{3.2.13}$$

For any fixed $k \geq 1$ and any string $\underline{x} = (x_1^k, x_2^k, \ldots, x_{N_k}^k) \in S_k^{N_k}$ there exists a point $y = y(\underline{x}) \in X$ which satisfies

$$d_{n_k}(f^{a_j}(y), x_{i_j}^k) < \frac{\varepsilon}{2^k}, \qquad \forall\, j = 1, 2, \ldots, N_k$$

where

$$a_j = \begin{cases} 0 & , \quad \text{if } j = 1 \\ (j-1)n_k + \sum_{r=1}^{j-1} p_{k,r} & , \quad \text{if } j = 2, \ldots, N_k \end{cases}$$

and $p_{k,r}$ are the transition time functions, bounded by m_k.

Define

$$C_k = \left\{ y(\underline{x}) \in X : \underline{x} = (x_1^k, x_2^k, \ldots, x_{N_k}^k) \in S_k^{N_k} \right\}$$

and $c_k = N_k n_k + \sum_{r=1}^{N_k-1} p_{k,r}$ (it is a function on C_k). Proceeding as before, it is not hard to check that for any fixed $\underline{s} = (s_1, \ldots, s_{N_k-1})$ (with all coordinates bounded by m_k) the subset $C_k(\underline{s}) \subset C_k$ with these prescribed transition times is a $(3\varepsilon, N_k n_k + \sum_{i=1}^{N_k-1} s_i)$-separated set. Using (3.2.12) and the pigeonhole principle, there exists $\underline{s_k} = (s_1^k, \ldots, s_{N_k-1}^k)$ so that the set

$$C_k(\underline{s}_k) = \left\{ y(\underline{x}) \in C_k \colon \underline{x} \in S_k^{N_k} \text{ and } p_{k,1} = s_1^k, \ \ldots\ , \ p_{k,N_k-1} = s_{N_k-1}^k \right\}$$

satisfies

$$\begin{aligned} \sum_{x\in C_k(\underline{s}_k)} \exp\{\sum_{i=0}^{c_k-1} \psi(f^i(x))\} &\geq \frac{1}{(m_k+1)^{N_k}} \sum_{x\in C_k} \exp\{\sum_{i=0}^{c_k-1} \psi(f^i(x))\} \\ &\geq \exp\left(n_k N_k [tP_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 6\gamma]\right) \\ &\quad \times e^{-c_k\left[\mathrm{var}(\psi,\frac{\varepsilon}{2^k}) + \frac{N_k-1}{c_k}|\psi|_\infty + \frac{N_k \log m_k}{c_k}\right]} \\ &\geq \exp\left(c_k[tP_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 7\gamma]\right) \end{aligned} \tag{3.2.14}$$

for every large $k \geq 1$, where $c_k = n_k N_k + \sum_{i=1}^{N_k-1} s_i^k$ is constant for all points of the set $C_k(\underline{s}_k)$. We used that $\log m_k / n_k \to 0$ (cf. (3.2.7)) and $(n_k N_k)/c_k \to 1$ as $k \to \infty$. As before we will denote $C_k(\underline{s}_k)$ simply by C_k.

We now construct points whose averages oscillate between $\int \varphi \, d\mu_1$ and $\int \varphi \, d\mu_2$. Define $T_1 = C_1$ and $t_1 = c_1$, and we define the families $(T_k)_{k\geq 1}$ and $(t_k)_k$ recursively. If $x \in T_k$ and $y \in C_{k+1}$ there exists a point $z := z(x,y) \in X$ and $0 \leq p_{k+1} \leq m_{k+1}$ such that

$$d_{t_k}(x,z) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{c_{k+1}}(f^{t_k+p_{k+1}}(z), y) < \frac{\varepsilon}{2^{k+1}}.$$

Define the set

$$T_{k+1} = \{z = z(x,y) \ \in X \ : x \in \ T_k, \ y \in \ C_{k+1}\}$$

and $t_{k+1} = t_k + p_{k+1} + c_{k+1}$ (it is a function on T_k). Using the previous argument once more as above we conclude that there exists $0 \leq \underline{p}_{k+1} \leq m_{k+1}$ such that $T_{k+1}(\underline{p}_{k+1}) \subset T_{k+1}$ is a $(2\varepsilon, t_k + \underline{p}_{k+1} + c_{k+1})$-separated set. We will keep denoting $T_{k+1}(\underline{p}_{k+1})$ by T_{k+1} for notational simplicity. In particular, if $z = z(x,y) \in T_{k+1}$, then

$$d_{t_k}(x,z) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{c_{k+1}}\big(f^{t_k+\underline{p}_{k+1}}(z), y\big) < \frac{\varepsilon}{2^{k+1}}.$$

3.2.4 Construction of a fractal set with large topological pressure

Define

$$F_k = \bigcup_{z\in T_k} \overline{B_{t_k}(z, \frac{\varepsilon}{2^k})} \quad \text{and} \quad F = \bigcap_{k\geq 1} F_k.$$

The previous set F depends on ε, but we shall omit its dependence for notational simplicity. As $F_{k+1} \subset F_k$ for all $k \geq 1$ follows that F is the (non-empty) intersection of a sequence of compact and nested subsets. In the present subsection we will prove the following:

$$P_F(f,\psi,\varepsilon) \geq tP_{top}(f,\psi,\varepsilon) - \text{var}(\psi,\varepsilon) - 9\gamma. \tag{3.2.15}$$

Remark 3.2.6. Every point $x \in F$ can be uniquely represented by an itinerary $\underline{x} = (\underline{x_1}, \underline{x_2}, \underline{x_3}, \dots)$ where each $\underline{x_i} = (x_1^i, \dots, x_{N_k}^i) \in S_i^{N_i}$. We will keep denoting by $y(\underline{x_i}) \in C_i$ the point in C_i determined by the sequence $\underline{x}_i$ with a sequence $\underline{s}_i = (s_1^i, \dots, s_{N_i-1}^i)$ of transition times, and by $z_i(\underline{x}) = z(z_{i-1}(\underline{x}), y(\underline{x_i})) \in T_i$ the element constructed using the points $z_{i-1}(\underline{x}) \in T_{i-1}$ and $y(\underline{x_i}) \in C_i$, and with transition time $\underline{p}_i$.

We will use the following pressure distribution principle:

Proposition 3.2.7. *[52, Proposition 2.4] Let $f : X \to X$ a continuous map on a compact metric space X and let $Z \subset X$ be a Borel set. Suppose there are $\varepsilon > 0$, $s \in \mathbb{R}$, $K > 0$ and a sequence of probability measures $(\mu_k)_k$ satisfying:*

(i) $\mu_k \to \mu$ *and* $\mu(Z) > 0$, *and*

(ii) $\limsup_{k\to\infty} \mu_k(B_n(x,\varepsilon)) \le K \exp\{-ns + \sum_{i=0}^{n-1} \psi(f^i(x))\}$ *for every large* n *and every ball* $B_n(x,\varepsilon)$ *such that* $B_n(x,\varepsilon) \cap Z \ne \emptyset$.

Then, $P_Z(\psi,\varepsilon) \ge s$.

Assume first that $h_{\text{top}}(f) < \infty$ (hence $P_{top}(f,\psi) < \infty$, by the variational principle). We use the previous proposition to estimate $P_F(f,\psi,\varepsilon)$. Consider a sequence $(\mu_k)_k$ of measures on F as follows: take $\nu_k = \sum_{z\in T_k} \Psi(z)\,\delta_z$ and its normalization

$$\mu_k = \frac{1}{Z_k}\nu_k \quad \text{where} \quad Z_k = \sum_{z\in T_k} \Psi(z),$$

and for every $z = z(\underline{x}_1, \dots, \underline{x}_k) \in T_k$ and $\underline{x}_i = (x_1^i, \dots, \underline{x}_{N_i}^i) \in S_i^{N_i}$ we set

$$\Psi(z) = \prod_{i=1}^{k} \prod_{l=1}^{N_i} \exp S_{n_i}\,\psi(x_l^i).$$

We will prove that $(\mu_k)_k$ satisfies the hypothesis of Proposition 3.2.7. Given $n \ge 1$, let $B = B_n(q,\varepsilon/2)$ be a dynamical ball that intersects F, let $k \ge 1$ be such that $t_k \le n < t_{k+1}$, and let $0 \le j \le N_{k+1} - 1$ be so that

$$t_k + jn_{k+1} + \sum_{1\le i\le j} s_i^{k+1} \le n < t_k + (j+1)n_{k+1} + \sum_{1\le i\le j+1} s_i^{k+1}. \tag{3.2.16}$$

Lemma 3.2.8. *If* $\mu_{k+1}(B) > 0$, *then*

$$\nu_{k+1}(B) \le e^{S_n\psi(q) + n\,\mathrm{var}(\psi,\varepsilon) + (\sum_{i=1}^k N_i m_i + j\,m_{k+1})|\psi|_\infty}\, M_{k+1}^{N_{k+1}-j}.$$

Proof. If $\mu_{k+1}(B) > 0$, then $T_{k+1} \cap B \ne \emptyset$. Let $z = z(x,y) \in T_{k+1} \cap B$ determined by $x \in T_k$ and $y = y(\underline{x}_1, \dots, \underline{x}_{N_{k+1}}) \in C_{k+1}$ and let $\underline{p}_{k+1}$ be so that

$$d_{t_k}(z,x) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{c_{k+1}}(f^{t_k+\underline{p}_{k+1}}(z), y) < \frac{\varepsilon}{2^{k+1}}.$$

Since $z \in B_n(q,\varepsilon/2)$ and $n \ge t_k$, then $d_{t_k}(x,q) \le d_{t_k}(x,z) + d_{t_k}(z,q) < \varepsilon$. Using the definition of n and the fact that $d_n(z,q) < \frac{\varepsilon}{2}$ we have that

$$d_{n_{k+1}}(f^{t_k+(l-1)n_{k+1}+\sum_{0\le i\le l-1} s_i^{k+1}}(z), f^{t_k+(l-1)n_{k+1}+\sum_{0\le i\le l-1} s_i^{k+1}}(q)) < \frac{\varepsilon}{2}$$

for all $l = 1, \dots, j$. Moreover, by construction $d_{c_{k+1}}(f^{t_k+\underline{p}_{k+1}}(z), y) < \frac{\varepsilon}{2^{k+1}}$. This implies on the following estimates for blocks of size n_{k+1}:

$$d_{n_{k+1}}(f^{t_k+\underline{p}_{k+1}+(l-1)\,n_{k+1}+\sum_{0\le i\le l-1} s_i^k}(z), f^{(l-1)\,n_{k+1}+\sum_{0\le i\le l-1} s_i^k}(y)) < \frac{\varepsilon}{2^{k+1}}$$

for all $l = 1, \ldots, N_{k+1}$. Using that $y = y(\underline{x}_1, \ldots, \underline{x}_{N_{k+1}}) \in C_{k+1}$ we also have

$$d_{n_{k+1}}(f^{(l-1)\,n_{k+1}+\sum_{0\le i\le l-1} s_i^k}(y),\ x_l^{k+1}) < \frac{\varepsilon}{2^{k+1}}$$

for all $l = 1, \ldots, j$. Altogether the previous estimates imply

$$d_{n_{k+1}}(f^{t_k+\underline{p}_{k+1}+(l-1)\,n_{k+1}+\sum_{0\le i\le l-1} s_i^k}(z),\ x_l^{k+1}) < 2\,\varepsilon \tag{3.2.17}$$

for all $l = 1, \ldots, j$.

We remark that if $\hat{z} = z(\hat{x}, \hat{y}) \in T_{k+1} \cap B$ then $d_{t_k}(\hat{x}, q) < \varepsilon$ and, consequently, $d_{t_k}(\hat{x}, x) < 2\varepsilon$. Since T_k is $(t_k, 2\varepsilon)$ separated and $n \ge t_k$, then $x = \hat{x}$. Moreover, the previous estimates also ensure (cf. (3.2.17)) that $d_{n_{k+1}}(x_l^{k+1}, \hat{x}_l^{k+1}) < 4\ \varepsilon$ for all $l = 1, \ldots, j$. However, as x_i^{k+1} and $\hat{x}_i^{k+1}$ belong to S_{k+1}, which is a $(n_{k+1}, 4\varepsilon)$-separated set, then $x_i^{k+1} = \hat{x}_i^{k+1}$ for every $i = 1, \ldots, j$.

The previous argument implies that all elements $z = z(x, y) \in T_{k+1} \cap B$ with $x \in T_k$ and $y = (\underline{x}_1, \ldots, \underline{x}_{N_{k+1}}) \in C_{k+1}$ may only differ in the last $N_{k+1} - j$ elements of S_{k+1}. Therefore, by the choice of k and j in (3.2.16),

$$\begin{aligned}
\nu_{k+1}(B) &= \sum_{z \in T_{k+1}\cap B} \Psi(z) \\
&\le \Psi(x)\Big[\prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_l^{k+1})\Big] \sum_{l=j+1}^{N_{k+1}} \exp(S_{n_{k+1}}\psi(x_l^{k+1})) \\
&= \Psi(x)\Big[\prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_l^{k+1})\Big] \prod_{l=j+1}^{N_{k+1}} \sum_{\tilde{x}\in S_{k+1}} \exp(S_{n_{k+1}}\psi(\tilde{x})) \\
&= \Psi(x)\Big[\prod_{l=1}^{j} \exp S_{n_{k+1}}\psi(x_l^{k+1})\Big]\, M_{k+1}^{N_{k+1}-j} \\
&\le e^{S_n\psi(q)+n\,\mathrm{var}(\psi,\varepsilon)+(\sum_{i=1}^{k} N_i m_i + j\,m_{k+1})|\psi|_\infty}\, M_{k+1}^{N_{k+1}-j}
\end{aligned}$$

which proves the lemma. □

Lemma 3.2.9. $Z_k\,(M_{k+1})^j \ge \exp(n(t\,P_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 8\gamma))$ *for all* $k \gg 1$.

Proof. By the variational principle and the fact that ψ is bounded away from zero and infinity assumption (i) is equivalent to $P_{top}(f,\psi) < \infty$. A simple computation shows that $Z_k = {M_k}^{N_k}$ for every $k \ge 1$. Moreover, using

$$\begin{aligned}
n &< t_k + (j+1)(n_{k+1} + m_{k+1}) \\
&= \sum_{i=1}^{k} n_i N_i + \sum_{i=1}^{k} \big(\underline{p}_i + \sum_{l=1}^{N_i - 1} s_l^i\big) + (j+1)(n_{k+1} + m_{k+1}) \\
&\le \sum_{i=1}^{k} [(n_i + m_i)N_i + m_i] + (j+1)(n_{k+1} + m_{k+1})
\end{aligned}$$

equation (3.2.12), and that $m_i \ll n_i \ll N_i$ for every $1 \le i \le k$ we get

$$
\begin{aligned}
Z_k\, M_{k+1}^j &= M_1^{N_1} \dots M_k^{N_k} M_{k+1}^j \\
&\ge \exp\ \big((\sum_{i=1}^k N_i n_i + j n_{k+1})[tP_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 6\gamma]\,\big) \\
&\ge \exp(t(\sum_{i=1}^k N_i(n_i+m_i) + j(n_{k+1}+m_{k+1}))[tP_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 7\gamma]) \\
&\ge \exp(n[t\,P_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 8\gamma])
\end{aligned}
$$

for all large k, proving the lemma. □

Corollary 3.2.10. *The following holds:*

$$
\limsup_{k\to\infty} \mu_k(B_n(q,\varepsilon/2)) \le \exp(-n(t\,P_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 9\gamma) + S_n\psi(q)).
$$

Proof. By Lemmas 3.2.8 and 3.2.9 we get

$$
\begin{aligned}
\mu_{k+1}(B) &\le \frac{1}{Z_k\, M_{k+1}^{N_{k+1}}} e^{S_n\psi(q) + n\,\mathrm{var}(\psi,\varepsilon) + (\sum_{i=1}^k N_i m_i + j\, m_{k+1})|\psi|_\infty}\, M_{k+1}^{N_{k+1}-j} \\
&= \frac{1}{Z_k\, M_{k+1}^{j}} e^{S_n\psi(q) + n\,\mathrm{var}(\psi,\varepsilon) + (\sum_{i=1}^k N_i m_i + j\, m_{k+1})|\psi|_\infty} \\
&\le \exp(-n(t\,P_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 9\gamma) + S_n\psi(q))
\end{aligned}
$$

for all large k, proving the corollary. □

Now, an argument similar e.g. to [11, p.1200] ensures that any accumulation point μ of μ_k satisfies $\mu(F) = 1$. Since the hypothesis of Proposition 3.2.7 are satisfied we conclude that $P_F(f,\psi,\varepsilon) \ge t\,P_{top}(f,\psi,\varepsilon) - \mathrm{var}(\psi,\varepsilon) - 9\gamma$ proving equation (3.2.15).

Finally, by the variational principle for the topological entropy, in the case that $\sup_{\mu\in\mathcal{M}_1(f)} h_\mu(f) = h_{\mathrm{top}}(f) = +\infty$ (hence $P_{top}(f,\psi) = +\infty$) the argument follows with minor modifications. Indeed, one can repeat the previous arguments and prove that for any $K > 0$ and $t \in (0,1)$ there exist invariant probability measures μ_1, μ_2 so that μ_1 is ergodic, $h_{\mu_1}(f) + \int \psi\, d\mu_1 > K$, $h_{\mu_2}(f) + \int \psi\, d\mu_2 > tK$ and $\int \varphi\, d\mu_1 \ne \int \varphi\, d\mu_2$. The same argument as before shows that for any given $\varepsilon, \gamma > 0$ there exists a fractal set $F \subset X_{\varphi,f}$ such that

$$
P_{X_{\varphi,f}}(f,\psi,\varepsilon) \ge P_F(f,\psi,\varepsilon) \ge tK - \mathrm{var}(\psi,\varepsilon) - 9\gamma,
$$

leading to the conclusion that $P_{X_{\varphi,f}}(f,\psi) \ge K$. Since $K > 0$ is arbitrary and ψ is bounded above and below, then

$$
P_{X_{\varphi,f}}(f,\psi) = P_{top}(f,\psi) = h_{X_{\varphi,f}}(f) = h_{\mathrm{top}}(f) = +\infty
$$

as claimed.

3.2.5 The set F is formed by points with historic behavior

In order to complete the proof of Theorem B it suffices to prove that $F \subset X_{\varphi,f}$.

Proposition 3.2.11. $F \subset X_{\varphi,f}$.

Proof. Let $x \in F$, and set $\chi(k) = 1$ if k odd, and $\chi(k) = 2$ otherwise. By Remark 3.2.6 let $y_k := y(\underline{x_k}) \in C_k$ and $z_k = z_k(\underline{x}) \in T_k$. First we prove that points in C_k have time averages close to $\int \varphi d\mu_{\chi(k)}$. More precisely, we claim that

$$\Big\| \frac{1}{c_k} \sum_{j=0}^{c_k-1} \varphi(f^j(y_k)) - \int \varphi d\mu_{\chi(k)} \Big\| \to 0 \quad \text{as} \quad k \to \infty. \tag{3.2.18}$$

Recalling that $c_k = N_k n_k + \sum_{i=1}^{N_k-1} s_i^k$ and $0 \le s_i^k \le m_k$ for every i, one can write

$$\begin{aligned}
&\Big\| \sum_{j=0}^{c_k-1} \varphi(f^j(y_k)) - c_k \int \varphi d\mu_{\chi(k)} \Big\| \\
&\qquad \le \Big\| \sum_{j=1}^{N_k} \sum_{i=0}^{n_k-1} \varphi(f^{i+(j-1)n_k+\sum_{i=1}^{j-1} s_i^k}(y_k)) - n_k N_k \int \varphi d\mu_{\chi(k)} \Big\| \\
&\qquad + 2m_k(N_k-1)\|\varphi\|_\infty \\
&\qquad \le \sum_{j=1}^{N_k} \sum_{i=0}^{n_k-1} \|\varphi(f^{i+(j-1)n_k+\sum_{i=1}^{j-1} s_i^k}(y_k)) - \varphi(f^i(x_j^k))\| \\
&\qquad + \Big\| \sum_{j=1}^{N_k} [\sum_{j=0}^{n_k-1} \varphi(f^i(x_j^k)) - n_k \int \varphi \, d\mu_{\chi(k)}] \Big\| \\
&\qquad + 2m_k(N_k-1)\|\varphi\|_\infty \\
&\qquad \le N_k n_k \Big(\mathrm{var}(\varphi, \frac{\varepsilon}{2^k}) + \zeta_k \Big) + 2m_k(N_k-1)\|\varphi\|_\infty.
\end{aligned}$$

Using that $\lim_{k\to\infty} \frac{n_k N_k}{c_k} = 1$ and $\lim_{k\to\infty} \frac{m_k N_k}{c_k} = 0$ we conclude that

$$\Big\| \frac{1}{c_k} \sum_{j=0}^{c_k-1} \varphi(f^j(y_k)) - \int \varphi d\mu_{\chi(k)} \Big\| \le \frac{N_k n_k}{c_k} \Big(\mathrm{var}(\varphi, \frac{\varepsilon}{2^k}) + \zeta_k \Big) + \frac{2m_k(N_k-1)}{c_k} \|\varphi\|_\infty$$

tends to zero as $k \to \infty$, which proves the claim.

Now, take any point $x \in F$. By definition for every $k \ge 1$ there exists $z_k = z(z_{k-1}, y_k) \in T_k$ so that $d_{t_k}(x, z_k) \le \frac{\varepsilon}{2^k}$. Using that $t_k = c_k + \underline{p}_k + t_{k-1}$ and triangular inequality we get

$$d_{c_k}(f^{t_k-c_k}(x), y_k) \le d_{t_k}(f^{t_k-c_k}(x), f^{t_k-c_k}(z_k)) + d_{c_k}(f^{t_k-c_k}(z_k), y_k) < \frac{\varepsilon}{2^{k-1}}.$$

In particular,

$$\Big\|\frac{1}{c_k}\sum_{j=0}^{c_k-1}\varphi(f^{t_k-c_k+j}(p)) - \int \varphi d\mu_{\chi(k)}\Big\|$$
$$\leq \operatorname{var}\left(\varphi, \frac{\varepsilon}{2^{k-1}}\right) + \Big\|\frac{1}{c_k}\sum_{j=0}^{c_k-1}\varphi(f^j(y_k)) - \int \varphi d\mu_{\chi(k)}\Big\|$$

tends to zero as $k \to \infty$. Using that $\lim_{k\to\infty}\frac{c_k}{t_k} = 1$ and dividing the t_k-time average in their first $t_k - c_k$ summands and the second c_k summands, a simple computation shows

$$\Big\|\frac{1}{t_k}\sum_{j=0}^{t_k-1}\varphi(f^j(x)) - \frac{1}{c_k}\sum_{j=0}^{c_k-1}\varphi(f^{j+t_k-c_k}(x))\Big\| \leq 2\frac{t_k-c_k}{t_k}\|\varphi\|_\infty \to 0$$

as $k \to \infty$. Altogether we get that $\lim_{k\to\infty}\ \big\|\frac{1}{t_k}\sum_{j=0}^{t_k-1}\varphi(f^j(p)) - \int\varphi d\mu_{\chi(k)}\big\| = 0$, which proves the proposition. □

3.2.6 Proof of Theorem B

We will consider the case of topological entropy and metric mean dimension, as the argument that proves that the historic set carries full topological pressure is completely analogous. We note that

$$0 \leq \underline{mdim}(f) \leq \overline{mdim}(f) \leq h_{\text{top}}(f) \leq +\infty$$

and that $\overline{mdim}(f) = 0$ whenever $h_{\text{top}}(f) < +\infty$. For that reason we distinguish the following cases:

Case 1: $0 = mdim(f) < h_{\text{top}}(f) < +\infty$.

It is immediate that $\text{mdim}_{X_{\varphi,f}}(f) = \text{mdim}(f) = 0$. It remains to prove that $h_{X_{\varphi,f}}(f) = h_{\text{top}}(f)$. Given $\varepsilon, \gamma, t > 0$ as before, Corollary 3.2.10 ensures that $(\mu_k)_k$ satisfies the hypothesis of Proposition 3.2.7 with $s = t\ h_{top}(f,\varepsilon) - 8\gamma$ and $K = 1$. Since $F \subset X_{\varphi,f}$, then $h_{X_{\varphi,f}}(f,\varepsilon) \geq h_F(f,\varepsilon) \geq t\ h_{top}(f,\varepsilon) - 8\gamma$ and so $h_{X_{\varphi,f}}(f) \geq t\ h_{top}(f) - 8\gamma$. As $\gamma > 0$ and $t \in (0,1)$ were chosen arbitrary we conclude that $h_{X_{\varphi,f}}(f) = h_{top}(f)$.

Case 2: $0 = mdim(f) < h_{\text{top}}(f) = +\infty$.

The argument that $h_{X_{\varphi,f}}(f) = +\infty$ is explained at the end of Subsection 3.2.4.

Case 3: $0 < \overline{mdim}(f) < h_{\text{top}}(f) = +\infty$.

As the proof that $h_{X_{\varphi,f}}(f) = +\infty$ was discussed in Case 2, we are left to prove that $\underline{\text{mdim}}_{X_{\varphi,f}}(f) = \underline{\text{mdim}}(f)$ and $\overline{\text{mdim}}_{X_{\varphi,f}}(f) = \overline{\text{mdim}}(f)$. If $\underline{\text{mdim}}(f) = 0$ it is immediate. Otherwise, given $\varepsilon, \gamma, t > 0$ as before, Corollary 3.2.10 implies

$$\frac{h_{X_{\varphi,f}}(f,\varepsilon)}{-\log\varepsilon} \geq \frac{h_F(f,\varepsilon)}{-\log\varepsilon} \geq \frac{t\ (h_{top}(f,\varepsilon) - 7\gamma)}{-\log\varepsilon}.$$

and, consequently, $\underline{\mathrm{mdim}}_{X_{\varphi,f}}(f) \geq t\ \underline{\mathrm{mdim}}(f)$. Since that $t \in (0,1)$ is arbitrary, then $\underline{\mathrm{mdim}}_{X_{\varphi,f}}(f) = \underline{\mathrm{mdim}}(f)$. The proof that $\overline{\mathrm{mdim}}_{X_{\varphi,f}}(f) = \overline{\mathrm{mdim}}(f)$ is identical. This proves the theorem.

3.3 Proof of Corollary A

The proof of the corollary relies on the genericity of the gluing orbit property on chain recurrent classes of the non-wandering set, obtained in Proposition 4.2.2.

Let $\widetilde{\mathcal{R}_0}$ be as in the proof of Proposition 4.2.2. Fix an arbitrary $f \in \widetilde{\mathcal{R}_0}$ and let $\Omega(f) = \bigcup_{i=1}^{k} \Gamma_i$ be the decomposition of the non-wandering set of f in chain recurrent classes. If $h_{\text{top}}(f) = 0$ then we have nothing to prove, and we take $\mathcal{R}_f = C^0(X, \mathbb{R}^d)$. Otherwise, by assumption there exists $1 \leq j \leq k$ so that $h_{\text{top}}(f) = h_{\text{top}}(f\mid_{\Gamma_j}) > 0$ and $f\mid_{\Gamma_j}$ has the periodic gluing orbit property.

We claim that there exists a Baire generic subset $\mathcal{R}_f \subset C^0(X, \mathbb{R}^d)$ so that $\Gamma_j \cap X_{\varphi,f} \neq \emptyset$ for every $\varphi \in \mathcal{R}_f$. Observe that the latter implies on the corollary because

$$\mathfrak{R} := \bigcup_{f \in \widetilde{\mathcal{R}_0}} \{f\} \times \mathcal{R}_f \subset \mathrm{Homeo}(X) \times C^0(X, \mathbb{R}^d)$$

becomes a C^0-Baire generic subset and, by Theorem B, the set $\Gamma_j \cap X_{\varphi,f} \neq \emptyset$ is a full topological entropy subset of Γ_j.

Hence, it remains to prove the claim. As $h_{\text{top}}(f\mid_{\Gamma_j}) > 0$ and $f\mid_{\Gamma_j}$ satisfies the periodic gluing orbit property then there are countably many distinct periodic points in Γ_j. By Lemma 3.2.1, $\Gamma_j \cap X_{\varphi,f} \neq \emptyset$ if and only if there exists a pair of periodic points in Γ_j with different Birkhoff averages with respect to φ. Since periodic points are generically permanent (recall the proof of Proposition 4.2.2) it is clear that the latter defines a C^0-open and dense subset $\mathcal{R}_f \subset C^0(X, \mathbb{R}^d)$. This proves the claim and finishes the proof of the corollary.

Chapter 4

The set of points with non-trivial pointwise rotation set

The main goal of this chapter is to prove Theorems C and D, concerning on the set of points in $\mathbb{T}^2$ with non-trivial pointwise rotation set for typical homeomorphisms.

4.1 Volume preserving homeomorphisms

Our starting point for the proof of Theorem C is that specification is generic among volume preserving homeomorphisms. More precisely, for any compact Riemannian manifold M of dimension at least 2, there exists a residual subset $\mathcal{R}_2 \subset \text{Homeo}_\lambda(M)$ such that every homeomorphism in $\mathcal{R}_2$ satisfies the specification property [22]. Together with the fact that $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ is open in $\text{Homeo}_\lambda(\mathbb{T}^d)$ this ensures:

Corollary 4.1.1. *There is a residual $\mathcal{R}_3 \subset Homeo_{0,\lambda}(\mathbb{T}^d)$ such that every $f \in \mathcal{R}_3$ satisfies the specification property (hence the gluing orbit property).*

Theorem 4.1.2. *[23, Theorem 1] The set of all homeomorphisms with a stable rotation set is open and dense set $\mathcal{O} \subset Homeo_0(\mathbb{T}^2)$. Moreover, the rotation set of every such homeomorphism is a convex polygon with rational vertices, and in the area-preserving setting this polygon has nonempty interior.*

Given $f \in \text{Homeo}_0(\mathbb{T}^2)$ recall that $\rho(f)$ is called *stable* if there exists $\delta > 0$ so that $\rho(g) = \rho(f)$ for every $g \in \text{Homeo}_0(\mathbb{T}^2)$ so that $d_{C^0}(f, g) < \delta$. By [23] and [39] we obtain the following

Theorem 4.1.3. *The set of all homeomorphisms with a stable rotation set is open and dense in $\mathcal{D} \subset Homeo_0(\mathbb{T}^2)$ and for every $f \in \mathcal{D}$ the rotation set of $\rho(F)$ is a convex polygon* *with rational vertices. Moreover, there exists an open and dense $\mathcal{O} \subset Homeo_{0,\lambda}(\mathbb{T}^2)$ so that for every $f \in \mathcal{D}$ the rotation set $\rho(F)$ is a convex polygon with rational vertices and nonempty interior.*

We are now in a position to prove Theorem C.

4.1.1 Proof of Theorem C

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f and consider the observable (displacement function) $\varphi_F : \mathbb{T}^2 \to \mathbb{R}^2$ given by $\varphi_F(x) = F(\widetilde{x}) - \widetilde{x}$, where $\widetilde{x} \in \pi^{-1}(x)$. Since

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi_F(f^j(x)) = \frac{F^n(\widetilde{x}) - \widetilde{x}}{n}$$

we have that $\rho(F, \widetilde{x})$ coincides with the accumulation points of $(\frac{1}{n}\sum_{j=0}^{n-1} \varphi_F(f^j(x)))_{n\geq 1}$. Thus, the non-trivial pointwise rotation sets of x can be represented by

$$\mathbb{F}_f := \{x \in \mathbb{T}^2 : \rho(F, \widetilde{x}) \text{ is not trivial, where } \pi(\widetilde{x}) = x\}.$$

Take the residual subset

$$\mathfrak{R}_1 := \mathcal{R}_3 \cap \mathcal{O} \subset \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^2).$$

We claim that $\mathbb{F}_f$ is residual in $\mathbb{T}^2$ for every $f \in \mathfrak{R}_1$. Indeed, any $f \in \mathfrak{R}_1$, f satisfies the gluing orbit property and, if F is a lift of f, $\mathrm{int}(\rho(F)) \neq \emptyset$. The latter ensures that $\varphi_F \notin \overline{Cob}$ (recall Lemma 3.2.1) and $X_{\varphi_F, f} \neq \emptyset$. Theorem C is now a consequence of Theorems A and B.

Moreover, since that $\rho(F, \widetilde{x}) \subset \rho(F)$, for all $x \in \mathbb{T}^2$, we are done.

4.2 Dissipative homeomorphisms

In this section is to prove the Theorem D. Before, we introduce an useful concept.

4.2.1 Continuous maps with the gluing orbit property

Here we prove the genericity of the gluing orbit property on chain recurrent classes of the non-wandering set, a result of independent interest inspired by [6]. The following lemma is an easy consequence of compactness.

Lemma 4.2.1. *Let f be a continuous map on a compact metric space X such that $\overline{Per(f)} = \Omega(f)$, and let $\Gamma \subset \Omega(f)$ be a chain recurrent class. For any $\delta > 0$ there exists $m = m(\delta) \geq 1$ and a set of periodic orbits $\mathfrak{L} = \{\theta_1, \theta_2, \dots, \theta_m\} \subset \Gamma \cap Per(f)$ such that for any points $x, y \in \Gamma$ there exists $n < m$ and a δ-pseudo orbit $(x_i)_{i=1,\dots,n}$ so that $x_0 = x, x_n = y$ and $x_i \in \mathfrak{L}$ for every $0 < i < n$.*

Proof. Let $\delta > 0$. Using that every chain recurrent class is compact, there are points $z_1, z_2, \dots, z_l \in \Gamma$ such that $\Gamma \subset B(z_1, \delta/4) \cup B(z_2, \delta/4) \cup \dots \cup B(z_l, \delta/4)$. If $B(z_i, \delta/4) \cap B(z_j, \delta/4) \neq \emptyset$ $(i < j)$ by denseness of periodic points we can choose a point $z_i^j \in Per(f) \cap B(z_i, \delta/4) \cap B(z_j, \delta/4)$. Since chain recurrence classes are compact and isolated, reducing δ if necessary we may assume such periodic points belong to Γ. As $Per(f)$ is dense in Γ, then every periodic orbit for the flow is not isolated, meaning each of these are accumulated by distinct periodic orbits. Consider the collection of periodic points z_i^j chosen above and denote this by $\mathfrak{L}_0 = \{\theta_1^0, \theta_2^0, \dots, \theta_{m_0}^0\}$.

By construction, for any two points $x, y \in \Gamma$ there exist $1 \leq n \leq m_0$ and a sequence $(x_i)_{i=1,\dots,n-1}$ of periodic points in $\mathfrak{L}_0$ in such a way that

$$d(f(x), x_1) < \frac{\delta}{2}, \quad d(x_{n-1}, f^{-1}(y)) < \frac{\delta}{2} \quad \text{and} \quad d(x_i, x_{i+1}) < \frac{\delta}{2}, \quad \text{for all } \ 1 \leq i \leq n-2.$$

We will build a finite set $\mathfrak{L} \subset \mathfrak{L}_0$ so that every two points in Γ can be connected by a δ-pseudo-orbit formed by points in $\mathfrak{L}$. Let p_i denote the period of θ_i for every $1 \leq i \leq m_0$. So, consider the set

$$\mathfrak{L} = \left\{ f^j(\theta_i^0) : \theta_i^0 \in \mathfrak{L}_0 \ \text{ and } \ 0 \leq j \leq p_i - 1 \right\}$$

and write it, for simplicity, as $\mathfrak{L} = \{\theta_1, \theta_2, \dots, \theta_m\}$ where $m = m_0 + \sum_{i=1}^{m_0} p_i$.

Now consider the δ-pseudo-orbit $\{(x_i)\}$ connecting $x_0 = x$ to $x_n = y$ defined by:

$$\{(x_i)_i\} = \{x, x_1, f(x_1), \dots, f^{p_1 - 1}(x_1), x_2, \dots, f^{p_2 - 1}(x_2), \dots, x_{l-1}, \dots, f^{p_{l-1} - 1}(x_{l-1}), f^{-1}(y), y\}$$

and times

$$\{(t_i)_i\} = \{1, \underbrace{1, \dots, 1}_{p_1}, \dots, \underbrace{1, \dots, 1}_{p_{l-1}}, 1, 1\}.$$

The condition $n < m$ stated in the lemma is given by $n = 2 + \sum_{i=1}^{l-1} p_i$. By construction $n < m$. □

The latter implies on the following consequence:

Proposition 4.2.2. *Let X be a compact Riemannian manifold of dimension at least 2. There exists a Baire residual subset $\mathcal{R}_0 \subset Homeo_0(X)$ so that if $f \in \mathcal{R}_0$ and $\Gamma \subset \Omega(f)$ is a chain recurrent class, then the restriction $f\mid_\Gamma$ satisfies the (periodic) gluing orbit property.*

Proof. It follows from [12, 43] that there exists a residual subset $\widetilde{\mathcal{R}_0} \subset \mathrm{Homeo}(X)$ such that every $f \in \widetilde{\mathcal{R}_0}$ has the periodic shadowing property and $\overline{\mathrm{Per}(f)} = \Omega(f)$.

We claim that for any $f \in \widetilde{\mathcal{R}_0}$ and any chain recurrent class $\Gamma \subset \Omega(f)$ we have that $f\mid_\Gamma$ has the periodic gluing orbit property. Let $(\Gamma_i)_i$ denote the chain recurrent classes of f. Let $0 < \varepsilon < \min_{i\neq j} d(\Gamma_i, \Gamma_j)/2$ and let $\delta = \delta(\varepsilon) > 0$ by given by the periodic shadowing property. Let $\mathfrak{L} = \{\theta_1, \theta_2, \dots, \theta_m\}$ (depending on $\delta = \delta(\varepsilon)$) be given by Lemma 4.2.1 and set $K = K(\delta) = \sum_{1\le i\le m} \pi_i$, where $\pi_i \geq 1$ is the prime period of θ_i. Now, consider arbitrary points $x_1, x_2, \dots, x_k \in \Gamma$ and integers $n_1, \dots, n_k \geq 0$. By Lemma 4.2.1, for every $1 \le s \le k$ there exists a δ-pseudo orbit $(y_i^s)_{i=0,\dots,l_s}$ connecting the point $f^{n_{s-1}}(x_{s-1})$ and x_s and a δ-pseudo orbit $(y_i^{k+1})_{i=0,\dots,l_{k+1}}$ connecting the point $f^{n_k}(x_k)$ and x_1, all formed by at most m periodic points in $\mathfrak{L}$. Finally, consider the δ-pseudo-orbit $(x_i)_i$ connecting x_1 and $f^{n_k}(x_k)$ defined by:

$$\{x_1, f(x_1), \dots, f^{n_1-1}(x_1), y_0^1, y_1^1, \dots, y_{l_1-1}^1, x_2, f(x_2), \dots, f^{n_2-1}(x_2),$$

$$y_0^2, y_1^2, \dots, y_{l_2-1}^2, \dots, x_k, f(x_k), \dots, f^{n_k}(x_k), y_0^{k+1}, y_1^{k+1}, \dots y_{l_{k+1}}^{k+1}, x_1\}$$

Using the periodic shadowing property for f there exists a periodic point $z \in X$ so that $d(f^j(z), f^j(x_1)) < \varepsilon$ for every $0 \le j \le n_1$ and

$$d(f^{j+p_1+n_1+\dots+p_{i-1}+n_k}(z), f^j(x_i)) < \varepsilon, \quad \forall \;\; i \in \{2, \dots, k\}, \quad \forall \;\; j \in \{0, 1, \dots, n_i\}$$

where each p_s is bounded above by K. Since $0 < \varepsilon < \min_{i\neq j} d(\Gamma_i, \Gamma_j)/2$ we conclude that $z \in \Gamma$. This proves $f\mid_\Gamma$ satisfies the periodic gluing orbit property. Since $\mathrm{Homeo}_0(X)$ is open in $\mathrm{Homeo}(X)$ it is sufficient consider $\mathcal{R}_0 := \widetilde{\mathcal{R}_0} \cap \mathrm{Homeo}_0(X)$. □

In order to prove Theorem D consider the set $\mathcal{A} := \{f \in \mathrm{Homeo}_0(\mathbb{T}^2) : \mathrm{int}(\rho(F)) \neq \emptyset\}$, which does not depend on the lift F. Misiurewicz and Ziemian proved that $\mathcal{A}$ is open in $\mathrm{Homeo}_0(\mathbb{T}^2)$ [34, Theorem B].

The Theorem 1.2.5 and 4.1.3 is enough to prove the following:

Lemma 4.2.3. *Take $f \in \mathcal{A}$ and let F be a lift of f. There exists a $\Gamma \subset \Omega(f)$ chain recurrent class such that $\rho(F\mid_{\pi^{-1}(\Gamma)})$ is non-trivial. Moreover, $h_{top}(f\mid_\Gamma) > 0$.*

Proof. Suppose, by contradiction, that $\text{int}(\rho(F)) \neq \emptyset$ and that $\rho(F\mid_{\pi^{-1}(\Gamma)})$ is trivial for every chain recurrent class $\Gamma \subset \Omega(f)$. Since rational points in the $\text{int}(\rho(F))$ are realizable by periodic point of f, due Theorem 1.2.3, given a small disk $D \subset \text{int}(\rho(F)) \neq \emptyset$, there is $x \in \mathbb{T}^2$ and $\tilde{x} \in \pi^{-1}(x)$ such that $\rho(F, \tilde{x}) = D$ (by Theorem 1.2.5). In consequence $D \subset \rho(F\mid_{\pi^{-1}(\Gamma)})$ where Γ denotes the chain recurrent class of f containing the point x, leading to a contradiction. The previous argument shows that there is a chain recurrent class $\Gamma \subset \Omega(f)$ such that $\rho(F\mid_{\pi^{-1}(\Gamma)})$ has non-empty interior. By Theorem 1.2.5, this implies the conclusion of the lemma. $\square$

4.2.2 Proof of Theorem D

Let $\mathcal{R}_0$ be given by Proposition 4.2.2 and take the residual subset $\mathfrak{R}_2 = \mathcal{R}_0 \cap \mathcal{A} \cap \mathcal{D}$. Given $f \in \mathfrak{R}_2$, the variational principle for topological entropy together with the Poincaré recurrence theorem imply that the topological entropy is supported on the non-wandering set and $h_{top}(f) = \max\{h_{top}(f\mid_\Gamma)\}$, where the maximum is taken among the set of chain recurrent classes Γ in $\Omega(f)$.

We claim that there is a chain recurrent class Γ such that $X_{\varphi_F, f} \cap \Gamma$ is Baire residual in Γ. Indeed, as $\text{int}(\rho(F)) \neq \emptyset$, Lemma 4.2.3 implies that there is a chain recurrent class Γ such that $\text{int}(\rho(F\mid_\Gamma)) \neq \emptyset$. In particular $h_{top}(f\mid_\Gamma) > 0$, by Theorem 1.2.5. Since $\text{int}(\rho(F\mid_\Gamma)) \neq \emptyset$, the restriction of the displacement function $\varphi_F = F - Id$ on $\pi^{-1}(\Gamma)$ is not cohomologous to constant consequently $X_{\varphi_F, f} \cap \Gamma \neq \emptyset$.

Theorems A and B imply that $X_{\varphi_F, f} \cap \Gamma$ is Baire residual and has full topological entropy and full metric mean dimension in the chain recurrence class Γ. This proves the theorem.

Chapter 5

Rotation sets on $\mathbb{T}^d$ are generically convex

The main goal of this chapter is to prove Theorem E. If $p \in \mathbb{T}^d$ is a periodic point of prime period $k \geq 1$ (with respect to f) we denote by $\mu_p := \frac{1}{k}\sum_{j=0}^{k-1} \delta_{f^j(p)}$ the *periodic measure* associated to p. We will use the following:

Lemma 5.0.4. *Assume that $\Lambda \subset \mathbb{T}^d$ is a compact f-invariant set. If $f\mid_\Lambda$ satisfies the periodic gluing orbit property then periodic measures are dense in $\mathcal{M}_1(f\mid_\Lambda)$ (in the weak* topology).*

Proof. The proof is a simple modification of the arguments in [48] (where it is considered the case where f satisfies the specification property). We will include a brief sketch for completeness.

Let $(\psi_n)_{n\geq 1}$ be countable and dense in $C^0(\Lambda, \mathbb{R})$ and consider the metric d_* on $\mathcal{M}(\Lambda)$ given by $d_*(\nu, \mu) = \sum_{n\geq 1} \frac{1}{2^n}\big|\int \psi_n d\nu - \int \psi_n d\mu\big|$. This metric is compatible with the weak* topology in $\mathcal{M}(\Lambda)$. The compactness of $\mathcal{M}_1(f\mid_\Lambda)$ and the ergodic decomposition theorem, ensures that for any $\eta \in \mathcal{M}_1(f\mid_\Lambda)$ and $\zeta > 0$ there exists a probability vector $(\alpha_i)_{1\leq i\leq k}$ and ergodic measures $(\eta_i)_{1\leq i\leq k}$ so that $d_*(\eta, \hat\eta) < \zeta/2$, where $\hat\eta = \sum_{i=1}^k \alpha_i\eta_i$. It is enough to construct a periodic point $p \in \Lambda$ such that $d_*(\mu_p, \hat\eta) < \zeta/2$. By definition of weak* topology, one can choose $\varepsilon > 0$ so that if $d_n(x, y) < \varepsilon$, then $d_*(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}, \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(y)}) < \zeta/10$. Let $m(\varepsilon) > 0$ be given by the gluing orbit property. Choose $N \geq 1$ large and for any $1 \leq i \leq k$:

- pick $x_i \in \Lambda$ so that $d_*(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^n(x_i)}, \eta_i) < \zeta/10$ for every $n \geq N$,
- let $n_i \geq N$ be so that $\big|\frac{n_i}{\sum_{j=1}^k n_j} - \alpha_i\big| < \frac{\zeta}{10k}$ and $\frac{(k+1)m(\varepsilon)}{\sum_{j=1}^k n_j} \leq \frac{\zeta}{10}$.

By the periodic gluing orbit property there are positive integers $0 \leq m_i \leq m(\varepsilon)$ and a periodic point $p \in \Lambda$ of period $\kappa(p) = \sum_{i=1}^{k}(n_i + m_i)$ satisfying

$$d_{n_i}(f^{\sum_{j<i}(n_j+m_j)}(p), x_i) < \varepsilon \quad \text{for every } 1 \leq i \leq k.$$

Then, by triangular inequality, it is not hard to check that

$$\begin{aligned}
d_*(\mu_p, \hat{\eta}) &\leq d_*\Big(\frac{1}{\kappa(p)} \sum_{i=1}^{\kappa(p)} \delta_{f^i(p)}, \frac{1}{\kappa(p)} \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}\Big) \\
&\quad + d_*\Big(\frac{1}{\kappa(p)} \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}, \frac{1}{\sum_{j=1}^{k} n_j} \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}\Big) \\
&\quad + d_*\Big(\frac{1}{\sum_{j=1}^{k} n_j} \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}, \sum_{i=1}^{k} \frac{\alpha_i}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}\Big) \\
&\quad + d_*\Big(\sum_{i=1}^{k} \frac{\alpha_i}{n_i} \sum_{j=0}^{n_i-1} \delta_{f^j(x_i)}, \sum_{i=1}^{k} \alpha_i \eta_i\Big) \\
&\leq \frac{2\,k\,m(\varepsilon)}{\kappa(p)} + \frac{3\zeta}{10} \leq \frac{\zeta}{2},
\end{aligned}$$

which proves the lemma. □

5.0.3 Proof of Theorem E

Let $d \geq 2$ be an integer and let $\mathfrak{R}_3 := \mathcal{R}_3$ be the C^0-residual subset in $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ formed by homeomorphisms with the specification property (cf. Corollary 4.1.1). Given $f \in \mathcal{R}_3$ and a lift F recall that

$$\rho_{erg}(F) \subseteq \rho_p(F) \subseteq \rho(F) \subseteq \rho_{inv}(F),$$

where $\rho_{inv}(F)$ is convex.

We claim that $\rho(F) \supseteq \rho_{inv}(F)$ for every lift F of a homeomorphism $f \in \mathcal{R}_3$. Given an arbitrary $v \in \rho_{inv}(F)$ and $\eta \in \mathcal{M}_{inv}(f)$ so that $v = \int \varphi_F \, d\eta$. By Lemma 5.0.4 there exists a sequence $(p_n)_n$ of periodic points so that $\mu_{p_n} \to \eta$ as $n \to \infty$. In particular, since φ_F is continuous, if $\kappa(p_n) \geq 1$ denotes the prime period of p_n and $\tilde{p}_n \in \pi^{-1}(p_n)$, then

$$\frac{F^{\kappa(p_n)}(\tilde{p}_n) - \tilde{p}_n}{\kappa(p_n)} = \frac{1}{\kappa(p_n)} \sum_{j=0}^{\kappa(p_n)-1} \varphi_F(f^j(p_n)) \to \int \varphi_F \, d\eta = v \quad \text{as } n \to \infty.$$

This ensures that $v \in \rho(F)$. Therefore, $\rho(F) = \rho_{inv}(F)$ is convex and proves item (1) in the theorem. The proof of item (2) is completely analogous, using the rotation set on each chain recurrent class instead of the generalized rotation set, Proposition 4.2.2 instead of Corollary 4.1.1 and taking $\mathfrak{R}_4 := \mathcal{R}_0$.

Chapter 6

Flows with reparametrized gluing orbit property, suspension flows and rotation set

In this chapter we are interested in proves the Theorems F, G and H.

6.1 Proof of the Theorem F

In this section we prove the Theorem F, it is a generalization of the Theorem A, since that reparametrized gluing orbit property implies gluing orbit property. This Theorem implies the Corollary B. The proof follows a strategy of Li and Wu in [25]. Let $(X_t)_t : M \to M$ be a continuous flow with the reparametrized gluing orbit property on a compact Ψ-invariant $\Delta \subset X$ and let $\varphi : M \to \mathbb{R}^d$ be a continuous function. If I_φ is trivial we are done. Thus, suppose that I_φ is non-trivial.

Let $D \subset \Delta$ be a countable and dense set. Given $\varepsilon > 0$ fixed let $K(\varepsilon)$ be given by the gluing orbit property on Δ.

For $w \in \mathcal{L}_\varphi$, $\delta > 0$ and $n \in \mathbb{N}$ set

$$P(w, \delta, t) = \left\{ x \in \Delta : \left\| \frac{1}{t} \int_0^t \varphi(X_r(x))\, d\, r - w \right\| < \delta \right\}.$$

Clearly, for $w \in \mathcal{L}_\varphi$ and any $\delta > 0$ the set $P(w, \delta, t)$ is not empty for sufficiently large t.

By property of gluing, since that $I_\varphi \neq \emptyset$ we have that there exists $u, v \in \mathcal{L}_\varphi$ distinct, let $\{\delta_k\}_{k\geq 1} \searrow 0$ be a sequence of positive real numbers and $\{t_k\}_{k\geq 1} \nearrow \infty$ be a sequence of integers with $t_k \gg K_k$, meaning here $\lim_{k\to\infty} \frac{K_k}{t_k} = 0$, where $K_k := K(\varepsilon/2^k)$, so that $P(u, \delta_{2j-1}, t_{2j-1}) \neq \emptyset$ and $P(v, \delta_{2j}, t_{2j}) \neq \emptyset$, for all $j \geq 1$. Given $q \in D$ and $k \geq 1$, let $W_0 = \{q\}$. For $j \geq 1$ let W_{2j-1} be a maximal $(t_{2j-1}, 8\varepsilon)$-separated subset of $P(u, \delta_{2j-1}, t_{2j-1}) \neq \emptyset$

and W_{2j} be a maximal $(t_{2j}, 8\varepsilon)$-separated subset of $P(v, \delta_{2j}, t_{2j}) \neq \emptyset$. Choose a sequence of integers $\{N_k\}_{k\geq 1}$ so that

$$\lim_{k\to\infty} \frac{t_k+K_k}{N_k} = 0, \quad \text{and} \\ \lim_{k\to\infty} \frac{N_1(t_1+K_1)+\cdots+N_k(t_k+K_k)}{N_k} = 0. \tag{6.1.1}$$

We need the following auxiliary construction. The reparametrized gluing orbit property ensures that for every $\underline{x_k} := (x_1^k, \ldots, x_{N_k}^k) \in (W_k)^{N_k}$ there exists a point $y = y(\underline{x_k}) \in X$, a reparametrization $\tau \in Rep(\epsilon)$ and transition time functions

$$p_j^k : W_k^{N_k} \times \mathbb{R}_+ \to \mathbb{N}, \qquad j = 1, 2, \ldots, N_k - 1$$

bounded by K_k so that

$$d(X_{\tau(\mathrm{a}_j+t)}(y), X_t(x_j^k)) < \frac{\varepsilon}{2^k}, \quad \text{for every } \ t \in [0, t_k] \text{ and } \ j = 1, 2, \ldots, N_k - 1, \tag{6.1.2}$$

where

$$a_j = \begin{cases} 0 & \text{if } j = 1 \\ (j-1)\, t_k + \sum_{r=1}^{j-1} p_r^k & \text{if } j = 2, \ldots, N_k \end{cases}. \tag{6.1.3}$$

For $k \geq 1$ and $j \in \{1, 2, \ldots, N_k - 2\}$ we have that $p_j^k = p_j^k(x_1^k, x_2^k, \ldots, x_{N_k}^k, \varepsilon)$ is a function that describes the time lag that the orbit of $y = y(\underline{x_k})$ takes to jump from a $\frac{\varepsilon}{2^k}$-neighborhood of $X_{t_k}(x_j^k)$ to a $\frac{\varepsilon}{2^k}$-neighborhood of x_{j+1}^k, and it is bounded above by K_k.

We order the family $\{W_k\}_{k\geq 1}$ lexicographically: $W_k \prec W_s$ if and only if $k \leq s$. We proceed to make a recursive construction of points in a neighborhood of q that shadow points N_k in the family W_k successively with bounded time lags in between. More precisely, we construct a family $\{L_k(q)\}_{k\geq 0}$ of sets (guaranteed by the gluing orbit property) contained in a neighborhood of q and a family of positive integers $\{l_k\}_{k\geq 0}$ (also depending on q) corresponding to the time during the shadowing process. Let:

- $L_0(q) = \{q\}$ and $l_0 = N_0 t_0 = 0$;
- $L_1(q) = \{z = z(q, y(\underline{x_1})) \in X : \underline{x_1} \in W_1^{N_1}\}$ and $l_1 = p_0^1 + s_1$ with $s_1 = N_1 n_1 + \sum_{r=1}^{N_1-1} p_1^r$, where $z = z(q, y(\underline{x_1}))$ satisfies $d(z, q) < \frac{\varepsilon}{2}$ and $d(X_{\tau(p_0^1+t)}(z), X_{\tau(t)}(y(\underline{x_1}))) < \frac{\varepsilon}{2}$, for every $t \in [0, s_1]$ and $y(\underline{x_1})$ is defined by (6.1.2), $\tau \in Rep(\frac{\varepsilon}{2})$ and $0 \leq p_0^1 \leq K(\frac{\varepsilon}{2^2})$ is given by the gluing orbit property;
- $L_k(q) = \{z = z(z_0, y(\underline{x_k})) \in X \colon \underline{x_k} \in W_k^{N_k} \text{ and } z_0 \in L_{k-1}\}$, and $l_k = l_{k-1} + p_0^k + s_k$,

with $s_k = N_k n_k + \sum_{r=1}^{N_k-1} p_0^k$, where the shadowing point z satisfies

$$d(X_{\tau(t)}(z), X_{\tau(t)}(z_0)) < \frac{\varepsilon}{2^k}, \ \forall\, t \in [0, l_{k-1}]$$

and

$$d(X_{\tau(l_{k-1}+p_0^k+t)}(z), X_{\tau(t)}(y(\underline{x_k}))) < \frac{\varepsilon}{2^k}, \ \forall\, t \in [0, s_k].$$

for $\tau \in Rep(\frac{\varepsilon}{2^k})$.

Note that since $\tau \in Rep(\frac{\varepsilon}{2^k}) \subset Rep(\varepsilon)$, then $\tau(t_k + p_j^k) - \tau(t_k) \le (1+\epsilon)K_k$.

The previous points $y = y(\underline{x_k})$ are defined as in (6.1.2). By construction, for every $k \ge 1$

$$l_k = \sum_{r=1}^{k} N_r n_r + \sum_{r=1}^{k} \sum_{t=0}^{N_r-1} p_t^k. \tag{6.1.4}$$

Remark 6.1.1. Note that l_k and s_k are functions (as these depend on p_j^k) and, by definition of N_k cf. (6.1.1), one has that $\frac{\|l_k\|}{N_k} \le \frac{\sum_{r=1}^k (n_r+m_r)}{N_k}$ tends to zero as $k \to \infty$.

For every $k \ge 0$, $q \in D$ and $\varepsilon > 0$ define

$$\mathcal{R}_k(q,\varepsilon) = \bigcup_{z \in L_k(q)} \widetilde{B}^{\tau}_{l_k}\Big(z, \frac{\varepsilon}{2^k}\Big) \quad \text{and} \quad \mathcal{R}(q,\varepsilon) = \bigcap_{k=0}^{\infty} \mathcal{R}_k(q,\varepsilon),$$

where $\widetilde{B}^{\tau}_{l_k}(x,\delta)$ is the set of points $y \in X$ so that $d(X_{\tau(\alpha)}(x), X_\alpha(y)) < \delta$ for all $0 \le \alpha \le l_k - 1$. Note that τ depend of point z chosen, but we will omit it.

Consider also the sets

$$\mathcal{R} = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} R(q, \frac{1}{j}) = \bigcup_{j=1}^{\infty} \bigcup_{q \in D} \bigcap_{k=0}^{\infty} \bigcup_{z \in L_k(q)} \widetilde{B}^{\tau}_{l_k}\Big(z, \frac{1}{j2^k}\Big). \tag{6.1.5}$$

The following lemma, identical to Propositions 2.2 and 2.3 in [25], ensures that $\mathcal{R}$ is a Baire generic subset of X.

Lemma 6.1.2. *$\mathcal{R}$ is a G_δ-set and it is dense in X.*

Proof. First we prove denseness. It is enough to show that $\mathcal{R} \cap B(x,r) \neq \emptyset$ for every $x \in X$ and $r > 0$. In fact, given $x \in X$ and $r > 0$, there exists $j \in \mathbb{N}$ with $2/j < r$ and $q \in D$ such that $d(x,q) < 1/j$. Given $y \in \mathcal{R}(q, \frac{1}{j})$ it holds that $d(q,y) < \frac{1}{j}$ because $\mathcal{R}(q, \frac{1}{j}) \subset B(q, \frac{1}{j})$. Therefore, $d(x,y) \le d(x,q) + d(q,y) < 2/j < r$. This ensures that $\mathcal{R} \cap B(x,r) \neq \emptyset$.

Now we prove that $\mathcal{R}$ is a G_δ-set. Fix $j \in \mathbb{N}$ and $q \in D$. For any $k \ge 1$, consider the open set

$$G_k(q,\varepsilon) := \bigcup_{z \in L_k(q)} B^{\tau}_{l_k}\Big(z, \frac{\varepsilon}{2^k}\Big)$$

where $B^{\tau}_{l_k}(x,\delta)$ is the set of points $y\in X$ so that $d(X_{\tau(\alpha)}(x), X_\alpha(y)) < \delta$ for all $0\le\alpha\le l_{k-1}-1$. Note that $G_k(q,\varepsilon)\subset \mathcal{R}_k(q,\varepsilon)$ for any $k\ge 1$. We claim that $\mathcal{R}_{k+1}(q,\varepsilon)\subset G_k(q,\varepsilon)$ for any $k\ge 1$. The claim implies that

$$\mathcal{R}(q,\varepsilon) := \bigcap_{k=0}^{\infty}\mathcal{R}_k(q,\varepsilon) = \bigcap_{k=0}^{\infty} G_k(q,\varepsilon)$$

and guarantees that $\mathcal{R}$ is a G_δ-set.

Now we proceed to prove the claim. We prove that $\mathcal{R}_{k+1}(q,\varepsilon)\subset G_k(q,\varepsilon)$ for any $k\ge 1$. Given $y\in R_{k+1}(q,\varepsilon)$, there exists $z\in L_{k+1}(q)$ such that $y\in \widetilde{B}^{\tau}_{l_{k+1}}(z,\frac{\varepsilon}{2^{k+1}})$. By definition of $L_{k+1}(q)$, there exists $z_0\in L_k(q)$ and $\tau\in Rep(\frac{\varepsilon}{2^k})$ such that $d(X_{\tau(t)}(z), X_{\tau(t)}(z_0)) < \frac{\varepsilon}{2^{k+1}}$ for all $t\in[0,l_k]$. Thus,

$$\begin{aligned} d(X_t(y), X_{\tau(t)}(z_0)) &\le d(X_t((y), X_{\tau(t)}(z)) + d(X_{\tau(t)}(z), X_{\tau(t)}(z_0)) \\ &< \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k} \end{aligned}$$

for all $t\in[0,l_k]$. Therefore, $y\in G_k(q,\varepsilon)$. This proves the claim and completes the proof of the lemma. □

We must show that $\mathcal{R}\subset I_\varphi$. It sufficient to prove that $\mathcal{R}(\varepsilon,q)\subset I_\varphi$ for any ε and any $q\in D$. For any $\eta>0$ put $\mathrm{var}(\varphi,\eta) := \sup\{|\ \varphi(x)-\varphi(y)\ |: d(x;y)<\eta\}$ By compactness, $\mathrm{var}(\varphi,\eta)\to 0$ as $\eta\to 0$. We need the follows lemmas whose the proof are quite similar to Lemmas 2.1 and 2.2 in [25].:

Lemma 6.1.3. *For every $k\ge 1$, the following hold:*
(i) if k is odd and $y=y(\underline{x_k})$, then

$$\Big\|\int_0^{s_k}\varphi(X_{\tau(r)}(y))\,dr - s_k u\Big\| \le N_k t_k\Big(\mathrm{var}(\varphi,\frac{\varepsilon}{2^k})+\delta_k\Big) + 2(N_k-1)K_k\|\varphi\|_\infty;$$

(ii) if k is even, then

$$\Big\|\int_0^{s_k}\varphi(X_{\tau(r)}(y))\,dr - s_k v\Big\| \le N_k t_k\Big(\mathrm{var}(\varphi,\frac{\varepsilon}{2^k})+\delta_k\Big) + 2(N_k-1)K_k\|\varphi\|_\infty.$$

Proof. Let $k\ge 1$ be fixed and assume that it is odd (the case it is even is completely analogous). By construction of W_k and relation (6.1.2) there exists $(x_1^k,\dots,x_{N_k}^k)\in W_k^{N_k}$ so that

$$d(X_{\tau(a_j+t)}(y), X_t(x_j^k)) < \frac{\varepsilon}{2^k},$$

and

$$\begin{aligned} \Big\|\int_0^{t_k}\varphi(X_{\tau(a_j+r)}(y))\,dr - t_k u\Big\| &\le \Big\|\textstyle\int_0^{t_k}\varphi(X_{\tau(a_j+r)}(y))\,dr - \int_0^{t_k}\varphi(X_r(x_j^k))\,dr\Big\| \\ &\quad + \Big\|\textstyle\int_0^{t_k}\varphi(X_r(x_j^k))\,dr - t_k u\Big\| \qquad (6.1.6)\\ &\le t_k\Big(\mathrm{var}(\varphi,\tfrac{\varepsilon}{2^k})+\delta_k\Big), \end{aligned}$$

for every $j = 1, 2, \dots, N_k - 1$.

On the other hand, as $\|u\| \le \|\varphi\|_\infty$, we also have that

$$\Big\| \int_0^{p_j^k} \varphi(X_{\tau(a_j+t_k+r)}(y)) - p_j^k u \Big\| \le K_k(\|\varphi\| + \|u\|) \le 2K_k\|\varphi\|_\infty. \tag{6.1.7}$$

for every $j = 1, 2, \dots, N_k - 1$.

Moreover, decomposing the time interval $[0, s_k - 1]$ according to shadowing times and transition times

$$[0, s_k - 1] = \bigcup_{j=1}^{N_k} [b_j, b_j + t_k - 1] \cup \bigcup_{j=1}^{N_k - 1} [b_j + t_k, b_j + t_k + p_j^k - 1].$$

For times on the intervals $[b_j, b_j + t_k - 1]$ and $[b_j + t_k, b_j + t_k + p_j^k - 1]$ and using (6.1.6) and (6.1.7), respectively, we get

$$\Big\| \int_0^{s_k} \varphi(X_{\tau(r)}(y))\, dr - s_k\, u \Big\| \le N_k t_k \Big(\operatorname{var}\Big(\varphi, \frac{\varepsilon}{2^k}\Big) + \delta_k \Big) + (N_k - 1) 2K_k \|\varphi\|_\infty$$

as desired. □

The next lemma is a step in the proof that Birkhoff averages of points in $\mathcal{R}$ oscillate between the vectors u and v.

Lemma 6.1.4. *For every $k \ge 1$ the following hold:*
(i) if k is odd and $z \in L_k(q)$, then $\frac{1}{l_k}\big\| \int_0^{l_k} \varphi(X_{\tau(r)}(z))\, dr - \ u\big\| \to 0$ as $k \to \infty$; and

(ii) if k is even and $z \in L_k(q)$, then $\frac{1}{l_k}\big\| \int_0^{l_k} \varphi(X_{\tau(r)}(z))\, dr - \ v\big\| \to 0$ as $k \to \infty$.

Proof. (i) Fix $k \ge 0$ odd. Let $z = z(z_0, y(\underline{x_k})) \in L_k(q)$, $y = y(\underline{x_k})$ and $l_k = l_{k-1} + p_0^k + s_k$ be given as on the definition of $L_k(q)$. Then,

$$d(X_{\tau(l_{k-1}+p_0^k+t)}(z), X_{\tau(t)}(y)) \le \frac{\varepsilon}{2^{k-1}},$$

for every $t \in [0, s_k]$.

Hence,

$$\begin{aligned}
\Big\| \int_0^{l_k} \varphi(X_{\tau(r)}(z)) - l_k\, u \Big\| &\le \Big\| \int_0^{s_k} \varphi(X_{\tau(l_{k-1}+p_0^k+r)}(z))\, dr - s_k\, u \Big\| \\
&+ 2(l_{k-1} + p_0^k)\|\varphi\|_\infty \\
&\le \Big\| \int_0^{s_k} \varphi(X_{\tau(l_{k-1}+p_0^k+r)}(z))\, dr - \int_0^{s_k} \varphi(X_{\tau(r)}(y))\, dr \Big\| \\
&+ \Big\| \int_0^{s_k} \varphi(X_{\tau(r)}(y)) - s_k\, u \Big\| \\
&+ 2(l_{k-1} + p_0^k)\|\varphi\|_\infty.
\end{aligned}$$

Dividing the previous expression by l_k and appealing the Lemma 6.1.3 we obtain that

$$\begin{aligned}\frac{1}{l_k}\Big\| \int_0^{l_k} \varphi(X_{\tau(r)}(z)) - u\Big\| \;&\le\; s_k \operatorname{var}\Big(\varphi, \frac{\varepsilon}{2^k}\Big) + \frac{2(l_{k-1}+p_0^k)\|\varphi\|_\infty}{l_k} \\ &+ \frac{N_k s_k\Big(\operatorname{var}(\varphi, \frac{\varepsilon}{2^k}) + \delta_k\Big) + 2(N_k-1)K_k\|\varphi\|_\infty}{l_k}\end{aligned}$$

which goes to zero as $k\to\infty$, because $\frac{t_k}{l_k}\le 1$, $\frac{N_k n_k}{l_k}\le 1$ and

$$\frac{l_k+p_0^k}{l_k} \le \frac{N_0\, t_0 + \sum_{j=1}^{k} N_j(t_j+K_j) + K_k}{l_k} \to 0$$

when $k\to\infty$.

The proof of item (ii) is analogous. $\square$

We are now able to prove that $\mathcal{R}(q,\varepsilon)\subset I_\varphi$. If $x\in\mathcal{R}(q,\varepsilon)$, then $x\in \widetilde{B}^{\tau}_{l_{2k}}(z,\varepsilon/2^{2k})$ for some $z\in L_{2k}(q)$. Note that $z=z(z_0, y(\underline{x_{2k}}))$ satisfies $d(X_t(x),X_{\tau(t)}(z)) < \frac{\varepsilon}{2^{2k}}$ for all $t\in[0,l_{2k-1}-1]$ and $d(X_t(x),X_{\tau(t)}(z)) \le \frac{\varepsilon}{2^{2k}}$ for all $t\in[l_{2k-1}, l_{2k}-1]$.

On the other hand, there exists $z_0\in L_{2k-1}$ so that $d(X_{\tau(t)}(z),X_{\tau(t)}(z_0)) < \frac{\varepsilon}{2^{2k}}$ for all $t\in[0,l_{2k-1}-1]$. Consequently, $d(X_t(x),X_{\tau(t)}(z_0)) < \frac{\varepsilon}{2^{k-1}}$, for all $t\in[0,l_{2k-1}-1]$.

Therefore,

$$\begin{aligned}&\Big\|\int_0^{l_{2k-1}} \varphi(X_r(x))\,dr - l_{2k-1}u\Big\| \le \\ &\Big\|\int_0^{l_{2k-1}} \varphi(X_r(x))\,dr - \int_0^{l_{2k-1}} \varphi(X_{\tau(r)}(z_0))\,dr\Big\| + \Big\|\int_0^{l_{2k-1}} \varphi(X_{\tau(r)}(z_0))\,dr - l_{2k-1}u\Big\| \\ &\le\; l_{2k-1}\operatorname{var}\Big(\varphi,\frac{\varepsilon}{2^{2k-1}}\Big) + \Big\|\int_0^{l_{2k-1}} \varphi(X_{\tau(r)}(z_0))\,dr - l_{2k-1}u\Big\|.\end{aligned}$$

By Lemma 6.1.4 we have that $\Big\|\frac{1}{l_{2k-1}}\int_0^{l_{2k-1}} \varphi(X_r(x))\,dr - u\Big\| \to 0$ as $k\to\infty$.

In the similar way we can proves that $\Big\|\frac{1}{l_{2k}}\int_0^{l_{2k}} \varphi(X_r(x))\,dr - v\Big\| \to 0$ as $k\to\infty$.

This proves that $x\in I_\varphi$ and therefore $\mathcal{R}(q,\varepsilon)\subset I_\varphi$. The proof that $\mathcal{R}\subset I_\varphi$ is now complete.

6.2 Points with historic behavior for suspension flows

In this section we are interested to proves the Theorem G. For this, we need of the following Theorem.

Theorem 6.2.1. *[7, Theorem F] Let M be a metric space and let $f : M \to M$ with gluing orbit property. Assume the roof function $R : M \to \mathbb{R}_+$ is bounded from above and below, is uniformly continuous and the constants*

$$C_\xi = \sup_{n\geq 1} \sup_{y\in B(x,n,\xi)} \mid S_n R(x) - S_n R(y) \mid < \infty, \quad \textit{satisfy } \lim_{\xi\to 0} C_\xi = 0,$$

where $S_n R = \sum_{j=0}^{n-1} R \circ f^j$. Then the suspension flow $(Y_t)_t$ over f has the gluing orbit property.

Consider the set

$$\widehat{I}_\varphi := \left\{ x \in M : \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \psi_1 \circ f^s(x)}{\sum_{i=0}^{n-1} \psi_2 \circ f^s(x)} \text{ does not exists} \right\}$$

Inspired in [3], we can prove the following Theorem, using the techniques similar to Theorem A. Essentially the difference is that Hopf Theorem must be used for quotients of Birkhoff average instead of Birkhoff Theorem.

Theorem 6.2.2. *Let $(X_t)_t$ a suspension flow over $f : M \to M$ where f satisfies gluing orbit property, $\psi_1 : M \to \mathbb{R}^d$ and $\psi_2 : M \to \mathbb{R}^d$ continuous with $\inf\ \psi_2 > 0$. Then $\widehat{I}_\varphi$ is either empty or Baire residual.*

We are able to prove the Theorem G

Proof. Let $(X_t)_t$ a suspension flow over $f : M \to M$ and $\varphi : M \to \mathbb{R}^d$ continuous. Given $v \in M$ and $t > 1$, write $t = R^n(v) + q$ with $n \in \mathbb{N}$ and $0 \leq q < R(v)$, where $R : M \to \mathbb{R}_+$ is a measurable roof function given by $R^n(v) = \sum_{j=0}^{n-1} R \circ f^j(v)$. Thus,

$$\begin{aligned}
\lim_{t\to+\infty} \frac{1}{t} \int_0^t \varphi \circ X_r(v,s)\, dr &= \lim_{n\to+\infty} \frac{1}{R^n(v)+q} \int_0^{R^n(v)+q} \varphi \circ X_r(v,s)\, dr \\
&= \lim_{n\to+\infty} \frac{1}{R^n(v)} \int_0^{R^n(v)} \varphi \circ X_r(v,s)\, dr \\
&= \lim_{n\to+\infty} \frac{1}{R^n(v)} \sum_{j=0}^{n-1} \int_0^{R(f^j(v))} \varphi \circ X_r(f^j(v),s)\, dr \\
&= \lim_{n\to+\infty} \frac{Z^n(v)}{R^n(v)},
\end{aligned}$$

where $Z(v) = \int_0^{R(v)} \varphi \circ X_r(f^n(x),s)\, dr$ and $Z^n(v) = \sum_{j=0}^{n-1} \int_0^{R(f^j(v))} \varphi \circ X_r(f^j(v))\, dr$.

Therefore, we can write the irregular set as follows

$$I_\varphi = \left\{ (v,s) \in M_r : \lim_{n\to+\infty} \frac{Z^n(v)}{R^n(v)} \text{ does not exists} \right\}.$$

Using the Theorems 6.2.1 we have that $(X_t)_t$ satisfies the gluing orbit property. Thus, we can use the Theorem 6.2.2 with $\psi_1 = \int_0^{R(\cdot)} \varphi \circ X_r(\cdot)\, dr$ and $\psi_2 = R$, note that $\inf R > 0$. Therefore, I_φ is either empty or residual. $\square$

6.3 Suspension flows and rotation set

In this section we are interested to prove the Theorem H.

Proof. Suppose that roof function $r : \Sigma \to (0, \infty)$ is coboundary, in particular R is coboundary. Let $t \geq 1$ and $n = n(x,s)$ such that $R^n(x) \leq t < R^{n+1}(x)$.

As $R^n(x) \leq t < R^{n+1}(x) \leq R^n(x) + \|R\|_\infty$, we have that

$$1 \leq \frac{t}{R^n(x)} \leq 1 + \frac{\|R\|_\infty}{R^n(x)}$$

and consequently,

$$\lim_{t\to\infty} \frac{t}{R^n(x)} = 1.$$

Let $(x,s) \in \widetilde{M}_r$ so that $\rho(F,x)$ is a rational vertices of $\rho(F)$ and suppose (from the previous paragraph) that $t \simeq R^n(x)$, then

$$\frac{Y_t(x,s) - (x,s)}{t} \simeq \frac{1}{R^n(x)} \left(F^n(x) - x, R^n(x)\right) = \left(\frac{F^n(x) - x}{n} \frac{n}{R^n(x)}, 1\right).$$

The term $\frac{n}{R^n(x)}$ can be written by $\frac{1}{\frac{1}{n}\sum_{j=0}^{n-1} R(f^j(x))}$. Since R is coboundary there are $c \in \mathbb{R}$ and χ bounded such that $R = c + \chi - \chi \circ f$, thus

$$\frac{1}{n}\sum_{j=0}^{n-1} R(f^j(x)) = \frac{1}{n}\left(n\, c + \chi - \chi \circ f^n\right).$$

In particular,

$$\frac{1}{n}\sum_{j=0}^{n-1} R(f^j(x)) \longrightarrow c, \text{as } n \to \infty.$$

Therefore, $\frac{n}{R^n(x)} \longrightarrow \frac{1}{c}$, as $n \to \infty$.

So,

$$\begin{aligned} \rho((Y_t)_t, (x,s)) &= \lim_{t\to\infty} \frac{Y_t(x,s) - (x,s)}{t} \\ &= \lim_{n\to\infty} \left(\frac{F^n(x) - x}{n} \frac{n}{R^n(x)}, 1\right) \\ &= \left(\rho(F,x)\frac{1}{c}, 1\right). \end{aligned}$$

Note that such limit exists, because $\frac{n}{R^n(x)} \longrightarrow \frac{1}{c}$, as $n \to \infty$ and x was chosen so that $\rho(F, x)$ exists.

So, doing this for every vertex of $\rho(F)$ and knowing that $\rho((Y_t)_t)$ is connected, we get that $\rho((Y_t)_t) = \left(\frac{\rho(F)}{c}, 1\right)$ (is possible that $\rho((Y_t)_t)$ is a class). This is, $\rho((Y_t)_t)$ is a polygon. $\square$

Chapter 7

Some comments and further questions

To finish we will make some comments on related concepts and future perspectives. First, the general concept of multifractal analysis is to decompose the phase space in subsets of points which have a similar dynamical behavior and to describe the size of each of such subsets from the geometrical or topological viewpoint. We refer the reader to the introduction of [37] and references therein for a historical overview. The study of the topological pressure or Hausdorff dimension of the level and the irregular sets can be traced back to Besicovitch. Such a multifractal analysis program has been carried out successfully to deal with self-similar measures and Birkhoff averages [37, 38, 42, 54], among other applications. We expect that our methods can be applied in other related problems as the multifractal analysis of level sets for Birkhoff averages.

A different question that can be endorsed concerns the concept of localized entropy. In [32], studied the directional $H(v)$ entropy (in the direction of a rotation vector v) introduced in [26] (we refer the reader to [26, 32] for the definition). They prove that, if the localized entropy satisfies some mild continuity assumptions, the localized entropy associated to locally maximal invariant set of $C^{1+\alpha}$-diffeomorphisms is entirely determined by the exponential growth rate of periodic orbits whose rotation vectors are sufficiently close to v (cf. [32, Theorem 5] for the precise statement). While it is not hard to check that any fixed rotation vector v there exist points whose pointwise rotation set coincides with v in the case of maps with the gluing orbit property, we expect that the inequality $H(v) \leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \#Per(v, n, \varepsilon)$ holds.

One different question concerns the Hopf ratio ergodic theorem. More precisely, although we did not pursue this here, it is most likely that our results can describe the set of points with historic behavior for quotients of Birkhoff sums in the spirit of [2, 52], with possible applications to the case of suspension flows over continuous maps with the gluing orbit property, considered in [7].

Finally, the convexity of the rotation set played a key role on the rotation theory for homeomorphisms on the 2-torus. Hence, we expect Theorem E to contribute for the development of the rotation theory for generic conservative homeomorphisms on tori. In particular, taking into account [39], an interesting open question is whether the rotation set of a C^0-generic homeomorphisms on $\mathbb{T}^d$ homotopic to the identity is a rational polyhedron.

We work in the case of flows trying to generalize the theorems done here in the discrete case. We are most interested in the study of flows for the rotation set. In the case of the Theorem H, the reciprocal we believe to be true. Working on it.

Appendix: Lifts for continuous flows on $\mathbb{T}^d$

Let $d \in \mathbb{N} \setminus \{0\}$. The map $\pi : \mathbb{T}^d \to \mathbb{R}^d$ given by $x \mapsto x(\mod \mathbb{Z}^d)$ is called the natural projection. A lift of a map $f : \mathbb{T}^d \to \mathbb{T}^d$ is a map $F : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies $f \circ \pi = \pi \circ F$. The own of this section is to show how construct a lift for homotopic to identity continuous map on $\mathbb{T}^d$, with $d \geq 1$ and for continuous flows on $\mathbb{T}^3$.

Let X be a topological space. A *covering space* of X is a topological space $\mathfrak{C}$ together with a continuous surjective map $p : \mathfrak{C} \to X$ such that for every $x \in X$, there exists an open neighborhood U of x, such that $p^{-1}(U)$ is a countable union of disjoint open sets $C_1, \ldots, C_k$ subset in $\mathfrak{C}$, so that $p \mid C_i : C_i \to U$ is a homeomorphism for every $i \in \{1, \ldots, k\}$, i.e., each of which is mapped homeomorphically onto U by p. The map p is called the *covering map* and the space X is called the *base space* of the covering, and the space $\mathfrak{C}$ is called the *total space* of the covering. A covering space is the *universal covering space* if it is simply connected.

A exemple of covering map is the natural projection $\pi : \mathbb{R}^d \longrightarrow \mathbb{T}^d$, with $d \in \mathbb{N}$ and the covering space of $\mathbb{T}^d$ is $\mathbb{R}^d$.

Consider $f : \mathbb{T}^d \to \mathbb{T}^d$ a continuous map, we will show how to get a lift $F : \mathbb{R}^d \to \mathbb{R}^d$ from f. For this, we use the following auxiliar Proposition 6.7 page 135 due Elon L. Lima, [16].

Lemma 7.0.1. *Let X a space metric compact and $\gamma : [0,1] \longrightarrow X$ a continuous curve. Given $x_0 \in \pi^{-1}(\gamma(0))$ there exists a unique $\widehat{\gamma} : [0,1] \longrightarrow \widehat{X}$, where $\widehat{X}$ is universal covering of X, such that $\pi(\widehat{\gamma}(t)) = \gamma(t), \quad \forall\ t \ \in [0,1]$ and $\widehat{\gamma}(0) = x_0$.*

Let $d \geq 1$, $x \in \mathbb{T}^d$ and $y := f(x)$. Let $\alpha : [0,1] \longrightarrow \mathbb{T}^d$ a curve such that $\alpha(0) = x$ and $\alpha(1) = y$, by Lemma 7.0.1 there exist a curve $\widetilde{\alpha} : [0,1] \longrightarrow \mathbb{R}^d$, lift, which is a copy of α on $\mathbb{R}^d$ such that $\widetilde{\alpha}(0) = \widetilde{x}$ and $\widetilde{\alpha}(1) = \widetilde{y}$. Since π is continuous and onto, follows that $\pi(\widetilde{\alpha}) = \alpha$. Therefore, we will define a function $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ that satisfies $F(\widetilde{x}) = \widetilde{y}$.

Note that under these circumstances

$$\pi(F(\widetilde{x})) = y = f(x) = f(\pi(\widetilde{x})).$$

Let $\mathcal{U}_0$ a open neighborhood of x. For every point $z_0 \in \mathcal{U}_0 \cap \mathbb{T}^d$ we can repeat the idea of the previous paragraph to define α connecting x to z_0. Thus we obtain a function F defined in the whole set $\mathcal{U}_0$ such that if $x \in \mathcal{U}_0$ and $y := f(x)$, then $F(\widetilde{x}) = \widetilde{y}$.

Since $\mathbb{T}^d$ is compact we can define F in the whole $\mathbb{T}^d$. Finally, since $\mathbb{R}^d$ is a universal covering of $\mathbb{T}^d$ we can repeat construction above using Lemma 7.0.1 and getting $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ (lift of f), given by $F(\widetilde{x}) = \widetilde{y}$.

Remark 7.0.2. Hold that $f \circ \pi(y) = \pi \circ F(y)$. In fact, let $y \in \mathbb{R}^d$ and $\pi(y) =: x \in \mathbb{T}^d$. Then there exist a curve $\widetilde{\alpha}$ such that $\widetilde{\alpha}(0) = y$ and $\widetilde{\alpha}(1) = F(y)$ and a curve $\alpha = \pi(\widetilde{\alpha})$ such that $\alpha(0) = x$ and $\alpha(1) = f(x)$. Therefore,

$$f(\pi(y)) = f(\pi(\widetilde{\alpha}(0))) = f(\alpha(0)) = \alpha(1) = \pi(\widetilde{\alpha}(1)) = \pi(F(y)).$$

Let a continuous flow $(X_t)_t$ on $\mathbb{T}^3$ we will construct o lift $(Y_t)_t$ on $\mathbb{R}^3$: Let $x \in \mathbb{T}^3$, $t \in \mathbb{R}$ and $y := X_t(x)$. So, we can consider $\alpha : [0,1] \longrightarrow \mathbb{T}^3$ a curve such that $\alpha(0) = x$ and $\alpha(1) = y$, by Lemma 7.0.1 there exist a curve $\widetilde{\alpha} : [0,1] \longrightarrow \mathbb{R}^3$, which is a copy of α on $\mathbb{R}^3$ such that $\widetilde{\alpha}(0) = x$ and $\widetilde{\alpha}(1) = y$. Since $\pi : \mathbb{R}^3 \longrightarrow \mathbb{T}^3$ is continuous and onto, follows that $\pi(\widetilde{\alpha}) = \alpha$. Fixed t we can define a function $Y_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $Y_t(\widetilde{\alpha}(0)) = \widetilde{\alpha}(1)$. Note that hold

$$\pi(Y_t(x)) = X_t(\pi(x)), \text{ for all } x \in \mathbb{R}^3$$

and

$$Y_t(x+v) = Y_t(x) + v, \text{ for all } x \in \mathbb{R}^3 \text{ and } v \in \mathbb{Z}^3.$$

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