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Tese de Doutorado

Pesin's Entropy Formula for C^1 non-uniformly expanding maps.

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Orientador: **Prof. Dr. Vitor D. Martins de Araújo**

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Pesin's Entropy Formula for C^1 non-uniformly expanding maps.

Tese de Doutorado submetida ao Colegiado de Pós-graduação em Matemática UFBA/UFAL como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

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Dedico este trabalho
aos meus pais, meus irmãos,
à Lara e à minha noiva Milena,
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“Nele estão escondidos todos os
tesouros da sabedoria e da ciência.”
Colossenses 2:3.

Resumo

Provamos a existência de estados de equilíbrio com propriedades especiais para uma classe de homeomorfismos locais positivamente expansivos e potenciais contínuos, definidos em espaço métrico compacto. Além disso, formulamos uma generalização de classe C^1 da Fórmula de Entropia de Pesin: toda medida ergódica weak-SRB-like satisfaz a Fórmula de Entropia de Pesin para transformações de classe C^1 não uniformemente expansoras. Mostramos que para transformação expansora fraca, tal que Leb-q.t.p x tenha frequência positiva de tempos hiperbólicos, medidas weak-SRB-like, existem e satisfazem a Fórmula de Entropia de Pesin e são estados de equilíbrio para o potencial $\psi = -\log|\det Df|$. Em particular, isso é válido para qualquer transformação expansora de classe C^1 e neste caso o conjunto de medidas de probabilidade invariantes que satisfazem a Fórmula de Entropia de Pesin é o fecho convexo na topologia fraca* das medidas ergódicas weak-SRB-like.

Palavras-chave: Teoria Ergódica, expansão não uniforme, Medidas SRB\física-fraca, Estados de Equilíbrio e Fórmula de Entropia de Pesin.

Abstract

We prove existence of equilibrium states with special properties for a class of distance expanding local homeomorphisms on compact metric spaces and continuous potentials. Moreover, we formulate a C^1 generalization of Pesin's Entropy Formula: all ergodic weak-SRB-like measures satisfy Pesin's Entropy Formula for C^1 non-uniformly expanding maps. We show that for weak-expanding maps such that Leb-a.e x has positive frequency of hyperbolic times, then all the necessarily existing ergodic weak-SRB-like measures satisfy Pesin's Entropy Formula and are equilibrium states for the potential $\psi = -\log|\det Df|$. In particular, this holds for any C^1-expanding map and in this case the set of invariant probability measures that satisfy Pesin's Entropy Formula is the weak*-closed convex hull of the ergodic weak-SRB-like measures.

Keywords: Ergodic theory, non-uniform expansion, SRB\physical-like measures, equilibrium states, Pesin's entropy formula.

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Capítulo 1

Introdução e descrição dos resultados

Os estados de equilíbrio, um conceito originalmente da Mecânica Estatística, é uma classe especial de medidas de probabilidade. Na definição clássica, dadas uma transformação contínua $T : X \to X$ definida em um espaço métrico compacto X e uma função contínua $\phi : X \to \mathbb{R}$ chamada de potencial, dizemos que uma medida de probabilidade T-invariante é ϕ-estado de equilíbrio (ou estado de equilíbrio para o potencial ϕ) se

$$P_{\text{top}}(T, \phi) = h_\mu(T) + \int \phi \, d\mu, \text{ onde } P_{\text{top}}(T, \phi) = \sup_{\lambda \in \mathcal{M}_T} \left\{ h_\lambda(T) + \int \phi \, d\lambda \right\},$$

e $\mathcal{M}_T$ é o conjunto de todas as medidas de probabilidade T-invariantes. Nesta relação, $P_{\text{top}}(T, \phi)$ é chamada *pressão topológica* e a igualdade acima do lado direito é conhecida por *Princípio Variacional*; veja por exemplo [47, 29] para as definições de entropia $h_\mu(T)$ e pressão topológica. Postergamos as definições formais para os próximos capítulos.

Dependendo do sistema dinâmico e potencial estudados (T, ϕ), os estados de equilíbrio podem ter propriedades que fornecem informação útil sobre o sistema. Sinai, Ruelle e Bowen [38, 13, 12, 34] foram os pioneiros da teoria dos estados de equilíbrio para sistemas dinâmicos uniformemente hiperbólicos. Eles estabeleceram uma importante conexão entre estados de equilíbrio e medidas que são caracterizadas por fornecer informações assintóticas sobre um conjunto de trajetórias, isto é, dada uma medida μ, esperamos que sua *bacia*

$$B(\mu) = \left\{ x \in X : \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j x) \xrightarrow[n \to +\infty]{} \int \phi d\mu, \ \forall \phi \in C^0(X, \mathbb{R}) \right\}$$

seja grande do ponto de vista da medida de Lebesgue ou alguma outra medida de referência relevante. Essas medidas são chamadas *medidas SRB* em homenagem a Sinai, Ruelle e Bowen, e também *medidas físicas* devido à origem de muitos conceitos na Mecânica Estatística.

Apesar do substancial progresso de vários autores em estender essa teoria, uma visão global ainda está longe de ser completa. Por exemplo, a estratégia básica utilizada por Sinai-Ruelle-Bowen foi (semi-)conjugar a dinâmica a um subshift de tipo finito, através de uma partição Markov, e usar as fortes propriedades topológicas e ergódicas das Cadeias de Markov Topológicas para obter resultados para a dinâmica uniformemente hiperbólica original.

No entanto, a existência de partições de Markov é conhecida apenas em alguns casos e, muitas vezes, tais partições podem não ser finitas. Além disso, existem transformações que não admitem estados de equilíbrio ou medidas físicas/SRB (como por exemplo, a transformação identidade).

Em muitas situações com algum tipo de expansão, é possível garantir que os estados de equilíbrio sempre existam. No entanto, os estados de equilíbrio geralmente não são únicos e, pela forma como são obtidos, não fornecem informações adicionais para o estudo da dinâmica. No contexto das funções uniformemente expansoras, Ruelle mostrou em [35] que existe um único estado de equilíbrio que é uma medida física/SRB se o potencial for Hölder contínuo e a dinâmica f for transitiva.

Neste sentido, surgem algumas questões naturais:

1. Quando um sistema dinâmico tem um estado de equilíbrio?

2. Se tais medidas existem, que propriedades estatísticas essas probabilidades exibem em relação à medida de referência (possivelmente não-invariante)?

3. Quando existe, o estado de equilíbrio é único?

A existência de estados de equilíbrio é uma propriedade relativamente mais simples que muitas vezes pode ser estabelecida através de argumentos de compacidade. Entretanto, as propriedades estatísticas e a unicidade do estado de equilíbrio são geralmente mais sutis e requerem uma melhor compreensão da dinâmica. Em nosso contexto, não esperamos ter uma unicidade de estado de equilíbrio, porque consideramos:

- sistemas dinâmicos com baixa regularidade: contínua ou de classe C^1 somente; e

- apenas potenciais contínuos (com exceção do Teorema A).

Do Formalismo Termodinâmico a resposta para a primeira questão é conhecida e afirmativa para todas as transformações expansivas (também chamadas transformações Ruelle expansiva), em um espaço métrico compacto X e para todo potencial contínuo; veja [37, 29].

Inspirado na definição de medida observável introduzida por Keller em [25], Catsigeras-Henrich apresentam em [18] a noção de medidas "SRB-like" ou pseudo-físicas (uma generalização da noção de medida física/SRB) e mostram que tais medidas sempre existem para toda transformação contínua definida em um espaço métrico compacto. Além disto, eles mostram que a definição de medida "SRB-like" é ótima em certo sentido, se o objetivo for descrever as estatísticas assintóticas de Lebesgue quase toda órbita do sistema. Mais precisamente, dado um ponto $x \in X$ denotamos o conjunto

$$p\omega(x) = \left\{ \mu \in \mathcal{M}_T; \exists n_j \xrightarrow[j\to+\infty]{} +\infty \text{ tal que } \sigma_{n_j}(x) = \frac{1}{n_j}\sum_{i=0}^{n_j-1} \delta_{T^i(x)} \xrightarrow[j\to+\infty]{w^*} \mu \right\}.$$

Fixado uma medida de referência ν (não necessariamente a medida de Lebesgue) para o espaço métrico X, dizemos que $\mu \in \mathcal{M}_T$ é ν-SRB-like se, e só se, $\nu(A_\varepsilon(\mu)) > 0$ para todo $\varepsilon > 0$, onde

$$A_\varepsilon(\mu) = \{x \in X;\ \text{dist}(p\omega(x), \mu) < \varepsilon\}$$

é a bacia ε-fraca de atração de μ e dist é uma métrica que induz a topologia fraca* no espaço das medidas de probabilidades Boreliana.

Mostramos que, em nosso contexto, as medidas "ν-SRB-like" são particularmente interessantes, porque podem ser vistas como medidas que surgem naturalmente como pontos de acumulação de medidas ν-SRB.

Teorema A. *Seja $T : X \to X$ uma transformação aberta, expansiva e topologicamente transitiva definida em um espaço métrico compacto X. Considere $(\phi_n)_{n\geq1}$ uma sequência de potenciais Hölder contínuas, $(\nu_n)_{n\geq1}$ uma sequência de medidas conformes associados a (T, ϕ_n) e $(\mu_n)_{n\geq1}$ uma sequência de medidas ν_n-SRB. Assuma que:*

1. $\phi_{n_j} \xrightarrow[j\to+\infty]{} \phi$ *na topologia da convergência uniforme;*

2. $\nu_{n_j} \xrightarrow[j\to+\infty]{w^*} \nu$ *na topologia fraca* de convergência de medidas;*

3. $\mu_{n_j} \xrightarrow[j\to+\infty]{w^*} \mu$ *também na topologia fraca*.*

Então ν é medida conforme para (T, ϕ) e μ é ν-SRB-like. Em particular μ é ϕ-estado de equilíbrio. Além disso, se T é topologicamente exata então $\nu(X \backslash A_\varepsilon(\mu)) = 0$ para todo $\varepsilon > 0$.

Relacionando agora as transformações que expandem distância e as medidas "ν-SRB-like", obtemos uma resposta positiva para a segunda pergunta.

Teorema B. *Seja $T : X \to X$ uma transformação aberta, expansiva e topologicamente transitiva, definida em um espaço métrico compacto X e $\phi : X \to \mathbb{R}$ contínua. Então, para cada (necessariamente existente) medida ϕ-conforme $\nu = \nu_\phi$ todas as (necessariamente existentes) medidas ν_ϕ-SRB-like são ϕ-estado de equilíbrio.*

Esse resultado, em particular, estende o principal resultado obtido por Catsigeras-Henrich em [19], onde foram estudadas apenas as medidas "SRB-like" com respeito à medida de Lebesgue para dinâmicas C^1 expansoras do círculo.

Em [17], Catsigeras, Cerminara e Henrich estendem a noção de medidas SRB-like, as medidas "weak-SRB-like" ou weak pseudo-física. Diz-se que uma medida $\mu \in \mathcal{M}_T$ é ν-weak-SRB-like se, e só se,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \nu(A_{\varepsilon,n}(\mu)) = 0 \ \ \forall \varepsilon > 0,$$

onde $A_{\varepsilon,n}(\mu) = \{x \in X : \ \mathrm{dist}(\sigma_n(x), \mu) < \varepsilon\}$ é a bacia ε-fraca de μ em tempo n.

Observamos que se μ é um ϕ-estado de equilíbrio e se "relaciona bem" com a medida de referência ν então μ é combinação convexa fraca* de medidas ergódicas weak-SRB-like. Além disso, se μ é o único ϕ-estado de equilíbrio então μ possui cota superior de grandes desvios, isto é, para qualquer vizinhança fraca* $\mathcal{V}$ de μ em $\mathcal{M}_1$ (espaço das medidas de probabilidade Boreliana), então a probabilidade $\nu(\{x \in X : \ \sigma_n(x) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)} \in \mathcal{M}_1 \backslash \mathcal{V}\})$ decresce exponencialmente com n em uma taxa que depende do "tamanho" de $\mathcal{V}$.

Corolário C. *Sejam $T : X \to X$ uma transformação aberta, expansiva e topologicamente transitiva, definida em um espaço métrico compacto X, $\phi : X \to \mathbb{R}$ um potencial contínuo e ν uma medida ϕ-conforme. Se μ é um ϕ-estado de equilíbrio tal que $h_\nu(T, \mu) < \infty$ então quase toda componente ergódica μ_x de μ é uma medida de probabilidade ν-weak-SRB-like. Além disso, se μ é o único ϕ-estado de equilíbrio então μ é uma medida ν-SRB tal que $\nu(B(\mu)) = 1$ e μ satisfaz a seguinte cota* *superior de grandes desvios: para toda vizinhança fraca* $\mathcal{V}$ de μ temos*

$$\limsup_{n\to+\infty}\frac{1}{n}\log\nu(\{x\in X:\ \sigma_n(x)\in\mathcal{M}_1\backslash\mathcal{V}\})\leq -I(\mathcal{V})$$

onde $I(\mathcal{V})=\sup\{r>0:\mathcal{K}_r(\phi)\subset\mathcal{V}\}$.

Considere agora M uma variedade Riemanniana compacta finita sem bordo. Na década de 1970, Pesin mostrou uma relação entre dois importantes conceitos matemáticos: os expoentes de Lyapunov e a entropia para sistemas dinâmicos diferenciáveis.

Mais precisamente, em [28], Pesin mostrou que se μ é uma medida f-invariante para um difeomorfismo de classe C^2 (ou $C^{1+\alpha}$, $\alpha>0$) definido em uma variedade compacta que é absolutamente contínua com respeito a medida de Lebesgue, então

$$h_\mu(f)=\int\Sigma^+d\mu, \tag{1.0.1}$$

onde Σ^+ denota a soma dos expoentes de Lyapunov positivos em pontos regulares, contando suas multiplicidades. Dizemos que uma medida f-invariante μ satisfaz a Fórmula de entropia de Pesin se μ satisfaz a equação (1.0.1).

Ledrappier e Young em [26], no mesmo contexto que Pesin, generalizaram a fórmula de entropia de Pesin, mostrando que uma medida invariante satisfaz a Fórmula de Entropia de Pesin se, e somente se, tiver medidas condicionais absolutamente contínuas em relação a medida de Lebesgue nas folhas instáveis.

Podemos também citar, Liu, Qian e Zhu que obtiveram em [27, 30] generalizações da fórmula de entropia de Pesin para endomorfismos de classe C^2.

Para funções de classe C^1, a ausência de distorção limitada impossibilita o uso de muitos métodos já conhecidos para obter a Fórmula de Entropia de Pesin para dinâmicas de classe C^2 (ou $C^{1+\alpha}$, $\alpha>0$). Recentemente, Xueting Tian mostrou em [44] a fórmula de entropia de Pesin para dinâmica entre C^1 e $C^{1+\alpha}$.

Entretanto, podemos encontrar diversos resultados já obtidos para sistemas dinâmicos de classe C^1:

- Ali Tahzibi em [40] mostrou a fórmula de entropia de Pesin para difeomorfismos C^1 genéricos que preserva área em superfícies de dimensão 2;

- Bessa-Varandas em [10] mostraram que vale o resultado anterior para tempo contínuo (fluxos incompreensíveis) em variedades de dimensão 3;

- Hao Qiu em [31] provou que se f é Anosov transitiva então C^1-genericamente existe uma única medida SRB que satisfaz a fórmula de entropia de Pesin;

- Em 2012, Sun e Tian exibem em [39] condições para que uma medida invariante μ satisfaça a fórmula de entropia de Pesin para difeomorfismos C^1 com decomposição dominada ao longo da órbita para μ quase todos os pontos. Além disso, entre outras coisas, eles estendem o resultado obtido em [40] para qualquer dimensão;

- Catsigeras e Enrich estabelecem relação entre as medidas “SRB-like” e a fórmula de entropia de Pesin para funções C^1-expansoras no círculo em [19]. Posteriormente, Catsigeras, Cerminara e Enrich estenderam essa relação para difeomorfismos C^1 com decomposição dominada em [16]. Os mesmos autores, em [17] estendem o resultado para difeomorfismos C^1 Anosov para medidas “weak-SRB-like”;

- Yang e Cao, mais recentemente, mostraram em [14], que se f é um difeomorfismo C^1 que admite atrator Λ com decomposição dominada $T_\Lambda M = E \oplus F$ tal que para qualquer medida suportada em Λ temos que todos os expoentes de Lyapunov ao longo de E são não positivos e todos os expoentes de Lyapunov ao longo de F são não negativos, então existe medida suportada em Λ que satisfaz a fórmula de entropia de Pesin.

Apesar de muito progresso nessa direção, ainda há muitas questões em aberto. Nesse sentido, foi investigada a Fórmula de entropia de Pesin para difeomorfismos locais de classe C^1 não uniformemente expansor.

Teorema D. *Seja $f : M \to M$ um difeomorfismo local de classe C^1 não uniformemente expansor. Então toda medida de probabilidade expansora weak-SRB-like satisfaz a fórmula de entropia de Pesin. Além disso, todas as medidas de probabilidades ergódicas SRB-like são expansoras.*

Além disso, foram estudadas a existência de medidas ergódicas “weak-SRB-like” e algumas de suas propriedades para sistemas dinâmicos com alguma propriedade de expansão. Sabe-se que nem todas as dinâmicas admitem medidas ergódicas “weak-SRB-like”, como pode ser visto em [18, Exemplo 5.4]. Porém, Catsigeras, Cerminara e Enrich mostram em [17] que medida ergódica “weak-SRB-like” sempre existe para difeomorfismos Anosov de classe C^1 e, mais recentemente em [20], Catsigeras e Troubetzkoy mostram que, para funções contínuas C^0-genérica do intervalo, todas as medidas ergódicas são SRB-like.

Relacionando os conceitos das medidas “weak-SRB-like”, Fórmula de Entropia de Pesin e estado de equilíbrio obtemos

Corolário E. *Seja $f : M \to M$ um difeomorfismo local de classe C^1 não uniformemente expansor e $\psi = -\log|\det Df|$. Se $P_{top}(f, \psi) = 0$ então, toda medida de probabilidade weak-SRB-like é ψ-estado de equilíbrio e todas as (necessariamente existentes) medidas expansora ergódicas weak-SRB-like satisfazem a Fórmula de Entropia de Pesin.*

Ademais, supondo propriedades de expansão mais fortes obtemos:

Corolário F. *Seja $f : M \to M$ um difeomorfismo local de classe C^1 expansor fraco e não uniformemente expansor. Então,*

1. *Todas as (necessariamente existentes) medidas de probabilidade weak-SRB-like são ψ-estados de equilíbrio e, em particular satisfazem a Fórmula de Entropia de Pesin;*

2. *Existe medida de probabilidade ergódica weak-SRB-like;*

3. *Se $\psi < 0$ não existe medida de probabilidade atômica weak-SRB-like;*

4. *Se $\mathcal{D} = \{x \in M : \ \|Df(x)^{-1}\| = 1\}$ é finito e $\psi < 0$ quase todas as componentes ergódicas de um ψ-estado de equilíbrio são medidas weak-SRB-like. Além disso, toda medida de probabilidade weak-SRB-like μ, suas componentes ergódicas μ_x são medidas de probabilidades expansoras weak-SRB-like para μ-quase todo x no subconjunto expansor.*

Em particular, vemos que uma construção análoga ao resultado de existência de medida física/SRB e atômica para uma aplicação quadrática, obtida por Keller em [23], não é possível entre transformações C^1 uniformemente expansoras, embora genericamente tais medidas sejam singulares com respeito a qualquer forma de volume.

Corolário G. *Não existe medida atômica SRB para uma transformação expansora C^1 de uma variedade compacta.*

Pelo trabalho de Ávila e Bochi em [7], é sabido que C^1-genericamente as transformações de classe C^1 de uma variedade compacta sem bordo de dimensão finita não admitem medida de probabilidade absolutamente contínua em relação a medida de Lebesgue. Esse é um dos empecilhos que assegura que não é possível obter um análogo aos resultados obtidos por Pesin, Ledrappier e Young em [28, 26] de modo geral. Mas podemos nos perguntar se é possível estabelecer condições necessárias e suficientes para que um análogo à Fórmula de Entropia de Pesin seja satisfeita. Neste sentido, obtemos a seguinte caracterização para dinâmica C^1-expansora.

Corolário H. *Seja $f : M \to M$ uma função expansora de classe C^1. Então uma medida de probabilidade f-invariante μ satisfaz a Fórmula de entropia de Pesin se, e somente se, suas componentes ergódicas μ_x são weak-SRB-like μ-q.t.p. $x \in M$. Além disso, todas as medidas de probabilidade weak-SRB-like satisfazem a Fórmula de entropia de Pesin e, em particular, são ψ-estados de equilíbrio. Adicionalmente, se existe uma única medida de probabilidade μ weak-SRB-like, então μ é SRB,* $\mathrm{Leb}(B(\mu)) = 1$, *μ é o único ψ-estado de equilíbrio, é ergódico e possui cota superior de grandes desvios.*

Concluímos a introdução com a seguinte versão fraca de estabilidade estatística para dinâmica C^1-expansora: os pontos de acumulação de toda sequência de medidas weak-SRB-like são combinações lineares convexas generalizadas de medidas ergódicas weak-SRB-like.

Corolário I. *Sejam $\{f_n : M \to M\}_{n\geq 1}$ uma sequência de funções expansoras de classe C^1 tal que $f_n \to f$ na topologia C^1 e $f : M \to M$ é uma função expansora de classe C^1. Considere para cada $n \geq 1$, μ_n uma medida weak-SRB-like associada a f_n. Então todo ponto de acumulação da sequência $(\mu_n)_{n\geq 1}$ é um estado de equilíbrio para o potencial $\psi = -\log|\det Df|$ (em particular, satisfaz a Fórmula de Entropia de Pesin) e quase todas as suas componentes ergódicas são medidas weak-SRB-like.*

1.1 Comentários e Questões futuras

Nesta seção listamos algumas questões baseadas nos estudos realizados na presente tese e que serão objeto de trabalhos futuros.

Questão 1.1.1. *Nas mesmas hipóteses do Corolário C será possível obter uma cota inferior de grandes desvios que tenha a mesma taxa encontrada no Corolário C? Talvez seja necessário considerar que a dinâmica é topologicamente misturadora.*

Questão 1.1.2. *Será possível obter uma propriedade estatística para as medidas weak-SRB-like, como no caso do Corolário C, para transformação expansora fraca não uniformimente expansora?*

Questão 1.1.3. *Nas hipóteses do Corolário I é possível garantir que μ é uma medida weak-SRB-like?*

Alguns exemplos que surgiram naturalmente durante o desenvolvimento deste trabalho também motivaram algumas questões.

O primeiro exemplo é atribuído a Bowen, e pode ser encontrado em maior detalhe para nosso contexto no Exemplo 5.5 em [18].

Exemplo 1.1.4. *Considere um difeomorfismo f em uma bola de $\mathbb{R}^2$ com dois pontos de sela A e B, de modo que uma componente conexa da variedade global instável $W^u(A)\setminus\{A\}$ é um arco que coincide com uma componente conexa da variedade global estável $W^s(B)\setminus\{B\}$ e, inversamente, a componente conexa $W^u(B)\setminus\{B\} = W^s(A)\setminus\{A\}$. Seja f, de modo que exista uma fonte $C \subset U$ onde U é o conjunto aberto com bordo $W^u(A) \cup W^u(B)$.*

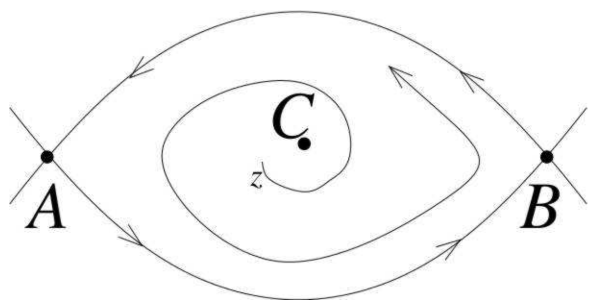

Figura 1.1: Olho de Bowen.

Se os autovalores da derivada de f em A e B forem adequadamente escolhidos conforme especificado em [42, 22], então a sequência $\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\}_{n\geq 0}$ para todos $x \in U\setminus\{C\}$ não é convergente. Neste caso existe pelo menos duas subsequências convergentes para diferentes combinações convexas dos deltas de Dirac δ_A e δ_B.

Assim, como observado em [18], as medidas de probabilidade SRB-like são combinações convexas de δ_A e δ_B e formam um segmento no espaço $\mathcal{M}$ das medidas de probabilidade. Este exemplo mostra que as medidas SRB-like não são necessariamente ergódicas.

Além disso, os autovalores de Df nos pontos de sela A e B podem ser adequadamente modificados para obter a convergência da sequência $\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\}_{n\geq 0}$ como indicado no Lema (i) da página 457 em [41]. Na verdade, fazendo uma pequena pertubação C^0 de f fora de uma pequena vizinhança dos pontos de sela A e B temos que o ω-limite das órbitas em $U\setminus\{C\}$ ainda contém os pontos A e B, obtendo que a sequência (2.4.1) converge para uma única medida $\mu = \lambda\delta_A + (1-\lambda)\delta_B$ para todo $x \in U\setminus\{C\}$, com uma constante fixa $0 < \lambda < 1$. Então, μ é a única medida SRB-like. Isso prova que o conjunto de medidas de probabilidade SRB-like não depende continuamente da função.

Este exemplo motiva algumas questões:

Questão 1.1.5. *Quando podemos afirmar que um sistema dinâmico* (f, ϕ) *tem medida de probabilidade ergódica* ν_ϕ*-SRB-like ou* ν_ϕ*-weak-SRB-like?*

Questão 1.1.6. *Pelo exemplo 1.1.4, sabemos que o conjunto das medidas de probabilidade* ν_ϕ*-SRB-like não depende, em geral, continuamente com a função. Será que adicionando a dinâmica propriedade de expansão forte (como nos casos do Corolário C e Corolário H) podemos ter variação contínua das medidas* ν_ϕ*-SRB-like ou* ν_ϕ*-weak-SRB-like?*

Questão 1.1.7. *Existe dependência contínua entre as medidas de probabilidades* ν_ϕ*-SRB-like (ou* ν_ϕ*-weak-SRB-like) e o potencial* ϕ*?*

Uma resposta positiva nesta direção é dada no Teorema A, no entanto, não sabemos em geral se a medida limite (veja Teorema A) é ergódica.

Questão 1.1.8. *No caso de uma resposta positiva na Questão 1.1.6, será que há estabilidade estocástica?*

O próximo exemplo é uma adaptação da transformação Intermitente (Manneville-Pomeau) em um homeomorfismo local do círculo.

Exemplo 1.1.9. *Considere* $I = [-1, 1]$ *e a transformação* $\hat{f} : I \to I$ *(veja a Figura 1.2) dada por*

$$\hat{f}(x) = \begin{cases} 2\sqrt{x} - 1 & \text{se } x \geq 0, \\ 1 - 2\sqrt{|x|} & \text{caso contrário.} \end{cases}$$

Esta transformação induz um homeomorfismo local contínuo $f : \mathbb{S}^1 \to \mathbb{S}^1$ *através da identificação* $\mathbb{S}^1 = I/\sim$*, onde* $-1 \sim 1$*. Esta é uma transformação que só não é diferenciável no ponto* 0*, possui frequência positiva de tempos hiperbólicos para Lebesgue quase todos os pontos, como pode ser visto em [1, seção 5].*

Este é um exemplo de transformação expansora fraca e não uniformemente expansora, mas não é um difeomorfismo local C^1 em todo ponto.

Este exemplo sugere que podemos pensar em uma generalização do Teorema B usando as mesmas ideias do Teorema D, substituindo a medida de Lebesgue por uma medida expansora ϕ-conforme ν para $\phi \in C^0(X, \mathbb{R})$ e o potencial ψ por $-\log J_\nu f$, onde $J_\nu f$ é o Jacobiano f com respeito a ν.

Questão 1.1.10. *Sejam* $T : X \to X$ *um homeomorfismo local definido em um espaço métrico compacto* X *e* $\phi : X \to \mathbb{R}$ *um potencial contínuo. Se existe medida expansora* ϕ*-conforme* ν *então todas as (necessariamente existente) medidas* ν*-SRB-like são estados de equilíbrio para o potencial* ϕ*?*

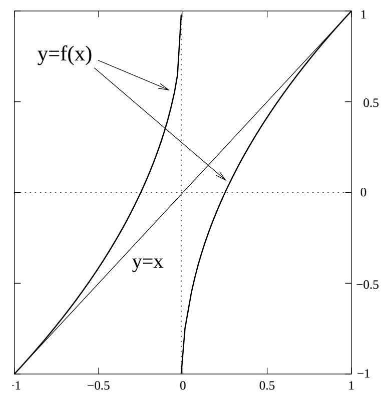

Figura 1.2: Gráfico de f.

Pode ser necessário considerar hipóteses adicionais sobre os potenciais, como por exemplo que os potenciais contínuos sejam *potenciais hiperbólicos* (veja a definição e resultados sobre o potencial hiperbólico Hölder contínuo em [33]).

Além disso, também podemos pensar em uma generalição do Teorema D para o caso de difeomorfismo local de classe C^1 afastado de conjunto crítico/singular com frequência positiva de tempos hiperbólicos (veja [2] para definição de conjunto crítico/singular).

Questão 1.1.11. *Seja $f : M \to M$ um difeomorfismo local de classe C^1 afastado de um conjunto crítico/singular $\mathcal{C}$ com recorrência lenta a $\mathcal{C}$ e tal que para alguns $0 < \sigma < 1$, $b, \delta > 0$ e $\theta > 0$ $\text{Leb}-q.t.p.$ x tem frequência positiva $\geq \theta$ de (σ, δ, b)-tempos hiperbólicos para f. Todas as medidas de probabilidade expansora weak-SRB-like satisfazem a Fórmula de Entropia de Pesin?*

1.2 Organização do trabalho

A tese está dividida em nove capítulos, incluindo o capítulo introdutório "Introdução e descrição dos resultados". Nos capítulos 2 e 4 denominados respectivamente "Statement of the results" e "Preliminary definitions and results" serão introduzidos os principais resultados da tese, grande parte das definições básicas e notações necessárias ao longo do texto, além de alguns resultados auxiliares sobre medidas de probabilidades ν-SRB-like, ν_ϕ-weak-SRB-like e entropia.

No capítulo 3 intitulado "Examples of application" apresentamos alguns exemplos onde os principais resultados desse trabalho podem ser aplicados.

No Capítulo 5, "Continuous variation of SRB-like measures" será provado o Teorema A.

No Capítulo 6, "Expanding maps on compact metric space" serão exibidas algumas propriedades das aplicações expansivas e a demonstração do Teorema B.

No Capítulo 7, "Entropy Formula" serão usadas as propriedades dos tempos hiperbólicos e das medidas weak-SRB-like para provar uma reformulação da Fórmula de Entropia para difeomorfismo local de classe C^1 não uniformemente expansor (o Teorema D).

No Capítulo 8, "Ergodic weak-SRB-like measure" será estudado a existência de medida ergódica weak-SRB-like, sua relação com a Fórmula de Entropia de Pesin e a prova do Corolário E.

No Capítulo 9, "Weak-Expanding non-uniformly maps" serão usados alguns resultados obtidos nos capítulos anteriores para provar os Corolários F, C, I e H.

Chapter 2

Statement of the results

Let $T : X \to X$ be a continuous transformation defined on a compact metric space (X, d).

2.1 Topological pressure

The *dynamical ball* of center $x \in X$, radius $\delta > 0$, and length $n \geq 1$ is defined by

$$B(x, n, \delta) = \{y \in X : d(T^j x, T^j y) \leq \delta,\ 0 \leq j \leq n-1\}.$$

Let ν be a Borel probability measure on X. We define

$$\begin{aligned} h_\nu(T, x) &= \lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{n} \log \nu(B(x, n, \delta)) \text{ and} \\ h_\nu(T, \mu) &= \mu - \operatorname{ess\,sup} h_\nu(T, x). \end{aligned} \tag{2.1.1}$$

Note that ν is not necessarily T-invariant, and if μ is ergodic T-invariant, we have for μ-a.e. $x \in X$, $h_\mu(T, x) = h_\mu(T)$, the usual metric entropy of T with respect to μ.

Let n be a natural number, $\varepsilon > 0$ and let K be a compact subset of X. A subset F of X is said to (n, ε) span K with respect to T if $\forall x \in K$ there exists $y \in F$ with $d(T^j x, T^j y) \leq \varepsilon$ for all $0 \leq j \leq n-1$, that is,

$$K \subset \bigcup_{x \in F} B(x, n, \varepsilon).$$

If K is a compact subset of X. Let

$$h(T; K) = \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log N(n, \varepsilon, K), \tag{2.1.2}$$

where $N(n, \varepsilon, K)$ denote the smallest cardinality of any (n, ε)-spanning set for K with respect to T.

Definition 1. The *topological entropy* of T is $h_{\text{top}}(T) = \sup\{h(T, K);\ K \subset X\}$, where the supremum is taken over the collection of all compact subset of X.

Let $\phi : X \to \mathbb{R}$ be a real continuous function that we call the *potential*.

Given an open cover α for X we define the pressure $P_T(\phi, \alpha)$ of ϕ with respect to α by

$$P_T(\phi, \alpha) := \lim_{n\to+\infty} \frac{1}{n} \log \inf_{\mathcal{U}\subset\alpha^n} \left\{ \sum_{U\in\mathcal{U}} e^{S_n\phi(U)} \right\}$$

where the infimum is taken over all subcovers $\mathcal{U}$ of $\alpha^n = \vee_{j=0}^{n-1} T^{-j}(\alpha)$, $S_n\phi(x) := \sum_{j=0}^{n-1} \phi(T^j x)$ and $S_n\phi(U) := \sup\{S_n\phi(x);\ x \in U\}$.

Definition 2. The *topological pressure* $P_{\text{top}}(T, \phi)$ of the potential ϕ with respect to the dynamics T is defined by

$$P_{\text{top}}(T, \phi) = \lim_{\delta\to 0} \left\{ \sup_{|\alpha|\le\delta} P_T(\phi, \alpha) \right\}$$

where $|\alpha|$ denotes the diameter of the open cover α.

For given $n > 0$ and $\varepsilon > 0$, a subset $E \subset X$ is called (n, ε)-separated if $x, y \in E$, $x \neq y$ then there exists $0 \le j \le n-1$ such that $d(T^j x, T^j y) > \varepsilon$.

An alternative way of defining topological pressure is through the notion of (n, ε)-separated set.

Definition 3. The topological pressure $P_{\text{top}}(T, \phi)$ of the potential ϕ with respect to the dynamics T is defined by

$$P_{\text{top}}(T, \phi) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log \sup \left\{ \sum_{x\in E} e^{S_n\phi(x)} \right\}$$

where the supremum is taken over all maximal (n, ε)-separated sets E.

We refer the reader to [47] for more details and properties of the topological pressure.

2.2 Expanding maps

A continuous mapping $T : X \to X$ is *expanding* (with respect to the metric d) if there exist constants $\lambda > 1$, $\eta > 0$ and $n \geq 1$, such that for all $x, y \in X$

$$\text{if } d(x,y) < 2\eta \text{ then } d(T^n(x), T^n(y)) \geq \lambda d(x,y). \tag{2.2.1}$$

In the sequel we will always assume (without loss of generality, see chapter 3 in [29]) that $n = 1$, that is

$$d(x,y) < 2\eta \implies d(T(x), T(y)) \geq \lambda d(x,y). \tag{2.2.2}$$

We refer the reader to [21, 6, 29] for more details and properties of expanding map.

2.3 Transfer operator

We consider the Ruelle-Perron-Fröbenius transfer operator $\mathcal{L}_{T,\phi} = \mathcal{L}_\phi$ associated to $T : X \to X$ and the continuous function (potential) $\phi : X \to \mathbb{R}$ as the linear operator defined on the space $C^0(X, \mathbb{R})$ of continuous functions $g : X \to \mathbb{R}$ by

$$\mathcal{L}_\phi(g)(x) = \sum_{T(y)=x} e^{\phi(y)} g(y).$$

The dual of the Ruelle-Perron-Fröbenius transfer operator is given by

$$\begin{array}{rccclcl} \mathcal{L}_\phi^* : & \mathcal{M}_1 & \to & \mathcal{M}_1 & & & \\ & \eta & \mapsto & \mathcal{L}_\phi^*\eta : & C^0(X,\mathbb{R}) & \to & \mathbb{R} \\ & & & & \psi & \mapsto & \displaystyle\int \mathcal{L}_\phi \psi \, d\,\eta. \end{array}$$

where $\mathcal{M}_1$ is the set of Borel probabilities in X.

2.4 Weak-SRB-like and SRB-like probability measures

For any point $x \in X$ consider,

$$\sigma_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)} \tag{2.4.1}$$

where δ_y is the Dirac delta probability measure supported in $y \in X$.

Definition 4. For each point $x \in M$, denote $p\omega(x) \subset \mathcal{M}_T$ the limit set with initial state x,

$$p\omega(x) := \left\{\mu \in \mathcal{M}_T; \exists n_j \xrightarrow[j\to+\infty]{} +\infty \text{ s.t. } \sigma_{n_j}(x) \xrightarrow[j\to+\infty]{w^*} \mu\right\}.$$

Definition 5. Fix a reference measure ν for the space X, we say that probability measure $\mu \in \mathcal{M}_T$ is ν-SRB (or ν-physical) if $\nu(B(\mu)) > 0$, where

$$B(\mu) = \{x \in X; p\omega(x) = \{\mu\}\} \text{ is the "ergodic basin" of } \mu.$$

Let $\mu \in \mathcal{M}_T$ and $\varepsilon > 0$. We will consider the following measurable sets in X:

$$A_{\varepsilon,n}(\mu) := \{x \in X : \text{ dist}(\sigma_n(x), \mu) < \varepsilon\}; \tag{2.4.2}$$

$$A_{\varepsilon}(\mu) := \{x \in X : \text{ dist}(p\omega(x), \mu) < \varepsilon\}. \tag{2.4.3}$$

We call $A_{\varepsilon,n}(\mu)$ the *ε-pseudo basin of μ up to time n*, $A_\varepsilon(\mu)$ the basin of *ε-weak statistical attraction* of μ and dist is a metric in $\mathcal{M}_1$ that induces the weak* topology (see definition in (4.1.1)).

Definition 6. Fix a reference probability measure ν for the space X. We say that a T-invariant probability measure μ is:

1. *ν-SRB-like* (or *ν-physical-like*), if and only if $\nu(A_\varepsilon(\mu)) > 0$ for all $\varepsilon > 0$;

2. *ν-weak-SRB-like* (or *ν-weak-physical-like*), if and only if

$$\limsup_{n\to+\infty} \frac{1}{n} \log \nu(A_{\varepsilon,n}(\mu)) = 0 \ \forall \varepsilon > 0.$$

When $\nu = \text{Leb}$ we say that μ is simply SRB-like (or weak-SRB-like).

Remark 2.4.1. *It is easy to see that every ν-SRB measure is also a ν-SRB-like measure. Moreover, the ν-SRB-like measures are particular case of ν-weak-SRB-like (see item B in [17]).*

We denote $\mathcal{W}^*_T(\nu) \subset \mathcal{M}_T$ the set of ν-weak-SRB-like probability measures. When $\nu = \text{Leb}$ we denote $\mathcal{W}^*_T(\text{Leb}) = \mathcal{W}^*_T$.

Definition 7. Given $T : X \to X$ and an arbitrary continuous function $\phi : M \to \mathbb{R}$, we say that a probability measure ν is conformal for T with respect to ϕ (or ϕ-conformal) if there exists $\lambda > 0$ so that $\mathcal{L}^*_\phi \nu = \lambda\nu$.

2.5 SRB-like measures as limits of SRB measures

The next result shows that we can see the ν-SRB-like measures as measures that naturally arise as accumulation points of ν_n-SRB measures.

Theorem A. *Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space X, $(\phi_n)_{n\geq 1}$ a sequence of Hölder continuous potentials, $(\nu_n)_{n\geq 1}$ a sequence of conformal measures associated to the pair (T, ϕ_n) and $(\mu_n)_{n\geq 1}$ a sequence of ν_n-SRB measures. Assume that*

1. $\phi_{n_j} \xrightarrow[j\to+\infty]{} \phi$ *(in the topology of uniform convergence);*

2. $\nu_{n_j} \xrightarrow[j\to+\infty]{w^*} \nu$ *(in the weak* topology);*

3. $\mu_{n_j} \xrightarrow[j\to+\infty]{w^*} \mu$ *(in the weak* topology).*

Then ν is a conformal measure for (T, ϕ) and μ is ν-SRB-like. In particular, μ is an equilibrium state for the potential ϕ. Moreover, if T is topologically exact then $\nu(X \setminus A_\varepsilon(\mu)) = 0$ for all $\varepsilon > 0$.

This is one of the motivations for the study of SRB-like measures as the natural extension of the notion of physical/SRB measure for C^1 maps. Additional justification is given by the results of this work.

It is worth noting that Hölder continuous potentials are only used in this work on the assumption of Theorem A and all continuous potentials can be approximated by Lipschitz potentials (and all Lipschitz continuous potential are α-Hölder continuous for $\alpha \in (0, 1]$) see Remark 5.2.4.

The next result extends, in particular, the main result obtained in [19, Theorem 2.3] valid only for expanding circle maps.

Theorem B. *Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space X and $\phi : X \to \mathbb{R}$ a continuous potential. For each (necessarily existing) conformal measure ν_ϕ all the (necessarily existing) ν_ϕ-SRB-like measures are equilibrium states for the potential ϕ.*

Definition 8. Let $T : X \to X$ be a continuous map and $\phi : X \to \mathbb{R}$ a continuous potential. Given $r > 0$ we consider the set in $\mathcal{M}_T$

$$\mathcal{K}_r(\phi) = \{\mu \in \mathcal{M}_T : h_\mu(T) + \int \phi d\mu \geq P_{\text{top}}(T, \phi) - r\}$$

where $\mathcal{M}_T$ is the set of all T-invariant Borel probability measures.

Corollary C. *Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space X, $\phi : X \to \mathbb{R}$ a continuous potential and ν a ϕ-conformal measure. If μ is a ϕ-equilibrium state such that $h_\nu(T, \mu) < \infty$ then every ergodic component μ_x of μ is a ν-weak-SRB-like measure. Moreover, if μ is the unique ϕ-equilibrium state then μ is ν-SRB, $\nu(B(\mu)) = 1$ and μ satisfies the following large deviation bound: for every weak* neighborhood $\mathcal{V}$ of μ we have*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \nu(\{x \in X : \ \sigma_n(x) \in \mathcal{M}_1 \setminus \mathcal{V}\}) \leq -I(\mathcal{V})$$

where $I(\mathcal{V}) = \sup\{r > 0 : \mathcal{K}_r(\phi) \subset \mathcal{V}\}$.

2.6 Non-uniformly expanding maps

We denote by $\|\cdot\|$ a Riemannian norm on the compact m-dimensional boundaryless manifold M, $m \geq 1$; by $d(\cdot, \cdot)$ the induced distance and by Leb a Riemannian volume form, which we call *Lebesgue measure* or *volume* and assume to be normalized: $\text{Leb}(M) = 1$. Note that Leb is not necessarily f-invariant.

Recall that a C^1-map $f : M \to M$ is *uniformly expanding* or just *expanding* if there is some $\lambda > 1$ such for some choice of a metric in M one has

$$\|Df(x)v\| > \lambda\|v\|, \text{ for all } x \in M \text{ and all } v \in T_xM\setminus\{0\}.$$

In what follows we write always $f : M \to M$ be C^1 local diffeomorphism and $\psi := -\log|\det Df|$.

Next corollary suggests that Theorem B should be extended to weaker forms of expansion.

Recall that a Borel set in a topological space is said to have total probability if it has probability one for every f-invariant probability measure.

Corollary 2.6.1. *Let $f : M \to M$ be C^1 local diffeomorphism and Y be a subset with total probability. If*

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < 0, \quad \forall x \in Y,$$

then all the (necessarily existing) SRB-like measures are equilibrium states for the potential $\psi = -\log|\det Df|$.

Proof. For a C^1-local diffeomorphism $f : M \to M$ we have that Leb is ψ-conformal. Moreover, we know by Theorem 1.15 in [3] that if

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < 0$$

for x in a total probability subset, then f is uniformly expanding. Thus, by Theorem B we concluded the proof. $\square$

The *Lyapunov exponents* of a C^1 local diffeomorphism f of a compact manifold M are defined by Oseledets Theorem which states that, for any f-invariant probability measure μ, for almost all points $x \in M$ there is $\kappa(x) \geq 1$, a filtration $T_xM = F_1(x) \supset F_2(x) \supset \cdots \supset F_{\kappa(x)}(x) \supset F_{\kappa(x)+1}(x) = \{0\}$, and numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_k(x)$ such $Df(x) \cdot F_i(x) = F_i(f(x))$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda_i(x)$$

for all $v \in F_i(x) \backslash F_{i+1}(x)$ and $0 \leq i \leq \kappa(x)$. The numbers $\lambda_i(x)$ are called *Lyapunov exponents* of f at the point x. For more details on Lyapunov exponents and non-uniform hyperbolicity see [9].

Definition 9. Let $f : M \to M$ be C^1 local diffeomorphism. We say that $\mu \in \mathcal{M}_f$ satisfies the Pesin Entropy Formula if

$$h_\mu(f) = \int \Sigma^+ d\mu,$$

where Σ^+ denotes the sum of the positive Lyapunov exponents at a regular point, counting multiplicities.

Let $0 < \sigma < 1$ we denote,

$$H(\sigma) = \left\{ x \in M;\ \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < \log \sigma \right\} \tag{2.6.1}$$

Definition 10. We say that $f : M \to M$ is *non-uniformly expanding* if there exists $\sigma \in (0, 1)$ such that $\mathrm{Leb}(H(\sigma)) = 1$.

A probability measure μ (not necessarily invariant) is *expanding* if there exists $\sigma \in (0, 1)$ such that $\mu(H(\sigma)) = 1$.

Next result relates non-uniform expansion and expanding measures with the Entropy Formula.

Theorem D. *Let $f : M \to M$ be non-uniformly expanding. Every expanding weak-SRB-like probability measure satisfies Pesin's Entropy Formula. In particular, all ergodic SRB-like probability measures are expanding.*

Now we add a condition ensuring that there exists expanding ergodic weak-SRB-like probability measure and that all weak-SRB-like measures become equilibrium states.

Corollary E. *Let $f : M \to M$ be non-uniformly expanding. If $P_{top}(f, \psi) = 0$, then all weak-SRB-like probability measures are ψ-equilibrium states and all the (necessarily existing) expanding ergodic weak-SRB-like measures satisfy Pesin's Entropy Formula.*

2.7 Weak and non-uniformly expanding maps

Now we strengthen the assumptions on non-uniform expansion to improve properties of weak-SRB-like measures.

Definition 11. We say that f is weak-expanding if, $\|Df(x)^{-1}\| \leq 1$ for all $x \in M$.

We say that a probability measure is atomic if it is supported on a finite set. We denote supp(μ) the support of probability measure μ.

Corollary F. *Let $f : M \to M$ be weak-expanding and non-uniformly expanding. Then,*

1. *all the (necessarily existing) weak-SRB-like probability measures are ψ-equilibrium states and, in particular, satisfy Pesin's Entropy Formula;*
2. *there exists some ergodic weak-SRB-like probability measure;*
3. *if $\psi < 0$ there is no atomic weak-SRB-like probability measure;*
4. *if $\mathcal{D} = \{x \in M : \|Df(x)^{-1}\| = 1\}$ is finite and $\psi < 0$ almost all ergodic components of a ψ-equilibrium state are weak-SRB-like measures. Moreover all weak-SRB-like probability measures μ its ergodic components μ_x are expanding weak-SRB-like probability measure for μ-a.e. $x \in M \backslash \mathcal{D}$.*

In particular, we see that an analogous result to the existence of an atomic physical measure for a quadratic map, as obtained by Keller in [23], is not possible in the C^1 expanding setting, although generically such measures must be singular with respect to any volume from.

Corollary G. *No atomic measure is a SRB measure for a C^1 uniformly expanding map of a compact manifold.*

We now restate the previous results in the uniformly expanding setting.

Corollary H. *Let $f : M \to M$ be a C^1-expanding map. Then an f-invariant probability measure μ satisfies Pesin's Entropy Formula if and only if its ergodic components μ_x are weak-SRB-like μ-a.e. $x \in M$. Moreover, all the (necessarily existing) weak-SRB-like probability measures satisfy Pesin's Entropy Formula, in particular, are ψ-equilibrium states. In addition, if μ is the unique weak-SRB-like probability measure then μ is SRB probability measure,* $\mathrm{Leb}(B(\mu)) = 1$*, is ergodic, μ is the unique ψ-equilibrium state and μ satisfies large deviation bound.*

Using this we get a weak statistical stability result in the uniformly C^1-expanding setting: all weak* accumulation points of weak-SRB-like measures are generalized convex linear combinations of ergodic weak-SRB-like measures, as follows.

Corollary I. *Let $\{f_n : M \to M\}_{n\geq 1}$ a sequence of C^1-expanding maps such that $f_n \to f$ in the C^1-topology and $f : M \to M$ be a C^1-expanding map. Let $(\mu_n)_{n\geq 1}$ a sequence of weak-SRB-like measures associated f_n. Then all accumulation points of $(\mu_n)_{n\geq 1}$ is an equilibrium state for the potential $\psi = -\log|\det Df|$ (in particular, satisfy Pesin's Entropy Formula) and almost all ergodic components are weak-SRB-like measures.*

Chapter 3

Examples of application

Here we present some examples of applications. In the first section we show the construction of a C^1 uniformly expanding map with many different SRB-like measures which in particular satisfies the assumptions of Theorem B and Corollary H.

In the second section we present a class of non uniformly expanding C^1 local diffeomorphisms satisfying the assumptions of Theorem D. In the third section we exhibit examples of weak-expanding and non uniformly expanding transformations which are not uniformly expanding in the setting of Corollary F.

3.1 Expanding map with several absolutely continuous invariant probability measures

We present the construction of a C^1-expanding map $f : \mathbb{S}^1 \to \mathbb{S}^1$ that has several SRB-Like measures. Moreover, such measures are absolutely continuous with respect to Lebesgue measure (which in particular means that f does not belong to a C^1 generic subset of $C^1(M, M)$, see [8]).

Example 3.1.1. *Following [11], we first adopt some notation for Cantor sets with positive Lebesgue measure. Let I be a closed interval and $\alpha_n > 0$ numbers with $\sum_{n=0}^{\infty} \alpha_n < |I|$, where $|E|$ denotes the one-dimensional Lebesgue measure of the subset E of I. Let $\underline{a} = a_1 a_2 \ldots a_n$ denote a sequence of 0's and 1's of length $n = n(\underline{a})$; we denote the empty sequence $\underline{a} = \emptyset$ with $n(\underline{a}) = 0$. Define $I_\emptyset = I = [a, b]$, $I^*_\emptyset = \left[\frac{a+b}{2} - \frac{\alpha_0}{2}, \frac{a+b}{2} + \frac{\alpha_0}{2}\right]$ and $I^*_{\underline{a}} \subset I_{\underline{a}}$ recursively as follows.*

Let $I_{\underline{a}0}$ and $I_{\underline{a}1}$ be the left and right intervals remaining when the interior of $I^*_{\underline{a}}$ is removed from $I_{\underline{a}}$; let $I^*_{\underline{a}}$ be the closed interval of length $\frac{\alpha_{n(\underline{a}k)}}{2^{n(\underline{a}k)}}$ and having the same *center as $I_{\underline{a}k}$ (k=0,1).*

The Cantor set K_I is given by $K_I = \bigcap_{m=0}^{\infty} \bigcup_{n(\underline{a})=m} I_{\underline{a}}$.

This is the standard construction of the Cantor set except that we allow ourselves some flexibility in the lengths of the removed intervals. The measure of K_I is $\mathrm{Leb}(K_I) = |I| - \sum_{n=0}^{\infty} \alpha_n > 0$.

Suppose that another interval $J \supset I$ is given together with $\beta_n > 0$ such that $\sum_{n=0}^{\infty} \beta_n < |J|$. One can then construct $J_{\underline{a}}$, $J_{\underline{a}}^*$ and K_J as above. Let us assume now that $\frac{\beta_n}{\alpha_n} \xrightarrow[n\to\infty]{} \gamma \geq 0$. Following the construction of Bowen [11], we get $g : I \to J$ a C^1 orientation preserving homeomorphism so that $g'(x) = \gamma$ for all $x \in K_I$ and $g'(x) > 1$ for all $x \in I$.

More precisely, let us take $J = [-1, 1]$ and choose $\beta_n > 0$ with $\sum_{n=0}^{\infty} \beta_n < 2$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$, $\left(e.g.\ \beta_n = \frac{1}{(n+100)^2}\right)$. Let $I = \left[\frac{\beta_0}{2}, 1\right]$ and $\alpha_n = \frac{\beta_{n+1}}{2}$. Then $\sum_{n=0}^{\infty} \alpha_n < 1 - \frac{\beta_0}{2}$ and $\gamma = \lim \frac{\beta_n}{\alpha_n} = 2$. We define a homeomorphism $G : (-I) \cup I \to J$ by $G(x) = \begin{cases} g(x) & \text{if } x \in I \\ -g(-x) & \text{if } x \in (-I) \end{cases}$, where $K_J = \bigcap_{n=0}^{+\infty} G^{-n}(J)$ and $G|_{K_J} : K_J \circlearrowleft$.

Consider now $c_1 = \left(\frac{3\beta_0}{2} - 2\right)\left(\frac{4}{\beta_0}\right)^3$, $c_2 = \left(\frac{9\beta_0}{4} - 3\right)\left(\frac{4}{\beta_0}\right)^2$; $f_1 : [-\frac{\beta_0}{2}, -\frac{\beta_0}{4}] \to [-1, -\frac{\beta_0}{4}]$ given by

$$f_1(x) = c_1\left(x + \frac{\beta_0}{2}\right)^3 - c_2\left(x + \frac{\beta_0}{2}\right)^2 + 2\left(x + \frac{\beta_0}{2}\right) - 1$$

and $f_2 : [\frac{\beta_0}{4}, \frac{\beta_0}{2}] \to [\frac{\beta_0}{4}, 1]$ given by $f_2(x) = c_1\left(x - \frac{\beta_0}{2}\right)^3 + c_2\left(x - \frac{\beta_0}{2}\right)^2 + 2\left(x - \frac{\beta_0}{2}\right) + 1$. Then, we have

1. *$f_1\left(-\frac{\beta_0}{2}\right) = -1$, $f_1\left(-\frac{\beta_0}{4}\right) = -\frac{\beta_0}{4}$, $f_2\left(\frac{\beta_0}{4}\right) = \frac{\beta_0}{4}$ and $f_2\left(\frac{\beta_0}{2}\right) = 1$*

2. *$f_1^+\left(\frac{\beta_0}{2}\right) = \lim_{h\to 0^+} \frac{f_1(\frac{\beta_0}{2}+h) - f_1(\frac{\beta_0}{2})}{h} = 2$, $f_2^+\left(\frac{\beta_0}{4}\right) = 2$, $f_1^-\left(\frac{\beta_0}{4}\right) = \lim_{h\to 0^-} \frac{f_1(\frac{\beta_0}{4}+h) - f_1(\frac{\beta_0}{4})}{h} = 2$ and $f_2^-\left(\frac{\beta_0}{2}\right) = 2$.*

Consider now $J_1 = \left[-\frac{\beta_0}{4}, \frac{\beta_0}{4}\right]$, $I_1 = \left[\frac{\beta_0^2}{8}, \frac{\beta_0}{4}\right]$ and choose $\beta_n' = \frac{\beta_0 \beta_n}{4} > 0$ with $\sum_{n=0}^{\infty} \beta_n' < \frac{\beta_0}{2}$ and $\frac{\beta_{n+1}'}{\beta_n'} = \frac{\beta_{n+1}}{\beta_n} \to 1$. Let $\alpha_n' = \frac{\beta_{n+1}'}{2}$, then $\sum_{n=0}^{\infty} \alpha_n' = \sum_{n=0}^{\infty} \frac{\beta_{n+1}'}{2} < \frac{1}{2}\left(\frac{\beta_0}{2} - \beta_0'\right) = \frac{1}{2}\left(\frac{\beta_0}{2} - \frac{\beta_0^2}{4}\right)$ and $\gamma = \lim \frac{\beta_n'}{\alpha_n'} = \lim \frac{2\beta_n}{\beta_{n+1}} = 2$.

Similarly to the above construction, we obtain a homeomorphism $G_1 : (-I_1) \cup I_1 \to J_1$ given by

$$G_1(x) = \begin{cases} g_1(x) & \text{if } x \in I_1 \\ -g_1(-x) & \text{if } x \in (-I_1) \end{cases},$$

where $K_{J_1} = \bigcap_{n=0}^{+\infty} G_1^{-n}(J_1)$ *is a Cantor set with positive Lebesgue measure,* $G_1|_{K_{J_1}} : K_{J_1} \circlearrowleft$ *and* $g_1 : I_1 \to J_1$ *is a* C^1 *orientation preserving homeomorphism so that* $g_1'(x) = 2$ *for all* $x \in K_{J_1}$ *and* $g_1'(x) > 1$ *for all* $x \in I_1$.

Similarly we obtain $f_3 : [0, \frac{\beta_0^2}{8}] \to [-1, -\frac{\beta_0}{4}]$ *and* $f_4 : [-\frac{\beta_0^2}{8}, 0] \to [\frac{\beta_0}{4}, 1]$ *such that* $f_3(0) = -1$, $f_3(\frac{\beta_0^2}{8}) = -\frac{\beta_0}{4}$, $f_3^-(\frac{\beta_0^2}{8}) = 2$, $f_3^+(0) = 2$, $f_4(\frac{-\beta_0^2}{8}) = \frac{\beta_0}{4}$, $f_4(0) = 1$, $f_4^+(-\frac{\beta_0^2}{8}) = 2$ *and* $f_4^-(0) = 2$

Finally, define the function $f : J \to J$ *by (see Figure 3.1 for its graph)*

$$f(x) = \begin{cases} G(x) & \text{if } x \in (-I) \cup I \\ f_1(x) & \text{if } x \in \left(-\frac{\beta_0}{2}, -\frac{\beta_0}{4}\right) \\ f_2(x) & \text{if } x \in \left(\frac{\beta_0}{4}, \frac{\beta_0}{2}\right) \\ G_1(x) & \text{if } x \in (-I_1) \cup I_1 \\ f_4(x) & \text{if } x \in \left(-\frac{\beta_0^2}{8}, 0\right) \\ f_3(x) & \text{if } x \in \left(0, \frac{\beta_0^2}{8}\right) \end{cases},$$

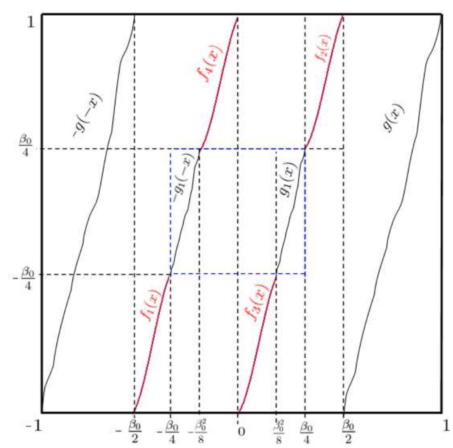

Figure 3.1: Graph of f.

Identifying -1 *and* 1 *and making a linear change of coordinates we obtain a* C^1 *expanding map of the circle* $f : \mathbb{S}^1 \circlearrowleft$.

Consider Leb_{K_J} *the normalized Lebesgue measure of the set* K_J, *this is,* $\mathrm{Leb}_{K_J}(A) = \mathrm{Leb}(A \cap K_J)/\mathrm{Leb}(K_J)$ *for all measurable* $A \subset J$. *Denote* $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ *and consider the homeomorphism* $h : K_J \to \Sigma_2^+$ *that associates each point* $x \in K_J$ *the sequence* $\underline{a} \in \Sigma_2^+$ *describing its location in the set* K_J, *this is,* $\underline{a}$ *is such that* $I_{\underline{a}} \cap K_J = \{x\}$. *Let* μ *be the Bernoulli measure in* Σ_2^+ *giving weight 1/2 to each digit.*

Claim 3.1.2. $\mu = h_* \operatorname{Leb}_{K_J}$

We show that this relation holds for an algebra of subsets which generates the Borel σ-algebra of Σ_2^+. For $\underline{a} = a_1a_2\dots a_n$ we have that $\mu([\underline{a}]) = 1/2^n$. Moreover, $h^{-1}([\underline{a}]) = I_{\underline{a}} \cap K_J$, so $\operatorname{Leb}_{K_J}(h^{-1}([\underline{a}])) = \operatorname{Leb}_{K_J}(I_{\underline{a}} \cap K_J)$. By construction, at each step all remaining intervals in the construction of K_J have the same length. Thus, the n-th stage contains 2^n intervals $I_{\underline{a}}$ (among them) all with the same Lebesgue measure. Since Leb_{K_J} is a probability measure, we have

$$2^n \operatorname{Leb}_{K_J}(I_{\underline{a}} \cap K_J) = 1 \Longrightarrow \operatorname{Leb}_{K_J}(I_{\underline{a}} \cap K_J) = \frac{1}{2^n}.$$

We conclude that $\operatorname{Leb}_{K_J}(I_{\underline{a}} \cap K_J) = \operatorname{Leb}_{K_J}(h^{-1}([\underline{a}])) = \mu([\underline{a}])$ for all $\underline{a} = a_1a_2\dots a_n$, $n \geq 1$ proving the Claim.

Clearly $h \circ f|_{K_J} = \sigma \circ h$ where σ is the standard left shift $\sigma : \Sigma_2^+ \circlearrowleft$. Since μ is σ-invariant, then Leb_{K_J} is $f|_{K_J}$-invariant. Consequently $(f, \operatorname{Leb}_{K_J})$ is mixing, $h_{top}(f) = \log 2 = h_{\operatorname{Leb}_{K_J}}(f) = h_\mu(\sigma)$ and $\operatorname{Leb}_{K_J} \ll \operatorname{Leb}$.

Hence $\lim_{n\to+\infty} \sigma_n(x) = \operatorname{Leb}_{K_J}$ for Leb_{K_J} $-$a.e, $x \in K_J$ and $|K_J| > 0$, so it follows that Leb_{K_J} is a SRB-like measure. By Theorem B, Leb_{K_J} is an equilibrium state for the potential $\psi = -\log|f'|$.

Similarly $\operatorname{Leb}_{K_{J_1}}$ is also a SRB-like measure, but distinct form Leb_{K_J}.

Note that this construction allows us to obtain countably many ergodic SRB-like measures by reapplying the construction to each removed subinterval of the first Cantor set.

Note also that, if we take a sequence of Hölder continuous potentials ϕ_n converging to $\psi = -\log|f'|$, choose ν_n a ϕ_n-conformal measure and a ν_n-SRB measure μ_n and weak* accumulation points ν, μ as in Theorem A, since f is topologically exact then $\nu(A_\varepsilon(\mu)) = 1$ for all $\varepsilon > 0$. Therefore ν is not Lebesgue measure on $\mathbb{S}^1$.

3.2 C^1 Non uniformly expanding maps

Next we present an example that is a robust (C^1 open) class of non-uniformly expanding C^1 local diffeomorphisms.

This family of maps was introduced in [2], (see also subsection 2.1 in [3, Chapter 1]) and maps in this class exhibit non-uniform expansion Lebesgue almost everywhere but are not uniformly expanding.

Example 3.2.1. *Let M be a compact manifold of dimensiom $d \geq 1$ and $f_0 : M \circlearrowleft$ is a C^1-expanding map. Let $V \subset M$ be some small compact domain, so that the restriction of f_0 to V is injective. Let f be any map in a sufficiently small C^1-neighborhood $\mathcal{N}$ of f_0 so that:*

1. *f is volume expanding everywhere: there exists $\sigma_1 > 1$ such that*

$$|\det Df(x)| \geq \sigma_1 \text{ for every } x \in M$$

2. *f is expanding outside V: there exists $\sigma_0 < 1$ such that*

$$\|Df(x)^{-1}\| < \sigma_0 \text{ for every } x \in M \backslash V;$$

3. *f is not too contracting on V : there is some small $\delta > 0$ such that*

$$\|Df(x)^{-1}\| < 1 + \delta \text{ for every } x \in V.$$

Then every map f in such a C^1-neighborhood $\mathcal{N}$ of f_0 is non-uniformly expanding (see a proof in subsection 2.1 in [3]). By Theorem D, every expanding weak-SRB-like probability measure satisfies Pesin's Entropy Formula, and every ergodic SRB-like measure is expanding.

Such classes of maps can be obtained through deformation of a uniformly expanding map by isotopy inside some small region.

3.3 Weak-expanding and non-uniformly expanding maps

Consider $\alpha > 0$ and the map $T_\alpha : [0,1] \to [0,1]$ defined as follows

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha} & \text{if } x \in [0,1/2) \\ x - 2^\alpha (1-x)^{1+\alpha} & \text{if } x \in [1/2,1] \end{cases} .$$

This defines a family of $C^{1+\alpha}$ maps of the unit circle $\mathbb{S}^1 := [0,1]/\sim$ into itself, known as *intermittent maps*. These applications are expanding, except at a neutral fixed point, the unique fixed point is 0 and $DT_\alpha(0) = 1$. The local behavior near this neutral point is responsible for various phenomena. The above family of maps provides many interesting results in ergodic theory. If $\alpha \geq 1$, i.e. if the order of tangency at zero is high enough, then the Dirac mass at zero δ_0 is the unique physical probability measure and so the Lyapunov exponent of Lebesgue almost all points vanishes (see [43]). This example shows that there exists systems which are weak-expanding but not non-uniformly expanding. Hence the assumption of weak expansion together with non-uniform expansion in Corollary F is not superfluous.

The following is an example of a map which is weak-expanding and non-uniformly expanding in the setting of Corollary F.

Example 3.3.1. *If, in the setting of the construction of the previous Example 3.2.1, the region V of a point p of a periodic orbit with period k, and the deformation weakens one of the eigenvalues of $Df_0^k(p)$ in such a way that 1 becomes and eigenvalue of $Df^k(p)$, then f is an example of a weak expanding and non uniformly expanding C^1 transformation.*

More precisely, consider the function $g_0(t) = \frac{t}{\log(1/t)}, 0 < t \leq 1/2$, $g_0(0) = 0$. It is easy to see that

- *$g_0'(t) > 0, 0 < t \leq 1/2$ and so g_0 is stricly increasing;*
- *g_0' is not of α-generalized bounded variation for any $\alpha \in (0,1)$ (see [24] for the definition of generalized bounded variation) and so g_0' is not C^α for any $0 < \alpha < 1$.*

Setting $g(t) = g_0(t)/g_0(1/2)$ we obtain $g : [0,1/2] \to [0,1]$ a C^1 strictly increasing function which is not $C^{1+\alpha}$ for any $\alpha \in (0,1)$. Now we consider the analogous map to T_α

$$T(x) = \begin{cases} x + xg(x) & \text{if } x \in [0,1/2) \\ x - (1-x)g(1-x) & \text{if } x \in [1/2,1] \end{cases}.$$

This is now a C^1 map of the circle into itself which is not $C^{1+\alpha}$ for any $0 < \alpha < 1$. Moreover, letting $f_0 = T \times E$ where $E(x) = 2x \mod 1$, we see that f_0 satisfies items (1-3) in Example 3.2.1 for V a small neighborhood of the fixed point $(0,0)$.

Hence f_0 is a C^1 non-uniformly expanding map. Moreover, since $T'(0) = 1$, we have that f_0 is also a C^1 weak expanding map with $\mathcal{D} = \{(0,0)\}$.

Now we extend this construction to obtain a weak expanding and non uniformly expanding C^1 map so that $\mathcal{D}$ is non denumerable.

Example 3.3.2. *Let $K \subset I = [0,1]$ be the middle third Cantor set and let $\beta_n(x) = d(x, K \cap [0,3^{-n}])$, where d is the Euclidean distance on I. Note that β_n is Lipschitz and $g_0' \circ \beta_n$ is bounded and continuous but not of α-generalized bounded variation for any $0 < \alpha < 1$, where g_0 was defined in Example 3.3.1. Indeed, since $(3^{-k}, 2 \cdot 3^{-k})$ is a gap of $K \cap [0,3^{-n}]$ for all $k > n$, then*

$$\begin{aligned} \frac{g_0'(\beta(3^{1-k}/2)) - g_0'(\beta(3^{-k}))}{3^{1-k}/2 - 3^{-k}} &= 2 \cdot 3^k g_0'\left(\frac{1}{2 \cdot 3^{k-1}}\right) \\ &= \frac{2 \cdot 3^k}{(1-k)\log 3 - \log 2}\left(\frac{1}{(1-k)\log 3 - \log 2} - 1\right) \end{aligned}$$

is not bounded when $k \nearrow \infty$. If $h : I \to \mathbb{R}$ is given by $h(x) = x + \left(\int_0^x g_0' \circ \beta_n\right)\left(\int_0^1 g_0' \circ \beta_n\right)^{-1}$, where the integrals are with respect to Lebesgue measure on the *real line, then $h(0) = 0$, $h(1) = 2$ and h induces a C^1 map $h_0 : \mathbb{S}^1 \to \mathbb{S}^1$ whose derivative is continuous but no C^α for any $0 < \alpha < 1$, such that $h_0'(x) = 1$ for each $x \in K \cap [0, 3^{-n}]$ and $h_0'(x) > 1$ otherwise.*

We set now $h_t(x) = x + \frac{\int_0^x (t+g_0'\circ\beta_n)}{\int_0^1 (t+g_0'\circ\beta_n)}$ for $t \in [0,1]$ which induces a C^1 map of $\mathbb{S}^1$ into itself with continuous derivative and $h_t' > 1$ for $t > 0$. We then define the skew-product map $f_0(x, y) = (E(x), h_{\sin \pi x}(y))$ for $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ which is a weak expanding map with $\mathcal{D} = \{0\} \times (K \cap [0, 3^{-n}])$.

For sufficiently big n it is easy to verify that f_0 satisfies items (1-3) in Example 3.2.1 for V a small neighborhood of the fixed point $(0, 0)$. Hence f_0 is also a C^1 non-uniformly expanding map which is not a $C^{1+\alpha}$ map for any $0 < \alpha < 1$.

Chapter 4

Preliminary definitions and results

The aim of this chapter is to fix notations, give definitions and state some facts which will be used throughout this work.

4.1 Invariant and ν-SRB-like measures

In this section, we revisit the definition and some results on the theory ν-SRB-like measures.

Let X be a compact metric space and $T : X \to X$ be continuous. Denote $\mathcal{M}_1$ the set of all Borel probability measures on X and $\mathcal{M}_T$ the set of all $T-$invariant Borel probability measures on X. In $\mathcal{M}_1$ fix the weak* metric

$$\operatorname{dist}(\mu, \nu) := \sum_{i=0}^{+\infty} \frac{1}{2^i} \left| \int \phi_i d\mu - \int \phi_i d\nu \right|, \tag{4.1.1}$$

where $\{\phi_i\}_{i\geq 0}$ is a countable family of continuous functions that is dense in the space $C^0(X, [0, 1])$.

The following technical lemma will be used in the proofs of Propositions A and B that are essential the proofs of Theorems B and D.

Lemma 4.1.1. *For all $\varepsilon > 0$ there is $\delta > 0$ such that if $d(T^i(x), T^i(y)) < \delta$ for all $i = 0, \dots, n-1$ then* $\operatorname{dist}(\sigma_n(x), \sigma_n(y)) < \varepsilon$

Proof. Given $\varepsilon > 0$ and a countable family of continuous functions $\mathcal{F} = \{\phi_k : k \in \mathbb{N}\}$ that is dense in the space $C^0(X, [0, 1])$, fix $N \geq 1$ such that $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon/2$. Let $\gamma = \varepsilon/4$ and consider $F = \{\phi_0, \dots, \phi_N\}$ the $(N + 1)$-first elements of $\mathcal{F}$.

Note that $V(\sigma_n(x), F, \gamma) \subset B_\varepsilon(\sigma_n(x))$, where

$$V(\sigma_n(x), F, \gamma) = \left\{\mu \in \mathcal{M}; \left|\int \phi_i d\mu - \int \phi_i d\sigma_n(x)\right| < \gamma,\ \forall i = 0, \ldots, N\right\}$$

is a neighborhood in the weak* topology.

In fact, given $\nu \in V(\sigma_n(x), F, \gamma)$ then $\left|\int \phi_i d\nu - \int \phi_i d\sigma_n(x)\right| < \gamma$ for all $i = 0, \ldots, N$ and

$$\begin{aligned} \operatorname{dist}(\nu, \sigma_n(x)) &= \sum_{i=0}^{N} \frac{1}{2^i}\left|\int \phi_i d\nu - \int \phi_i d\sigma_n(x)\right| + \sum_{i=N+1}^{\infty} \frac{1}{2^i}\left|\int \phi_i d\nu - \int \phi_i d\sigma_n(x)\right| \\ &< 2\gamma + \sum_{i=N+1}^{\infty} \frac{2\sup\|\phi_i\|}{2^i} < \frac{\varepsilon}{2} + \sum_{i=N}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

therefore $\nu \in B_\varepsilon(\sigma_n(x))$.

We will now show that there is $\delta > 0$ such that if $d(T^i(x), T^i(y)) < \delta$ for all $i = 0, \ldots, n-1$ then $\sigma_n(y) \in V(\sigma_n(x), F, \gamma) \subset B_\varepsilon(\sigma_n(x))$.

For each $k = 0, \ldots, N$ let $\delta(k) > 0$ be a constant of uniform continuity of ϕ_k and set $\delta = \min\{\delta(k);\ 0 \le k \le N\} > 0$. Thus, if $d(T^i(x), T^i(y)) < \delta$ for all $i = 0, \ldots, n-1$ we conclude that $\sigma_n(y) \in V(\sigma_n(x), F, \gamma)$, since $\int \phi_k d\sigma_n(z) = \frac{1}{n}\sum_{i=0}^{n-1} \phi_k(T^i(z))$, for all $k = 0, \ldots, N$ and

$$\left|\int \phi_k d\sigma_n(y) - \int \phi_k d\sigma_n(x)\right| \le \frac{1}{n}\sum_{i=0}^{n-1} |\phi_k(T^i(y)) - \phi_k(T^i(x))| < \frac{1}{n}\sum_{i=0}^{n-1} \gamma = \gamma.$$

□

Next results ensure the existence of ν-SRB-like measure and consequently, by Remark 2.4.1, that ν-weak-SRB-like measures do always exist.

Proposition 4.1.2. *Let $T : X \to X$ be a continuous map of a compact metric space X. For each reference probability measure ν there exist $\mathcal{W}_T(\nu) \subset \mathcal{M}_1$ the unique minimal non-empty and weak* compact set, such that $p\omega(x) \subset \mathcal{W}_T(\nu)$ for ν-a.e. $x \in X$. When $\nu = \mathrm{Leb}$ we denote $\mathcal{W}_T(\mathrm{Leb}) = \mathcal{W}_T$.*

Proof. In the case that ν coincides with the Lebesgue measure this corresponds to Theorem 1.5 in [18]. The proof presented here is the same as in Theorem 1.5 in [18], replacing Leb by ν.

Consider the family Υ of all the non-empty and weak* compact sets $A \subset \mathcal{M}_1$ such that $p\omega(x) \subset A$ for ν-a.e. $x \in M$.

The family Υ is not empty, since trivially $\mathcal{M}_T \in \Upsilon$. Define in Υ the partial order $A_1 \le A_2$ if and only if $A_1 \subset A_2$.

Let, $\{A_\alpha\}_{\alpha\in J} \subset \Upsilon$ is a chain if it is a totally ordered subset of Υ. Let us prove that $A := \cap_{\alpha\in J} A_\alpha$ belongs to Υ. For each fixed $\alpha \in J$, and for each $\varepsilon > 0$ define $B_0(\alpha) = \{x \in X; P\omega(x) \subset A_\alpha\}$ and $A^\varepsilon = \{x \in X; P\omega(x) \subset B_\varepsilon(A)\}$, where $B_\varepsilon(A) := \{\nu \in \mathcal{M}_1; \text{dist}(\nu, A) < \varepsilon\}$. To conclude that $A \in \Upsilon$, it is enough to prove that $\nu(A^\varepsilon) = 1$ for all $\varepsilon > 0$.

Claim 4.1.3. *For all $\varepsilon > 0$ there exists $\alpha \in J$ such that $A_\alpha \subset B_\varepsilon(A)$.*

If it did not exist then, by the property of finite intersections of compact sets, and since $\{A_\alpha\}_{\alpha\in I}$ is totally ordered, we would deduce that the set $\bigcap_{\alpha\in J}(A_\alpha \backslash B_\varepsilon(A)) \neq \emptyset$ would be non-empty, contained in A, but disjoint with its open neighborhood $B_\varepsilon(A)$.

We deduce that $B_0(\alpha) \subset B_\varepsilon(A)$. Since $A_\alpha \in \Upsilon$, we have that $\nu(B_0(\alpha)) = 1$ for all $\alpha \in I$. Thus $\nu(A^\varepsilon) = 1$ for all $\varepsilon > 0$, and therefore $A \in \Upsilon$. We have proved that each chain in Υ has a minimal element in Υ.

So, by Zorn's Lemma there exist minimal elements in Υ, namely, minimal non-empty and weak* compact sets $\mathcal{W}_T(\nu) \subset \mathcal{M}_1$ such that $p\omega(x) \subset \mathcal{W}_T(\nu)$ for ν almost all $x \in X$.

Finally, the minimal element $\mathcal{W}_T(\nu) \subset \Upsilon$ is unique since the intersection of two of them is also in Υ. □

Proposition 4.1.4. *Given ν a reference probability measure, a probability measure $\mu \in \mathcal{M}_1$ is ν-SRB-like if and only if $\mu \in \mathcal{W}_T(\nu)$.*

Proof. See proof of Proposition 2.2 in [19]. □

It is standard to check that $\mathcal{W}_T(\nu) \subset \mathcal{M}_T$. For more details on SRB-like measures, see [15, 18, 19] and see [17] for more details on weak-SRB-like measures.

4.2 Measure-theoretic entropy

For any Borel measurable finite partition $\mathcal{P}$ of M, and for any (not necessarily invariant) probability ν it is defined

$$H(\mathcal{P}, \nu) = -\sum_{P\in\mathcal{P}} \nu(P) \log \nu(P).$$

The *conditional entropy* of partition $\mathcal{P}$ given the partition $\mathcal{Q}$ is the number

$$H_\nu(\mathcal{P}/\mathcal{Q}) = -\sum_{P\in\mathcal{P}} \sum_{Q\in\mathcal{Q}} \nu(P \cap Q) \log \frac{\nu(P \cap Q)}{\nu(Q)}.$$

Lemma 4.2.1. *Given $S \geq 1$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for any finite partitions $\mathcal{P} = \{P_1, \ldots, P_S\}$ and $\tilde{\mathcal{P}} = \{\tilde{P}_1, \ldots, \tilde{P}_S\}$ such that $\nu(P_i \triangle \tilde{P}_i) < \delta$ for all $i = 1, \ldots, S$, then $H_\nu(\tilde{\mathcal{P}}/\mathcal{P}) < \varepsilon$.*

Proof. See Lemma 9.1.6 in [46] $\square$

Denote $\mathcal{P}^q = \bigvee_{j=0}^{q-1} f^{-j}(\mathcal{P})$, where for any pair of finite partitions $\mathcal{P}$ and $\mathcal{Q}$ it is defined $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q \neq \emptyset : P \in \mathcal{P},\ Q \in \mathcal{Q}\}$. If besides $\nu \in \mathcal{M}_f$ then

$$h(\mathcal{P}, \nu) = \lim_{q \to +\infty} \frac{1}{q} H(\mathcal{P}^q, \nu).$$

Finally, the measure-theoretic entropy $h_\nu(f)$ of an f-invariant measure ν is defined by $h_\nu(f) = \sup h(\mathcal{P}, \nu)$, where the sup is taken on all the Borel measurable finite partitions $\mathcal{P}$ of M.

We define the diameter $\operatorname{diam}(\mathcal{P})$ of a finite partition $\mathcal{P}$ as the maximum diameter of its pieces.

The following technical lemma will be used in the proof of the large deviation lemma that is essential for the proof of Theorem B.

Lemma 4.2.2. *Let $f : M \to M$ be a measurable function. For any sequence of not necessarily invariant probabilites ν_n, let $\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} (f^j)_* \nu_n$ and μ be a weak* accumulation point of (μ_n). Let $\mathcal{P}$ be a finite partition of M with S elements, with $\mu(\partial \mathcal{P}) = 0 = \mu_n(\partial \mathcal{P})$ for all $n \geq 1$. Then for, for any $\varepsilon > 0$, there is a subsequence of integers $n_i \nearrow \infty$ such that*

$$\frac{1}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) \leq \frac{\varepsilon}{4} + h_\mu(f) \ \ \forall i \geq 1.$$

Proof. Fix integers $q \geq 1$, and $n \geq q$. Write $n = aq + j$ where a, j are integer numbers such that $0 \leq j \leq q-1$. Fix a (not necessarily invariant) probability ν. From the properties of the entropy function H of ν with respect to the partition $\mathcal{P}$, we obtain

$$\begin{aligned} H(\mathcal{P}^n, \nu) &= H(\mathcal{P}^{aq+j}, \nu) \leq H(\mathcal{P}^{aq+q}, \nu) \\ &\leq H(\bigvee_{i=0}^{q-1} f^{-i}(\mathcal{P}), \nu) + H(\bigvee_{i=1}^{a} f^{-iq}(\mathcal{P}^q), \nu) \\ &\leq \sum_{i=0}^{q-1} H(\mathcal{P}, (f^i)_* \nu) + \sum_{i=1}^{a} H(\mathcal{P}^q, (f^{iq})_* \nu) \\ &\leq q \log S + \sum_{i=1}^{a} H(\mathcal{P}^q, (f^{iq})_* \nu) \ \forall q \geq 1, N \geq q. \end{aligned}$$

To obtain the inequality above recall that $H(\mathcal{P},\nu) \le \log S$ for all $\nu \in \mathcal{M}$ where S is the number of elements of the partition $\mathcal{P}$. The inequality above holds also for $f^{-l}(\mathcal{P})$ instead of $\mathcal{P}$, for any $l \ge 0$, because it holds for any partition with exactly S pieces. Thus

$$H(f^{-l}(\mathcal{P}^n),\nu) \le q\log S + \sum_{i=1}^{a} H(f^{-l}(\mathcal{P}^q),(f^{iq})_*\nu) = q\log S + \sum_{i=1}^{a} H(\mathcal{P}^q,(f^{l+iq})_*\nu).$$

Adding the above inequalities for $0 \le l \le q-1$, we obtain:

$$\sum_{l=0}^{q-1} H(f^{-l}(\mathcal{P}^n),\nu) \;\le\; q^2\log S + \sum_{l=0}^{q-1}\sum_{i=1}^{a} H(\mathcal{P}^q,(f^{l+iq})_*\nu).$$

Therefore,

$$\sum_{l=0}^{q-1} H(f^{-l}(\mathcal{P}^n),\nu) \;\le\; q^2\log S + \sum_{l=0}^{aq+q-1} H(\mathcal{P}^q,(f^{l})_*\nu). \tag{4.2.1}$$

On the other hand, for all $0 \le l \le q-1$,

$$\begin{aligned} H(\mathcal{P}^n,\nu) &\le H(\mathcal{P}^{n+l},\nu) \le \sum_{i=0}^{l-1} H(f^{-i}(\mathcal{P}),\nu) + H(f^{-l}(\mathcal{P}^n),\nu) \\ &\le q\log S + H(f^{-l}(\mathcal{P}^q),\nu). \end{aligned}$$

Therefore, adding the above inequalities for $0 \le l \le q-1$ and joining with the inequality (4.2.1), we obtain

$$qH(\mathcal{P}^n,\nu) \;\le\; 2q^2\log S + \sum_{l=0}^{aq+q-1} H(\mathcal{P}^q,(f^{l})_*\nu).$$

Recall that $n = aq + j$ with $0 \le j \le q-1$. So $aq + q \le n + q$ and then

$$\begin{aligned} qH(\mathcal{P}^n,\nu) &\le 2q^2\log S + \sum_{l=0}^{n-1} H(\mathcal{P}^q,(f^{l})_*\nu) + \sum_{l=n}^{aq+q-1} H(\mathcal{P}^q,(f^{l})_*\nu) \\ &\le 3q^2\log S + \sum_{l=0}^{n-1} H(\mathcal{P}^q,(f^{l})_*\nu). \end{aligned}$$

In the last inequality we have used that the number of non-empty pieces of $\mathcal{P}^q$ is at most S^q. Now we fix a sequence $n_i \nearrow \infty$ such that $\mu_{n_i} \xrightarrow[n\to+\infty]{w^*} \mu$ put $\nu = \nu_{n_i}$ and divide by n_i. Since H is convex we deduce:

$$\begin{aligned}
\frac{q}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) &\leq \frac{3q^2 \log S}{n_i} + \frac{1}{n_i} \sum_{l=0}^{n_i-1} H(\mathcal{P}^q, (f^l)_* \nu_{n_i}) + \frac{aq^2 \log S}{n_i} \\
&\leq \frac{3q^2 \log S}{n_i} + \frac{1}{n_i} \sum_{l=0}^{n_i-1} H(\mathcal{P}^q, (f^l)_* \nu_{n_i}) + \frac{aq^2 \log S}{n_i} \\
&\leq \frac{3q^2 \log S}{n_i} + H(\mathcal{P}^q, \mu_{n_i}).
\end{aligned}$$

Therefore, for all $i \geq i_0(q) = \max\left\{q, \frac{36q \log S}{\varepsilon}\right\}$

$$\frac{1}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) \leq \frac{\varepsilon}{12} + \frac{1}{q} H(\mathcal{P}^q, \mu_{n_i}) \ \ \forall i \geq i_0(q), \ \ \forall q \geq 1.$$

Since $\mu_{n_i} \xrightarrow[i\to+\infty]{w^*} \mu$ and $\mu_n(\partial\mathcal{P}) = \mu(\partial\mathcal{P}) = 0$ for all $n \in \mathbb{N}$, then $\lim\limits_{i\to+\infty} H(\mathcal{P}^q, \mu_{n_i}) = H(\mathcal{P}^q, \mu)$ because $\mu \in \mathcal{M}_f$. Thus there exists $i_1 > i_0(q)$ such that for all $i \geq i_1$

$$\frac{1}{q} H(\mathcal{P}^q, \mu_{n_i}) \leq \frac{1}{q} H(\mathcal{P}^q, \mu) + \frac{\varepsilon}{12}.$$

Thus, for all $i \geq i_1$

$$\frac{1}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) \leq \frac{\varepsilon}{6} + \frac{1}{q} H(\mathcal{P}^q, \mu).$$

Moreover, by definition $\lim\limits_{q\to+\infty} \frac{1}{q} H(\mathcal{P}^q, \mu) = h(\mathcal{P}, \mu)$ then there exists $q_0 \in \mathbb{N}$ such that $\frac{1}{q} H(\mathcal{P}^q, \mu) \leq h(\mathcal{P}, \mu) + \frac{\varepsilon}{12}$ for all $q \geq q_0$. Thus taking $i_2 := \max\{q_0, i_1\}$ we have

$$\frac{1}{n_i} H(\mathcal{P}^{n_i}, \nu_{n_i}) \leq h(\mathcal{P}, \mu) + \frac{\varepsilon}{4} \leq h_\mu(f) + \frac{\varepsilon}{4} \ \text{ for all } i \geq i_2. \tag{4.2.2}$$

This completes the proof using the subsequence $(n_i)_{i \geq i_2}$ as the sequence claimed in the statement of the lemma. □

Lemma 4.2.3. *Let $a_1, \ldots, a_\ell$ real numbers and let $p_1, \ldots, p_\ell$ non-negative numbers such that $\sum_{k=1}^{\ell} p_k = 1$. Denote $L = \sum_{k=1}^{\ell} e^{a_k}$. Then $\sum_{k=1}^{\ell} p_k(a_k - \log p_k) \leq \log L$. Moreover, the equality holds if, and only if, $p_k = \frac{e^{a_k}}{L}$ for all k.*

Proof. See Lemma 10.4.4 in [46]. □

Chapter 5

Continuous variation of SRB-like measures

Here we prove Theorem A showing that ν-SRB-like measures can be seen as measures that naturally arise as accumulation points of ν_n-SRB measures.

5.1 Measures with prescribed Jacobian. Conformal measures

Definition 12. A measurable function $J_\nu T : X \to [0, +\infty)$ is called the Jacobian of a map $T : X \to X$ with respect to a measure ν if for every Borel set $A \subset X$ on which T is injective

$$\nu(T(A)) = \int_A J_\nu T d\nu.$$

Next result guarantees the existence of measures with prescribed Jacobian.

Theorem 5.1.1. *Let $T : X \to X$ be a local homeomorphism of a compact metric space X and let $\phi : X \to \mathbb{R}$ be continuous. Then there exists a ϕ-conformal probability measure $\nu = \nu_\phi$ and a constant $\lambda > 0$, such that $\mathcal{L}_\phi^* \nu = \lambda \nu$. Moreover, the function $J_\nu T = \lambda e^{-\phi}$ is the Jacobian for T with respect to the measure ν.*

Proof. See Theorem 4.2.5 in [29]. □

5.2 SRB measures

We say that a continuous mapping $T : X \to X$ is open, if open sets have open images. This is equivalent to saying that if $f(x) = y$ and $y_n \xrightarrow[n\to+\infty]{} y$ then there exist $x_n \xrightarrow[n\to+\infty]{} x$ such that $f(x_n) = y_n$ for n large enough.

Definition 13. A continuous mapping $T : X \to X$ is called:

(a) *topologically exact* if for all non-empty open set $U \subset X$ there exists $N = N(U)$ such that $T^N U = X$.

(b) *topologically transitive* if for all non-empty open sets $U, V \subset X$ there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

Topological transitiveness ensures conformal measures give positive mass to any open subset.

Proposition 5.2.1. *Let $T : X \to X$ be an open expanding topologically transitive map and $\phi : X \to \mathbb{R}$ be continuous. Then every conformal measure $\nu = \nu_\phi$ is positive on non-empty open sets. Moreover for every $r > 0$ there exists $\alpha = \alpha(r) > 0$ such that for every $x \in X$, $\nu(B(x, r)) \geq \alpha$.*

Proof. See Proposition 4.2.7. in [29] □

For Hölder potentials it is known that there exists a unique ν-SRB probability measure.

Theorem 5.2.2. *Let $T : X \to X$ an open expanding topologically transitive map, $\nu = \nu_\phi$ a conformal measure associated to the a Hölder continuous function $\phi : X \to \mathbb{R}$. Then there exists a unique μ_ϕ ergodic invariant ν-SRB probability measure such that μ_ϕ is the unique equilibrium state for T and ϕ.*

Proof. The proof follows the results presented in Chapter 4 of [29]. □

The following result shows that positively invariant sets with positive reference measure have mass uniformly bounded away from zero.

Theorem 5.2.3. *In the same setting of Theorem 5.2.2, if G is an T-invariant set such that $\nu(G) > 0$, then there is a disk Δ of radius $\delta/4$ so that $\nu(\Delta \backslash G) = 0$.*

Proof. See the proof of Lemma 5.3 in [45]. □

Remark 5.2.4. *It is easy to see that each continuous potential in a compact metric space can be approximated in uniform convergence by Lipschitz continuous potentials (in particular, all Lipschitz continuous potentials are α-Hölder continuous for $0 < \alpha \leq 1$).*

In fact, let $\mathcal{A} = \{f : X \to \mathbb{R};\ f$ is Lipschitz continuous$\} \subset C(X, \mathbb{R})$ and observe that $\mathcal{A}$ is an subalgebra of the algebra $C(X, \mathbb{R})$, since $\mathcal{A}$ forms a vector space over $\mathbb{R}$ and given $f, g \in \mathcal{A}$, $x \in X$ then $(f \cdot g)(x) = f(x) \cdot g(x) \in \mathcal{A}$. Moreover, $f \equiv 1$ belongs to $\mathcal{A}$ and $\mathcal{A}$ separates points, since given $x, y \in X$, $x \neq y$, we may take $f : X \to \mathbb{R}$ given by $f(z) = d(z, \{x\})$, f is Lipschitz continuous (because, $\{x\}$ is a closed set) therefore $f \in \mathcal{A}$ and $f(x) \neq f(y)$. Thus, the Theorem of Stone-Weierstrass ensures that $\mathcal{A}$ is dense in $C(X, \mathbb{R})$.

5.3 Limits of SRB-measures

Now we are ready to prove Theorem A.

Proof of Theorem A. Let $\lambda_{n_j} \xrightarrow[j \to +\infty]{} \lambda$, where $\mathcal{L}^*{}_{\phi_{n_j}}(\nu_{n_j}) = \lambda_{n_j} \nu_{n_j}$ (it is easy to see that $\lambda > 0$ since $\lambda_n = \mathcal{L}^*_{\phi_n}(\nu_n)(1)$ for all n).

Then for each $\varphi \in C^0(X, \mathbb{R})$ we have

$$\int \varphi \, d(\mathcal{L}^*_{\phi_{n_j}} \nu_{n_j}) = \int \mathcal{L}_{\phi_{n_j}}(\varphi) \, d\nu_{n_j} \xrightarrow[j \to +\infty]{} \int \mathcal{L}_{\phi}(\varphi) \, d\nu = \int \varphi \, d(\mathcal{L}^*_{\phi} \nu)$$

thus, $\mathcal{L}^*_{\phi_{n_j}} \nu_{n_j} \xrightarrow[j \to +\infty]{w^*} \mathcal{L}^*_{\phi} \nu$. Moreover, $\lambda_{n_j} \nu_{n_j} \xrightarrow[j \to +\infty]{w^*} \lambda \nu$ and by uniqueness of the limit, it follows that

$$\lambda \nu = \lim_{j \to +\infty} \lambda_{n_j} \nu_{n_j} = \lim_{j \to +\infty} \mathcal{L}^*_{\phi_{n_j}} \nu_{n_j} = \mathcal{L}^*_{\phi} \nu.$$

Thus, ν is a ϕ-confomal measure.

Let μ_{n_j} be the ν_{n_j}-SRB measure and let $\mu = \lim_{j \to +\infty} \mu_{n_j}$. For each fixed $\varepsilon > 0$, consider $N = N(\varepsilon)$ such that

$$\operatorname{dist}(\mu_{n_j}, \mu_{n_m}) < \frac{\varepsilon}{4} \ \text{ and } \ \operatorname{dist}(\mu_{n_j}, \mu) < \frac{\varepsilon}{4} \ \ \forall j, m \geq N.$$

Thus, $A_{\varepsilon/4}(\mu_{n_j}) \subset A_{\varepsilon/2}(\mu_{n_m})$ and $A_{\varepsilon/2}(\mu_{n_j}) \subset A_{\varepsilon}(\mu)$ for all $j, m \geq N$.

In fact, for each $x \in A_{\varepsilon/4}(\mu_{n_j})$ we have $\operatorname{dist}(p\omega(x), \mu_{n_j}) < \varepsilon/4$. Then,

$$\operatorname{dist}(p\omega(x), \mu_{n_m}) \leq \operatorname{dist}(p\omega(x), \mu_{n_j}) + \operatorname{dist}(\mu_{n_j}, \mu_{n_m}) < \frac{\varepsilon}{2}$$

for all $j, m \geq N$. Therefore, $A_{\varepsilon/4}(\mu_{n_j}) \subset A_{\varepsilon/2}(\mu_{n_m})$ for all $j, m \geq N$. Analogously, $A_{\varepsilon/2}(\mu_{n_j}) \subset A_{\varepsilon}(\mu)$.

By Theorem 5.2.3, for each $j \geq 1$ there exists a disk Δ_{n_j} of radius $\delta_1/4$ around x_{n_j} such that $\nu_{n_j}(\Delta_{n_j} \backslash A_{\varepsilon/2}(\mu_{n_j})) = 0$. Let $x = \lim\limits_{j\to+\infty} x_{n_j}$ taking a subsequence if necessary. By compactness, the sequence $\left(\Delta_{n_j}\right)_j$ accumulates on a disc $\tilde{\Delta}$ of radius $\delta_1/4$ around x. Let Δ be the disk of radius $0 < s \leq \delta_1/8$ around x. Thus, there exists $N_0 \geq N$ such that $\Delta \subset \Delta_{n_j}$ for all $j \geq N_0$.

Consider $0 < s \leq \delta_1/8$ such that $\nu\Big(\partial\Big(\Delta \backslash A_{\varepsilon}(\mu)\Big)\Big) = 0$ and note that for all $j \geq N_0$,

$$0 = \nu_{n_j}\Big(\Delta_{n_j} \backslash A_{\varepsilon/2}(\mu_{n_j})\Big) \geq \nu_{n_j}\Big(\Delta \backslash A_{\varepsilon/2}(\mu_{n_j})\Big) \geq \nu_{n_j}\Big(\Delta \backslash A_{\varepsilon}(\mu)\Big),$$

since $\Delta \subset \Delta_{n_j}$ and $A_{\varepsilon/2}(\mu_{n_j}) \subset A_{\varepsilon}(\mu)$ for all $j \geq N_0$.

Thus, by weak* convergence

$$\nu\Big(\Delta \backslash A_{\varepsilon}(\mu)\Big) = \lim_{j\to+\infty} \nu_{n_j}\Big(\Delta \backslash A_{\varepsilon}(\mu)\Big) = 0. \tag{5.3.1}$$

Since ν is a conformal measure, it follows from (5.3.1) and Proposition 5.2.1 that $\nu\Big(A_{\varepsilon}(\mu)\Big) \geq \nu\Big(\Delta\Big) > 0$. As $\varepsilon > 0$ is arbitrary, we conclude that μ is ν-SRB-like.

Let $\mathcal{P}$ be a finite Borel partition of X with diameter not exceeding an expansive constant such that $\mu(\partial\mathcal{P}) = 0$. Then $\mathcal{P}$ generates the Borel sigma-algebra for every Borel probability T-invariant in X (see Lemma 2.5.5 in [29]) and by Kolmogorov-Sinai Theorem this implies that $\eta \mapsto h_{\eta}(T)$ is upper semi-continuous in μ.

Moreover, as $\int \phi_{n_j}\, d\mu_{n_j} \xrightarrow[j\to+\infty]{} \int \phi\, d\mu$, μ_{n_j} is an equilibrium state for (T, ϕ_{n_j}) (by Theorem 5.2.2) and by continuity of $\varphi \mapsto P_{\mathrm{top}}(T, \varphi)$ (see Theorem 9.7 in [47]) it follows that

$$h_{\mu}(T) + \int \phi d\mu \geq \lim_{j\to+\infty} \left(h_{\mu_{n_j}}(T) + \int \phi_{n_j} d\mu_{n_j}\right) = \lim_{j\to+\infty} P_{\mathrm{top}}(T, \phi_{n_j}) = P_{\mathrm{top}}(T, \phi).$$

This shows that μ is an equilibrium state for T with respect to ϕ.

Now we assume that T is topologically exact. We know that given $\varepsilon > 0$ there exists a disk Δ such that $\nu(\Delta \backslash A_{\varepsilon}(\mu)) = 0$. Since $\Delta \subset X$ is a non-empty open set, there exists $N > 0$ such that $X = T^N(\Delta)$. Moreover, $A_{\varepsilon}(\mu) = T(A_{\varepsilon}(\mu))$, thus

$$0 = \nu(T^N(\Delta \backslash A_{\varepsilon}(\mu)) \geq \nu(T^N(\Delta) \backslash A_{\varepsilon}(\mu)) = \nu(X \backslash A_{\varepsilon}(\mu)).$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\nu(A_{\varepsilon}(\mu)) = 1$ for all $\varepsilon > 0$.

□

Chapter 6

Expanding maps on compact metric spaces

Here we state the main results needed to obtain the proof of Theorem B. We start by presenting some results about expanding maps that will be used throughout this chapter. We close the chapter with the proof of Theorem B.

6.1 Basic properties of expanding open maps

In what follows X is a compact metric space.

Lemma 6.1.1. *If $T : X \to X$ is a continuous open map, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $T(B(x, \varepsilon)) \supset B(T(x), \delta)$ for every $x \in X$.*

Proof. See Lemma 3.1.2 in [29]. □

Remark 6.1.2. *If $T : X \to X$ is an expanding map, then by (2.2.1) and (2.2.2), for all $x \in X$, the restriction $T|_{B(x,\varepsilon)}$ is injective and therefore it has a local inverse map on $T(B(x, \varepsilon))$.*

If additionally $T : X \to X$ is an open map, then, in view of Lemma 6.1.1, the domain of the inverse map contains the ball $B(T(x), \delta)$. So it makes sense to define the restriction of the inverse map, $T_x^{-1} : B(T(x), \delta) \to B(x, \varepsilon)$ at each $x \in X$.

Lemma 6.1.3. *Let $T : X \to X$ be an open expanding map. If $x \in X$ and $y, z \in B(T(x), \delta)$ then $d(T_x^{-1}(y), T_x^{-1}(z)) \le \lambda^{-1} d(y, z)$. In particular $T_x^{-1}(B(T(x), \delta)) \subset B(x, \lambda^{-1}\delta) \subset B(x, \delta)$ and*

$$B(T(x), \delta) \subset T(B(x, \lambda^{-1}\delta)) \tag{6.1.1}$$

for all $\delta > 0$ small enough.

Proof. See Lemma 3.1.4 from [29]. □

Definition 14. Let $T : X \to X$ be an open expanding map. For every $x \in X$, every $n \geq 1$ and every $j = 0, 1, \dots, n-1$ write $x_j = T^j(x)$. In view of Lemma 6.1.3 the composition

$$T_{x_0}^{-1} \circ T_{x_1}^{-1} \circ \cdots \circ T_{x_{n-1}}^{-1} : B(T^n(x), \delta) \to X$$

is well-defined and will be denoted by T_x^{-n}.

Lemma 6.1.4. *Let $T : X \to X$ be an open expanding map. For every $x \in X$ we have:*

1. $T^{-n}(A) = \bigcup_{y \in T^{-n}(x)} T_y^{-n}(A)$ *for all* $A \subset B(x, \delta)$*;*

2. $d(T_x^{-n}(y), T_x^{-n}(z)) \leq \lambda^{-n} d(y, z)$ *for all* $y, z \in B(T^n(x), \delta)$*;*

3. $T_x^{-n}(B(T^n(x), r)) \subset B(x, \min\{\varepsilon, \lambda^{-n} r\})$ *for all* $r \leq \delta$*.*

Proof. The lemma follow from Lemma 6.1.3. □

For more details on the proofs in this subsection, see subsection 3.1 in [29].

6.2 Conformal measures. Weak Gibbs property

Here we cite results relating ϕ-conformal measures and the topological pressure of ϕ. Next results says ϕ-conformal measures for expanding dynamics with continuous potentials are almost Gibbs measures; see [25, 29] for more details.

Proposition 6.2.1. *Let $T : X \to X$ an open expanding topologically transitive map, $\nu = \nu_\phi$ a conformal measure associated the a continuous function $\phi : X \to \mathbb{R}$ and $J_\nu T = \lambda e^{-\phi}$ the Jacobian for T with respect to the measure ν. Given $\delta > 0$, for all $x \in X$ and all $n \geq 1$ there exists $\alpha(\varepsilon) > 0$ such that*

$$\alpha(\varepsilon) e^{-n\delta} \leq \frac{\nu((B(x, n, \varepsilon)))}{\exp\left(S_n\phi(x) - Pn\right)} \leq e^{n\delta}$$

for all $\varepsilon > 0$ small enough, where $P = \log \lambda$.

Proof. Given $\delta > 0$ there exist $\gamma > 0$ such that for all $x, y \in X$, with $d(x, y) < \gamma$ we have $|\phi(x) - \phi(y)| < \delta$. Fix $0 < \varepsilon < \gamma$, $x \in X$ and $n \geq 1$ arbitrary. Then

$$\nu(B(T^n x, \varepsilon)) = \nu(T^n(B(x, n, \varepsilon))) = \int_{B(x,n,\varepsilon)} J_\nu T^n \, d\nu. \tag{6.2.1}$$

Hence, since it gives uniform weight to balls of fixed radius and the by uniform continuity of ϕ, we obtain

$$\begin{aligned} \alpha(\varepsilon) &\leq \nu(B(T^n x, \varepsilon)) = \int_{B(x,n,\varepsilon)} J_\nu T^n \, d\nu = \int_{B(x,n,\varepsilon)} \lambda^n e^{-S_n \phi} \, d\nu \\ &\leq \lambda^n e^{-S_n \phi(x) + n\delta} \cdot \nu(B(x, n, \varepsilon)), \end{aligned}$$

and

$$1 \geq \nu(B(T^n x, \varepsilon)) = \int_{B(x,n,\varepsilon)} J_\nu T^n \, d\nu \geq \lambda^n e^{-S_n \phi(x) - n\delta} \cdot \nu(B(x, n, \varepsilon)).$$

Hence

$$\alpha(\varepsilon) e^{-n\delta} \leq \frac{\nu(B(x, n, \varepsilon))}{\exp\left(S_n \phi(x) - Pn\right)} \leq e^{n\delta}$$

where $P = \log \lambda$. □

Lemma 6.2.2. *Let $T : X \to X$ be an open expanding topologically transitive map, let $\phi : X \to \mathbb{R}$ be continuous and ν be a probability measure such that $\mathcal{L}_\phi^*(\nu) = \lambda \nu$. Then $P_{top}(T, \phi) \leq P = \log \lambda$.*

Proof. Given $\delta > 0$ consider $\varepsilon > 0$ small enough as in Proposition 6.2.1 and $n \geq 1$. Let $E_n \subset X$ be a maximal (n, ε)-separated set. Therefore $\{B(x, n, \varepsilon) : x \in E_n\}$ covers X, $\{B(x, n, \varepsilon/2) : x \in E_n\}$ is pairwise disjoint open sets. Thus by Proposition 6.2.1 we have,

$$\sum_{x \in E_n} e^{S_n \phi(x)} \leq \sum_{x \in E_n} \nu(B(x, n, \varepsilon/2)) \cdot \frac{e^{n(P+\delta)}}{\alpha(\varepsilon/2)} \leq \frac{e^{n(P+\delta)}}{\alpha(\varepsilon/2)}.$$

Therefore,

$$P_{\text{top}}(T, \phi) = \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{x \in E_n} e^{S_n \phi(x)} \leq P + \delta,$$

as $\delta > 0$ can be taken small enough, we conclude that $P_{\text{top}}(T, \phi) - P \leq 0$. □

6.3 Constructing an arbitrarily small initial partition with negligible boundary

Let ν be a Borel probability measure on the compact metric space X. We say that T is *ν-regular map* or ν is regular for T if $T_*\nu \ll \nu$, that is, if $E \subset X$ is such that $\nu(E) = 0$, then $\nu\Big(T^{-1}(E)\Big) = 0$. It follows that if T admits a Jacobian with respect a T-regular measure.

Lemma 6.3.1. *Let X be a compact metric space of dimension $d \geq 1$. If ν is a regular reference measure for T positive on non-empty open sets and $\delta > 0$, then there exists a finite partition $\mathcal{P}$ of X with* $\operatorname{diam}(\mathcal{P}) < \delta$ *such that every atom $P \in \mathcal{P}$ have non-empty interior and $\nu(\partial\mathcal{P}) = 0$.*

Proof. Let $\delta > 0$ and consider $0 < \delta_1 \leq \delta$ and $\mathcal{B} = \{B(\tilde{x}_l, \delta_1/8), l = 1, \ldots, q\}$ be a finite open cover of X by $\delta_1/8$- balls such that $\nu(\partial B(\tilde{x}_l, \delta_1/8)) = 0$ for all $l = 1, \ldots, q$. Note that such value of $\delta_1 > 0$ exists since the set of values of δ_1 such that $\nu(\partial B(\tilde{x}_l, \delta_1/8)) > 0$ for some $l \in \{1, \ldots, q\}$ is denumerable, because ν is a finite measure and X is a compact metric space soon separable.

From this we define a finite partition $\mathcal{P}$ of X as follows. We start by setting

$$P_1 := B\left(\tilde{x}_1, \frac{\delta_1}{8}\right), \ldots, P_k := B\left(\tilde{x}_k, \frac{\delta_1}{8}\right) \setminus (P_1 \cup \ldots \cup P_{k-1})$$

for all $k = 2, \ldots, S$ where $S \leq q$.

Note that if $P_k \neq \emptyset$ then P_k has non-empty interior (since X is separable), diameter smaller than $2\frac{\delta_1}{8} < \frac{\delta_1}{4}$ and the boundary ∂P_k is a (finite) union of pieces of boundaries of balls with zero ν-measure. We define $\mathcal{P}$ by the elements P_k constructed above which are non-empty.

Note that since T is ν-regular the boundary of $g(P)$ still has zero ν-measure for every atom $P \in \mathcal{P}$ and every inverse branch g of T^n, for any $n \geq 1$. $\square$

6.4 Expansive maps and existence of equilibrium states

A continuous transformation $T : X \to X$ of a compact metric space X equipped with a metric ρ is (positively) expansive if and only if

$$\exists \delta > 0 [\forall n \geq 0 \ \rho(T^n(x), T^n(y)) \leq \delta] \Longrightarrow x = y$$

and the number δ above is called an *expansive constant*.

Theorem 6.4.1. *If $T : X \to X$ is positively expansive, then the function $\mathcal{M}_T \ni \mu \mapsto h_\mu(T)$ is upper semi-continuous and consequently each continuous potential $\phi : X \to \mathbb{R}$ has an equilibrium state.*

Proof. See Theorem 2.5.6 in [29]. □

Theorem 6.4.2. *Expanding property implies positively expansive property.*

Proof. See Theorem 3.1.1 in [29]. □

Corollary 6.4.3. *If $T : X \to X$ is expanding, then each continuous potential $\phi : X \to \mathbb{R}$ has an equilibrium state. In particular, $\mathcal{K}_r(\phi) = \{\mu \in \mathcal{M}_T : h_\mu(T) + \int \phi d\mu \geq P_{top}(T, \phi) - r\} \neq \emptyset$ for all $r \geq 0$.*

Proof. The proof is immediate from Theorem 6.4.2. □

6.5 Large deviations

The statement of Theorem B is a consequence of the following more abstract result, inspired in Lemma 4.3 in [19] and in Proposition 6.1.11 in [25].

Proposition A. *Let $T : X \to X$ be an open expanding topologically transitive map and let $\phi : X \to \mathbb{R}$ be continuous. Fix $\nu = \nu_\phi$ a ϕ-conformal measure, $r > 0$ and consider the weak* distance* dist *defined in (4.1.1). Then, for all $0 < \varepsilon < r$, there exists $n_0 \geq 1$ and $\kappa > 0$ such that*

$$\nu\Big(\Big\{x \in X; \mathrm{dist}(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \varepsilon\Big\}\Big) < \kappa \exp\left[n(\varepsilon - r)\right] \ \ \forall n \geq n_0. \qquad (6.5.1)$$

Proof. We know by Theorem 5.1.1 and Proposition 5.2.1 that all the (necessarily existing) conformal measures ν are positive on non-empty open sets and $J_\nu T = \lambda e^{-\phi}$ is the Jacobian for T with respect to the measure ν.

Consider ν a conformal measure, fix $r > 0$ and let $0 < \varepsilon < r$. For $\varepsilon/6$, fix a constant $\gamma > 0$ of uniform continuity of ϕ, i.e., $|\phi(x) - \phi(y)| < \frac{\varepsilon}{6}$ whenever $d(x, y) < \gamma$. Consider $0 < \xi < \gamma$ and a partition $\mathcal{P}$ of X as in Lemma 6.3.1 such that $\mathrm{diam}(\mathcal{P}) < \frac{\xi}{4}$.

Let us choose one interior point in each atom $P \in \mathcal{P}$, and form the set $C_0 = \{w_1, \ldots, w_S\}$ of representatives of the atoms of $\mathcal{P} = \{P_l\}_{1 \leq l \leq S}$ (where $S = \#\{P \in \mathcal{P} : P \neq \emptyset\}$). Let $d_0 = \min\{d(w, \partial\mathcal{P}), w \in C_0\} > 0$ where $\partial\mathcal{P} = \bigcup_{l=1}^{S} \partial P_l$ is the boundary of $\mathcal{P}$.

Consider $\mathcal{A} := \{\mu \in \mathcal{M}; \mathrm{dist}(\mu, \mathcal{K}_r(\phi)) \geq \varepsilon\}$. Note that $\mathcal{A}$ is weak* compact, so it has a finite covering $B_1, \ldots, B_\kappa$ for minimal cardinality $\kappa \geq 1$, with open balls $B_i \subset \mathcal{M}$ of radius $\frac{\varepsilon}{3}$. For any fixed $n \geq 1$ write $C_{n,i} = \{x \in X; \sigma_n(x) \in B_i\}$, $C_n = \bigcup_{i=1}^{\kappa} C_{n,i}$, $\tilde{C}_{n,i} = \{x \in X; \sigma_n(x) \in \tilde{B}_i\}$ and $\tilde{C}_n = \bigcup_{i=1}^{\kappa} \tilde{C}_{n,i}$, where $\tilde{B}_i$ are open balls concentric with B_i of radius $\frac{2\varepsilon}{3}$ for $i = 1, \ldots, \kappa$.

Note that $C_{n,i} \subset \tilde{C}_{n,i}$. Moreover, $\{x \in X; \operatorname{dist}(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \varepsilon\} \subset C_n \subset \tilde{C}_n$.

Claim 6.5.1. *For each* $1 \leq i \leq \kappa$ *there exists* $n_i > 0$ *such that* $\nu(C_{n,i}) \leq \exp[n(\varepsilon - r)]$ *for all* $n \geq n_i$.

First, let us see that it is enough to prove the Claim 6.5.1 to finish the proof of the lemma. In fact, if Claim 6.5.1 holds, put $n_0 = \max_{1 \leq i \leq \kappa} n_i$.Then we obtain the following inequality for all $n \geq n_0$, as needed:

$$\nu(C_n) \leq \sum_{i=1}^{\kappa} \nu(C_{n,i}) \leq \kappa \exp[n(\varepsilon - r)].$$

Let us prove Claim 6.5.1 for $x \in C_{n,i}$ let $P \in \mathcal{P}$ be the atom such that $T^n(x) \in P$ and set $Q = T_x^{-n}(P)$. Then the family $\mathcal{Q}_n$ of all such sets Q is finite since both $\mathcal{P}$ and the number of inverse branches are finite. Moreover by the expression of $J_\nu T$ in terms of ϕ

$$\nu(Q \cap C_{n,i}) = \int_{T^n(Q \cap C_{n,i})} J_\nu T^{-n} d\nu = \int_{T^n(Q \cap C_{n,i})} \exp\left[\sum_{j=0}^{n-1} \phi \circ T^j - n \log \lambda\right] d\nu.$$

We note that if $\nu(C_{n,i}) = 0$, then Claim 6.5.1 becomes trivially proved. Consider the finite family of atoms $\{Q_1, \ldots, Q_N\} = \{Q \in \mathcal{Q}_n : \nu(Q \cap C_{n,i}) > 0\}$ which has $N = N(n, i)$ elements for some $N \geq 1$.

Note that $\nu(C_{n,i}) = \sum_{k=1}^{N} \nu(Q_k \cap C_{n,i})$. For each $k = 1, \ldots, N$, consider $x_k \in Q_k$ such that $T^n(x_k) = w_j$ for some $j = 1, \ldots, S$ (recall that w_j are interior points of each atom of the partition $\mathcal{P}$, so there is only one $j = j_{k,n}$ for each $x_k \in Q_k$ such that $T^n(x_k) = w_j$).

Since $\operatorname{diam}(\mathcal{P}^n) < \operatorname{diam}(\mathcal{P}) < \frac{\xi}{4} < \xi$ for all $n > 0$ (remember Lemma 6.1.4), then $|\phi(T^j(x_k)) - \phi(T^j(y))| < \frac{\varepsilon}{6}$ for all $y \in Q_k$ and $j = 0, \ldots, n-1$. Considering for each k, $y_k \in Q_k \cap C_{n,i}$ then,

$$\begin{aligned}
\nu(C_{n,i}) &= \sum_{k=1}^{N} \nu(Q_k \cap C_{n,i}) \le \sum_{k=1}^{N} \int_{T^n(Q_k \cap C_{n,i})} \exp\left[\sum_{j=0}^{n-1} \Big(\phi(T^j(y_k)) + \frac{\varepsilon}{6}\Big) - n\log\lambda\right] d\nu \\
&\le \sum_{k=1}^{N} \exp\left[\sum_{j=0}^{n-1} \Big(\phi(T^j(x_k)) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\Big) - n\log\lambda\right] \cdot \nu(T^n(Q_k \cap C_{n,i})) \\
&\le \sum_{k=1}^{N} \exp\left[\sum_{j=0}^{n-1} \Big(\phi(T^j(x_k)) + \frac{\varepsilon}{3}\Big) - n\log\lambda\right] \\
&\le \exp\left[n\Big(\frac{\varepsilon}{3} - \log\lambda\Big)\right] \sum_{k=1}^{N} \exp\left[\sum_{j=0}^{n-1} \phi(T^j(x_k))\right].
\end{aligned}$$

Defining

$$L := \sum_{k=1}^{N} \exp\left[\sum_{j=0}^{n-1} \phi(T^j(x_k))\right], \quad \lambda_k := \frac{1}{L} \exp\left[\sum_{j=0}^{n-1} \phi(T^j(x_k))\right]$$

then $\sum_{k=1}^{N} \lambda_k = 1$ and by Lemma 4.2.3

$$\log L = \left(\sum_{k=1}^{N} \lambda_k \sum_{j=0}^{n-1} \phi(T^j(x_k))\right) - \left(\sum_{k=1}^{N} \lambda_k \log\lambda_k\right).$$

Define the probability measures $\nu_n := \sum_{k=1}^{N} \lambda_k \delta_{x_k}$ and $\mu_n := \sum_{k=1}^{N} \lambda_k \sigma_n(x_k)$ so that we may rewrite, $\log L = n\int \phi d\mu_n + H(\mathcal{P}^n, \nu_n)$. We fix a weak* accumulation point μ of $(\mu_n)_n$ and take a subsequence $n_j \xrightarrow[j\to+\infty]{} +\infty$ such that $\mu_{n_j} \xrightarrow{w^*} \mu$ and

$$\limsup_{n\to+\infty} \frac{1}{n} \log\nu(C_{n,i}) = \lim_{j\to+\infty} \frac{1}{n_j} \log\nu(C_{n_j,i}). \tag{6.5.2}$$

We now make a small perturbation of the original partition so that the points $x_k \in Q_k$ are still given by the image of the same n th inverse branch of T of an atom of the perturbed partition and the boundaries of the new partition also have negligible μ measure.

Let $0 < \eta < \min\{\frac{d_0}{2}, \frac{\xi}{4}\}$ (note that d_0 does not depend on n) such that for all $l = 1, \dots, S$ and for each $n \ge 1$

$$\mu\Big(\partial B\Big(\tilde{x}_l, \frac{\xi}{4} + \eta\Big)\Big) = 0 = \mu_n\Big(\partial B\Big(\tilde{x}_l, \frac{\xi}{4} + \eta\Big)\Big) \tag{6.5.3}$$

where the centers are the ones from the construction of the initial partition $\mathcal{P}$ in the proof of Lemma 6.3.1.

Such value of η exists since the set of values of $\eta > 0$ such that some of the expressions in (6.5.3) is positive for some $l \in \{1, \ldots, S\}$ and some $n \geq 1$ is at most denumerable because the measures involved are probability measures. Thus we may take $\eta > 0$ satisfying (6.5.3) arbitrarily close to zero.

We consider now the finite open cover

$$\tilde{C} = \{B(\tilde{x}_l, \xi/4 + \eta) : l = 1, \ldots, S\} \tag{6.5.4}$$

of X and construct the partition $\tilde{\mathcal{P}}$ induced by $\tilde{C}$ by the same procedure as before and following the some order of construction $\mathcal{P}$, ($\tilde{x}_l, l \in \{1, \ldots, S\}$ are the same used in the construction of $\mathcal{P}$ in Lemma 6.3.1).

Note that $d(w_l, \partial B(\tilde{x}_l, \xi/4 + \eta)) > d_0 - \eta > d_0/2$ for all $l = 1, \ldots, S$ and $w_l \in C_0$ by construction. Therefore, each $w_l \in C_0$ is contained in some atom $P_{w_l} \in \tilde{\mathcal{P}}$. Moreover there cannot be distinct $w_l, w_k \in C_0$ such that $w_l \in P_{w_k}$, by the choice of η. Thus, the number of atoms of $\mathcal{P}$ is less than or equal to the number of atoms of $\tilde{\mathcal{P}}$. On the other hand, by construction the maximum number of elements $\tilde{\mathcal{P}}$ is $S = \mathrm{Card}(\mathcal{P})$, because $\mathrm{Card}(\tilde{C}) = S$. In this way we conclude that the partition $\tilde{\mathcal{P}}$ has the same number of atoms as $\mathcal{P}$.

We have

(1) $\mathrm{diam}(\tilde{\mathcal{P}}) < 2(\xi/4 + \eta) < \xi$;

(2) $\mu(\partial\tilde{\mathcal{P}}) = 0 = \mu_n(\partial\tilde{\mathcal{P}})$ for all $n \geq 1$;

(3) $w \in \mathrm{int}(\tilde{\mathcal{P}}(w))$, $\forall w \in C_0$.

Note that $H(\tilde{\mathcal{P}}^{n_j}, \nu_{n_j}) = H(\mathcal{P}^{n_j}, \nu_{n_j})$ by definition of the ν_{n_j} and by construction of the $\tilde{\mathcal{P}}$ as a perturbation of $\mathcal{P}$. Using items (1) and (2) we get, by Lemma 4.2.2, that there exists $j_0 > 0$ such that

$$\frac{1}{n_j} H(\mathcal{P}^{n_j}, \nu_{n_j}) = \frac{1}{n_j} H(\tilde{\mathcal{P}}^{n_j}, \nu_{n_j}) \leq h_\mu(T) + \frac{\varepsilon}{3}, \quad \forall j \geq j_0. \tag{6.5.5}$$

For the partition obtained above, we have that $x_k \in \tilde{Q}_k \cap Q_k$ and $Q_k \cap C_{n,i} \neq \emptyset$, where $\tilde{Q}_k$ belongs to $\tilde{\mathcal{Q}}_n$ for $k = 1, \ldots, N(n, i)$, where this family is defined similarly to $\mathcal{Q}_n$ changing only the atoms of the original partition $\mathcal{P}$ by the atoms of the perturbed partition $\tilde{\mathcal{P}}$.

Consider $y \in Q_k \cap C_{n,i}$. Then $\sigma_n(y) \in B_i$. As $d(T^j(x_k), T^j(y)) \leq \mathrm{diam}(\mathcal{P}) < \xi$, for all $j = 0, \ldots, n-1$ then, by Lemma 4.1.1, $d(\sigma_n(x_k), \sigma_n(y)) < \varepsilon/3$ and as $B_i \subset \tilde{B}_i$ concentrically, we conclude that $\sigma_n(x_k) \in \tilde{B}_i$.

Since the ball $\tilde{B}_i$ is convex and μ_n is a convex combination of the measures $\sigma_n(x_k)$, (recall that $\sum \lambda_k = 1$), we deduce that $\mu_n \in \tilde{B}_i$. Therefore, the weak* limit μ of any convergent subsequence of $\{\mu_n\}_n$ belongs to the weak* closure $\overline{\tilde{B}}_i$. Since the ball $\tilde{B}_i$ has radius $\frac{2\varepsilon}{3}$ we conclude that $\mu \notin \mathcal{K}_r(\phi)$. Then $\int \phi d\mu + h_\mu(T) < P_{\text{top}}(T, \phi) - r$, and therefore

$$\begin{aligned} \nu(C_{n,i}) &\leq \exp\Big[n\Big(\frac{\varepsilon}{3} - \log\lambda\Big)\Big] \cdot L = \exp\Big[n\Big(\frac{\varepsilon}{3} - \log\lambda\Big) + \log L\Big] \\ &= \exp\Big[n\Big(\frac{\varepsilon}{3} - \log\lambda + \int \phi d\mu_n + \frac{1}{n}H(\mathcal{P}^n, \nu_n)\Big)\Big]. \end{aligned}$$

We know that $\int \phi d\mu_{n_j} \xrightarrow[j\to\infty]{} \int \phi d\mu$ (because ϕ is continuous and by weak* convergence) and therefore, there exists $j_1 > 0$ such that $\int \phi d\mu_{n_j} \leq \int \phi d\mu + \varepsilon/3$ for all $j > j_1$.

By (6.5.5) there exists $j_0 > 0$ such that $\frac{1}{n_j}H(\mathcal{P}^{n_j}, \nu_{n_j}) \leq h_\mu(T) + \varepsilon/3, \forall j \geq j_0$. Taking $j_2 = \max\{j_1, j_0\}$ we have, $\forall j > j_2$

$$\begin{aligned} \nu(C_{n_j,i}) &\leq \exp\Big[n_j\Big(\frac{\varepsilon}{3} - \log\lambda + \int \phi d\mu + \frac{\varepsilon}{3} + h_\mu(T) + \frac{\varepsilon}{3}\Big)\Big] \\ &= \exp\Big[n_j\Big(\varepsilon - \log\lambda + \int \phi d\mu + h_\mu(T)\Big)\Big] \\ &\leq \exp\Big[n_j\Big(\varepsilon - r - \log\lambda + P_{\text{top}}(T, \phi)\Big)\Big] \leq \exp\Big[n_j\Big(\varepsilon - r\Big)\Big] \end{aligned}$$

where the last inequality follows from Lemma 6.2.2.

By (6.5.2) we conclude that there exist $n_0 > 0$ such that $\nu(C_{n,i}) \leq \exp\Big[n\Big(\varepsilon - r\Big)\Big]$ for all $n \geq n_0$ ending the proof. □

6.6 Proof of Theorem B

Given $r > 0$, consider the (non-empty) set $\mathcal{K}_r(\phi) \subset \mathcal{M}_T$. By the upper semicontinuity of the metric entropy (see Theorem 6.4.1), we have that $\mathcal{K}_r(\phi)$ is closed, hence, weak* compact. Since $\{\mathcal{K}_r(\phi)\}_r$ is decreasing with r, we have $\mathcal{K}_0(\phi) = \bigcap_{r>0} \mathcal{K}_r(\phi)$.

By the Variational Principle $h_\mu(T) + \int \phi\, d\mu \leq P_{\text{top}}(T, \phi)$ for all $\mu \in \mathcal{M}_T$. So, to prove Theorem B, we must prove that the set $\mathcal{W}_T(\nu)$ of ν-SRB-like measures satisfy $\mathcal{W}_T(\nu) \subset \mathcal{K}_r(\phi)$ for all $r > 0$, because $\mathcal{K}_0(\phi) = \Big\{\mu \in \mathcal{M}_T;\ h_\mu(T) + \int \phi d\mu = P_{\text{top}}(T, \phi)\Big\}$. Since $\mathcal{K}_r(\phi)$ is weak* compact, we have

$$\mathcal{K}_r(\phi) = \bigcap_{\varepsilon>0} \mathcal{K}_r^{\varepsilon}(\phi), \text{ where } \mathcal{K}_r^{\varepsilon}(\phi) = \left\{\mu \in \mathcal{M}_T; \ \mathrm{dist}(\mu, \mathcal{K}_r(\phi)) \le \varepsilon\right\}$$

with the weak* distance defined in (4.1.1). Therefore, it is enough to prove that $\mathcal{W}_T(\nu) \subset \mathcal{K}_r^{\varepsilon}(\phi)$ for all $0 < \varepsilon < r/2$ and for all $r > 0$. By Proposition 4.1.2 and since $\mathcal{K}_r^{\varepsilon}(\phi)$ is weak* compact, it is enough to prove the following

Lemma 6.6.1. *The basin of attraction of $\mathcal{K}_r^{\varepsilon}(\phi)$*

$$W^s(\mathcal{K}_r^{\varepsilon}(\phi)) := \left\{x \in X; p\omega(x) \subset \mathcal{K}_r^{\varepsilon}(\phi)\right\}$$

has full ν-measure.

Proof. By Proposition A since $0 < \varepsilon < r$, there exists $n_0 \ge 1$ and $\kappa > 0$ such that $\nu\Big(\Big\{x \in X; \sigma_n(x) \notin \mathcal{K}_r^{\varepsilon}(\phi)\Big\}\Big) \le \kappa e^{n(\varepsilon - r)}$ for any $n > n_0$. This implies that $\sum_{n=1}^{+\infty} \nu\Big(\Big\{x \in X; \sigma_n(x) \notin \mathcal{K}_r^{\varepsilon}(\phi)\Big\}\Big) < +\infty$. By the Borel-Cantelli Lemma it follows that

$$\nu\left(\bigcap_{n_r=1}^{+\infty} \bigcup_{n=n_r}^{+\infty} \left\{x \in X; \sigma_n(x) \notin \mathcal{K}_r^{\varepsilon}(\phi)\right\}\right) = 0.$$

In other words, for ν-a.e. $x \in X$ there exists $n_0 \ge 1$ such that $\sigma_n(x) \in \mathcal{K}_r^{\varepsilon}(\phi)$ for all $n \ge n_0$. Hence, $p\omega(x) \subset \mathcal{K}_r^{\varepsilon}(\phi)$ for ν-almost all the points $x \in X$, as required. □

The proof of Theorem B is complete.

Remark 6.6.2. *If the set $\mathcal{K}_r(\phi)$ is not closed, we may substitute $\overline{\mathcal{K}_r(\phi)}$ for $\mathcal{K}_r(\phi)$ in the proof of Theorem B, and by the same argument we conclude that $\mathcal{W}_T(\nu) \subset \bigcap_{r>0} \overline{\mathcal{K}_r(\phi)}$. Thus, in a more general context, where the Proposition A is valid and $\mathcal{K}_{1/n}(\phi) \neq \emptyset$ for all $n \ge 1$, we can say that ν-SRB-like measures are "almost ϕ-equilibrium states", since, given $\mu \in \mathcal{W}_T(\nu)$ then $\mu = \lim_{n\to+\infty} \mu_n$, $\mu_n \in \mathcal{K}_{1/n}(\phi)$ for all $n \ge 1$. Therefore, we can find a sequence of T-invariant probability measures so that $h_{\mu_n}(T) + \int \phi d\mu_n \ge P_{top}(T, \phi) - \frac{1}{n}$ for all $n \ge 1$ and $\mu_n \to \mu$ in the weak* topology.*

To obtain a ϕ-equilibrium state in the limit we need only assume that ϕ is uniformly approximated by continuous potentials, as follows.

Corollary 6.6.3. *Let $T : X \to X$ be an open expanding topologically transitive map of a compact metric space X, $(\phi_n)_{n\ge1}$ a sequence of continuous potentials, $(\nu_n)_{n\ge1}$ a sequence of conformal measures associated to the (T, ϕ_n) and μ_n a sequence of ν_n-SRB-like measures. Assume that*

1. *$\phi_{n_j} \xrightarrow[j\to+\infty]{} \phi$ in the topology of uniform convergence;*

2. *$\mu_{n_j} \xrightarrow[j\to+\infty]{w^*} \mu$ in the weak* topology.*

Then μ is an equilibrium state for the potential ϕ.

Proof. Let μ_{n_j} be a ν_{n_j}-SRB-like measure and let $\mu = \lim_{j\to+\infty} \mu_{n_j}$. Since any finite Borel partition $\mathcal{P}$ of X with diameter not exceeding an expansive constant and satisfying $\mu(\partial\mathcal{P}) = 0$ generates the Borel sigma-algebra for every Borel T-invariant probability measure in X (see Lemma 2.5.5 in [29]), then Kolmogorov-Sinai Theorem implies that $\eta \mapsto h_\eta(T)$ is upper semi-continuous.

Moreover, as $\int \phi_{n_j}\, d\mu_{n_j} \xrightarrow[j\to+\infty]{} \int \phi\, d\mu$, μ_{n_j} is an equilibrium state for (T, ϕ_{n_j}) (by Theorem B) and by continuity of $\varphi \mapsto P_{\text{top}}(T, \varphi)$ (see Theorem 9.7 in [47]) it follows that

$$h_\mu(T) + \int \phi d\mu \geq \lim_{j\to+\infty} \left(h_{\mu_{n_j}}(T) + \int \phi_{n_j} d\mu_{n_j} \right) = \lim_{j\to+\infty} P_{\text{top}}(T, \phi_{n_j}) = P_{\text{top}}(T, \phi).$$

This shows that μ is an equilibrium state for T with respect to ϕ. $\square$

Chapter 7

Entropy Formula

Here we state the main results needed to obtain the proof of Theorem D. Then we prove Theorem D in the last subsection.

7.1 Hyperbolic Times

The main technical tool used in the study of non-uniformly expanding maps is the notion of hyperbolic times, introduced in [4]. We now outline some the properties of hyperbolic times.

Definition 15. Given $\sigma \in (0,1)$, we say that h is a σ-hyperbolic time for a point $x \in M$ if for all $1 \leq k \leq h$,

$$\prod_{j=h-k}^{h-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \tag{7.1.1}$$

Remark 7.1.1. *Throughout this text the reader will find many quotes from works where f is assumed to be of class C^2 (or $f \in C^{1+\alpha}(M,M)$, $\alpha > 0$). But it is worth noting that these results cited are proven without using the bounded distortion assumption, and therefore the proofs are easily adapted to our context (in general, the proofs are the same).*

Proposition 7.1.2. *Given $0 < \sigma < 1$, there exists $\delta_1 > 0$ such that, whenever h is a σ-hyperbolic time for a point x, the dynamical ball $B(x,h,\delta_1)$ is mapped diffeomorphically by f^h onto the ball $B(f^h(x),\delta_1)$, with*

$$d(f^{h-k}(y), f^{h-k}(z)) \leq \sigma^{k/2} \cdot d(f^h(y), f^h(z))$$

for every $1 \leq k \leq h$ and $y,z \in B(x,h,\delta_1)$.

Proof. See Lemma 5.2 in [2] □

Remark 7.1.3. *For an open expanding and topologically transitive map T of a compact metric space X every time is a hyperbolic time for every point x, that is, every x satisfies the condition of Proposition 7.1.2.*

Definition 16. We say that the frequency of σ-hyperbolic times for $x \in M$ is positive, if there is some $\theta > 0$ such that all sufficiently for large $n \in \mathbb{N}$ there are $l \geq \theta n$ and integers $1 \leq h_1 < h_2 < \ldots < h_l \leq n$ which are σ-hyperbolic times for x.

The following Theorem ensures existence of infinitely many hyperbolic times Lebesgue almost every point for non-uniformly expanding maps. A complete proof can be found in Ref. [2], Sec. 5.

Theorem 7.1.4. *Let $f : M \to M$ be a C^1 non-uniformly expanding local diffeomorphism. Then there are $\sigma \in (0,1)$ and there exists $\theta = \theta(\sigma) > 0$ such that* Leb*-a.e. $x \in M$ has infinitely many σ-hyperbolic times. Moreover if we write $0 < h_1 < h_2 < h_3 < \ldots$ for the hyperbolic times of x then their asymptotic frequency satisfies*

$$\liminf_{N\to\infty} \frac{\#\{k \geq 1 : h_k \leq N\}}{N} \geq \theta \;\; \textit{for} \;\; \mathrm{Leb}-a.e.x \in M$$

The Lemma below shows that we can translate the density of hyperbolic times into the Lebesgue measure of the set of points which have a specific (large) hyperbolic time.

Lemma 7.1.5. *Let $B \subset M$, $\theta > 0$ and $g : M \to M$ be a local diffeomorphisms such that g has density $> 2\theta$ of hyperbolic times for every $x \in B$. Then, given any probability measure ν on B and any $n \geq 1$, there exists $h > n$ such that*

$$\nu(\{x \in B : h \textit{ is a hyperbolic time of } g \textit{ for } x\}) > \frac{\theta}{2}$$

Proof. See Lemma 3.3 in [5] □

The next result is the flexible covering lemma with hyperbolic preballs which will enable us to approximate the Lebesgue measure of a given set through the measure of families of hyperbolic preballs.

Lemma 7.1.6. *Let a measurable set $A \subset M$, $n \geq 1$ and $\varepsilon > 0$ be given with $m(A) > 0$. Let $\theta > 0$ be a lower bound for the density of hyperbolic times for Lebesgue almost every point. Then there are integers $n < h_1 < \ldots < h_k$ for $k = k(\varepsilon) \geq 1$ and families $\mathcal{E}_i$ of subsets of M, $i = 1, \ldots k$ such that*

1. *$\mathcal{E}_1 \cup \ldots \cup \mathcal{E}_k$ is a finite pairwise disjoint family of subsets of M;*
2. *h_i is a $\frac{\sigma}{2}$-hyperbolic time for every point in Q, for every element $Q \in \mathcal{E}_i$, $i = 1, \ldots, k$;*
3. *every $Q \in \mathcal{E}_i$ is the preimage of some element $P \in \mathcal{P}$ under an inverse branch of f^{h_i}, $i = 1, \ldots, k$;*
4. *there is an open set $U_1 \supset A$ containing the elements of $\mathcal{E}_1 \cup \ldots \mathcal{E}_k$ with $m(U_1 \setminus A) < \varepsilon$;*
5. *$m(A \triangle \cup_i \mathcal{E}_i) < \varepsilon$.*

Proof. See Lemma 3.5 in [5]. □

Remark 7.1.7. *This covering lemma is true replacing f by an open expanding and topologically transitive map T of a compact metric space X;* Leb *by a ϕ-conformal measure ν for a continuous potential $\phi : M \to \mathbb{R}$; recall Remark 7.1.3.*

We use this covering lemma to prove the following.

Proposition B. *Let $f : M \to M$ be a non-uniformly expanding map. For any $\mu \in \mathcal{W}_f^*$*

$$h_\mu(f) + \int \psi d\mu \geq 0. \tag{7.1.2}$$

Proof. Given $\mu \in \mathcal{W}_f^*$, consider $\delta_1 > 0$ as in Proposition 7.1.2 and as $f : M \to M$ is a C^1 local diffeomorphism in particular f is a regular map. Consider a partition $\mathcal{P}$ of M as in Proposition 6.3.1 such that $\text{diam}(\mathcal{P}) < \frac{\delta_1}{4}$ and $\mu(\partial \mathcal{P}) = 0$.

Since μ is f-invariant and $\mathcal{P}$ is a μ-mod0 partition such that $\mu(\partial \mathcal{P}) = 0$, then the function $\lambda \mapsto h(\mathcal{P}, \lambda)$ is upper semi-continuous in μ, that is, for each small enough $\tau > 0$ we can find $\delta_2 > 0$ such that

$$\text{if } \ \text{dist}(\mu, \tilde{\mu}) \leq \delta_2 \ \text{ then } \ h(\mathcal{P}, \tilde{\mu}) \leq h(\mathcal{P}, \mu) + \tau. \tag{7.1.3}$$

Fix $\tau > 0$ and $0 < \delta_2 < \tau$ as above. Since μ is weak-SRB-like probability measure, by definition, for any fixed value of $0 < \varepsilon < \delta_2/3$ there exists a subsequence of integers $n_l \to +\infty$ such that $\text{Leb}(A_{\varepsilon, n_l}(\mu)) > 0$ for all $l > 0$.

Take $\delta_2/3 > 0$ and fix $\gamma_0 > 0$ as in Lemma 4.1.1 and $\gamma_1 > 0$ of uniform continuity of ψ, i.e., $|\psi(x) - \psi(y)| < \delta_2/3$ if $d(x, y) < \gamma_1$. We denote $\gamma = \min\{\gamma_0, \gamma_1\}$.

Let us choose one interior point having density $\geq \theta$ of σ-hyperbolic times of f in each atom $P \in \mathcal{P}$ and form the set $W_0 = \{w_1, \ldots, w_S\}$ of representatives of the atoms of $\mathcal{P} = \{P_\ell\}_{1 \leq \ell \leq S}$, where $S = \#\{P \in \mathcal{P} : P \neq \emptyset\}$, and consider $d_0 = \min\{d(w, \partial\mathcal{P}), w \in W_0\} > 0$, where $\partial\mathcal{P} = \bigcup_{\ell=1}^{S} \partial P_\ell$ is the boundary of $\mathcal{P}$.

We use now the Lemma 7.1.6 for obtain the flexible covering of set $A_{\varepsilon,n_l}(\mu)$ with hyperbolic preballs. Take positive integers l, m and $\beta_l = \frac{1}{n_l}\operatorname{Leb}(A_{\varepsilon,n_l}(\mu)) > 0$ such that $\sigma^{\frac{m}{2}}\delta_1/4 < \gamma$. Then there are integers $n_l < n_l + m \leq h_1 < h_2 < \ldots < h_k$ with $k = k(l) \geq 1$ (here β_l takes the place of ε in Lemma 7.1.6) and families $\mathcal{E}_j$ of subsets of M, $j = 1, \ldots k$ so that

$$\begin{aligned}
\operatorname{Leb}(A_{\varepsilon,n_l}(\mu)) &= \sum_{j=1}^{k} \operatorname{Leb}(A_{\varepsilon,n_l}(\mu) \cap \mathcal{E}_j) + \sum_{j=1}^{k} \operatorname{Leb}(A_{\varepsilon,n_l}(\mu) \backslash \mathcal{E}_j) \\
&\leq \sum_{j=1}^{h_k} \sum_{Q \in \mathcal{E}_j} \operatorname{Leb}(A_{\varepsilon,n_l}(\mu) \cap Q) + \beta_l, \ \text{ hence} \\
\operatorname{Leb}(A_{\varepsilon,n_l}(\mu)) &\leq \frac{n_l}{n_l - 1} \sum_{j=1}^{h_k} \sum_{Q \in \mathcal{E}_j} \operatorname{Leb}(A_{\varepsilon,n_l}(\mu) \cap Q), \qquad (7.1.4)
\end{aligned}$$

where $\mathcal{E}_j = \mathcal{E}_{h_j}$ and $A_{\varepsilon,n_l}(\mu) \cap \mathcal{E}_j = \bigcup_{Q \in \mathcal{E}_j} A_{\varepsilon,n_l}(\mu) \cap Q$.

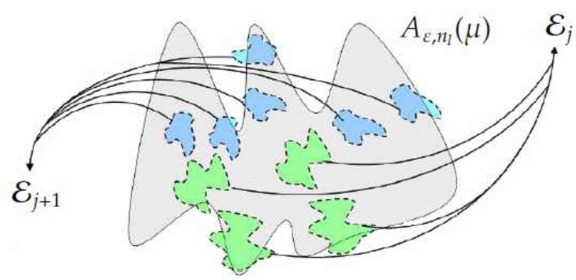

Figure 7.1: $\mathcal{E}_j$ is a family of all sets obtained as $f^{-h_j}(P)$ which intersect $A_{\varepsilon,n_l}(\mu)$ in points for which h_j is a hyperbolic time, where $P \in \mathcal{P}$. Analogously for $\mathcal{E}_{j+1}$.

As $\operatorname{diam}(\mathcal{P}) < \frac{\delta_1}{4}$, by Lemma 7.1.6, $f^{h_j}|_Q : Q \to f^{h_j}(Q)$ is diffeomorphism for all $Q \in \mathcal{E}_j$ $\forall 1 \leq j \leq k$, where h_j is a $\frac{\sigma}{2}$-hyperbolic time for every point in

Q, for every element $Q \in \mathcal{E}_j$, $j = 1, \ldots, k$ and every $Q \in \mathcal{E}_j$ is the preimage of some element $P \in \mathcal{P}$ under an inverse branch of f^{h_j}, $j = 1, \ldots, k$. Then

$$\mathrm{Leb}(Q \cap A_{\varepsilon,n_l}(\mu)) = \int_{f^{h_j}(Q \cap A_{\varepsilon,n_l}(\mu))} |\det Df^{-h_j}| d\,\mathrm{Leb} = \int_{f^{h_j}(Q \cap A_{\varepsilon,n_l}(\mu))} e^{S_{h_j}\psi} d\,\mathrm{Leb}\,.$$

Note that $S_{h_j}\psi(y) = S_{n_l}\psi(y) + S_{h_j - n_l}\psi(f^{n_l}(y))$ and since $h_j \geq n_l + m$ is a hyperbolic time for all $y \in \mathcal{E}_j$ we have $S_{h_j - n_l}\psi(f^{n_l}(y)) \leq 0$ and so $S_{h_j}\psi(y) \leq S_{n_l}\psi(y)$ for all $y \in \mathcal{E}_j$. Therefore,

$$\mathrm{Leb}(Q \cap A_{\varepsilon,n_l}(\mu)) \leq \int_{f^{h_j}(Q \cap A_{\varepsilon,n_l}(\mu))} e^{S_{n_l}\psi} d\,\mathrm{Leb} \quad \forall Q \in \mathcal{E}_j \;\; \forall j = 1, \ldots, k.$$

For each $Q \in \mathcal{E}_j$ such that $\mathrm{Leb}(Q \cap A_{\varepsilon,n_l}(\mu)) > 0$, consider $y_Q \in Q \cap A_{\varepsilon,n_j}(\mu)$ and let $x_Q \in Q$ be such that $f^{h_j}(x_Q) \in W_0$ (recall that elements of W_0 are interior points of each atom of the partition $\mathcal{P}$). We write W_{n_l} for the set of all points x_Q for all $Q \in \mathcal{E}_j$ such that $\mathrm{Leb}(Q \cap A_{\varepsilon,n_l}(\mu)) > 0$ for all $j = 1, \ldots, k$.

From Proposition 7.1.2, we know that $\max\{\mathrm{diam}(f^l(Q));\ Q \in \mathcal{E}_j,\ l = 0, \ldots n_l\} < \sigma^{\frac{1}{2}(h_j - n_l)}\delta_1/4 < \sigma^{\frac{m}{2}}\delta_1/4 < \gamma$, for all $j = 1, \ldots, k$, and by uniform continuity of ψ, $|\psi(f^i(x_Q)) - \psi(f^i(y))| < \delta_2/3$ for all $y \in Q$ and for all $i = 0, \ldots, n_l - 1$. Altogether we get

$$\mathrm{Leb}(A_{\varepsilon,n_l}(\mu) \cap Q) \leq \int_{f^{h_j}(Q \cap A_{\varepsilon,n_l}(\mu))} e^{S_{n_l}\psi} d\,\mathrm{Leb} \leq e^{\left(S_{n_l}\psi(y_Q) + n_l\frac{\delta_2}{3}\right)} \leq e^{\left(2n_l\frac{\delta_2}{3} + S_{n_l}\psi(x_Q)\right)}.$$

Thus, by (7.1.4) we can write $\mathrm{Leb}(A_{\varepsilon,n_l}(\mu)) \leq \frac{n_l}{n_l - 1} e^{\frac{2\delta_2}{3}n_l} \sum_{x \in W_{n_l}} e^{S_{n_l}\psi(x)}$.

Setting $L(n_l) := \sum_{x \in W_{n_l}} e^{S_{n_l}\psi(x)}$ and $\lambda(x) := \frac{1}{L(n_l)} e^{S_{n_l}\psi(x)}$ we can rewrite

$$\mathrm{Leb}(A_{\varepsilon,n_l}(\mu)) \quad \leq \quad \frac{n_l}{n_l - 1} \exp\left[n_l\left(2\frac{\delta_2}{3} + \frac{1}{n_l}\log L(n_l)\right)\right]. \tag{7.1.5}$$

Note that since $\sum_{x \in W_{n_l}} \lambda(x) = 1$ then by Lemma 4.2.3

$$\log L(n_l) = \left(\sum_{x \in W_{n_l}} \lambda(x) \sum_{j=1}^{n_l - 1} \psi(f^j x)\right) - \left(\sum_{x \in W_{n_l}} \lambda(x) \log \lambda(x)\right).$$

Defining the probability measures

$$\nu_{n_l} := \sum_{x \in W_{n_l}} \lambda(x)\delta_x \;\text{ and }\; \mu_{n_l} := \frac{1}{n_l}\sum_{i=0}^{n_l - 1}(f^i)_*(\nu_{n_l}) = \sum_{x \in W_{n_l}} \lambda(x)\sigma_{n_l}(x),$$

we can rewrite

$$\log L(n_l) = n_l \int \psi d\mu_{n_l} + H(\mathcal{P}^{n_l}, \nu_{n_l}). \tag{7.1.6}$$

Fix $(\mu_{n_l})_l$ and take a subsequence $n_{l_i} \xrightarrow[i\to+\infty]{} +\infty$ such that $\mu_{n_{l_i}} \xrightarrow[i\to+\infty]{w^*} \tilde{\mu}$ and

$$\lim_{i\to+\infty} \frac{1}{n_{l_i}} \log \mathrm{Leb}(A_{\varepsilon,n_{l_i}}(\mu)) = \limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}(A_{\varepsilon,n}(\mu)). \tag{7.1.7}$$

We keep the notation n_l for simplicity in what follows.

Note that, for each $x \in W_{n_l}$ there exists $y_Q \in Q \cap A_{\varepsilon,n_l}(\mu)$ such that $x, y_Q \in Q$, hence $\mathrm{dist}(f^i(x), f^i(y_Q)) < \gamma$ for all $i = 0, \ldots, n_l-1$. By Lemma 4.1.1 we have that $\mathrm{dist}(\sigma_{n_l}(x), \sigma_{n_l}(y_Q)) < \delta_2/3$, then by the triangular inequality

$$\mathrm{dist}(\sigma_{n_l}(x), \mu) \le \mathrm{dist}(\sigma_{n_l}(x), \sigma_{n_l}(y_Q)) + \mathrm{dist}(\sigma_{n_l}(y_Q), \mu) < \frac{\delta_2}{3} + \varepsilon < \delta_2,$$

because $y_Q \in A_{\varepsilon,n_l}(\mu)$ and $\varepsilon < \delta_2/3$. Thus, for any $\varphi \in C^0(M, \mathbb{R})$,

$$\begin{aligned}
\left|\int \varphi d\mu_{n_l} - \int \varphi d\mu\right| &= \left|\sum_{x\in W_{n_l}} \lambda(x) \int \varphi d\sigma_{n_l}(x) - \sum_{x\in W_{n_l}} \lambda(x) \int \varphi d\mu\right| \\
&\le \sum_{x\in W_{n_l}} \lambda(x) \left|\int \varphi d\sigma_{n_l}(x) - \int \varphi d\mu\right| < \delta_2.
\end{aligned}$$

Therefore, $\mathrm{dist}(\mu, \tilde{\mu}) \le \delta_2$, and so

$$\int \psi d\mu_{n_l} \le \int \psi d\mu + \delta_2 \quad \text{and} \quad \int \psi d\tilde{\mu} \le \int \psi d\mu + \delta_2. \tag{7.1.8}$$

We now make a small perturbation of $\mathcal{P}$ so that the points $x \in W_{n_l}$ still belong to the same atom of the h_j-th refinement the perturbed partition that intersect $A_{\varepsilon,n_l}(\mu)$ for each $Q \in \mathcal{E}_j$ and $j = 1, \ldots, k$ and $l \ge 1$.

Now fix $\delta > 0$ given by the statement of Lemma 4.2.1 with the choice of S and $\varepsilon = \delta_2/4$. Let $0 < \eta < \min\{\frac{d_0}{2}, \frac{\delta_1}{4}\}$ (by construction, d_0 does not depend on n_l) such that for all $i = 1, \ldots, S$ and for each $l \ge 1$

$$\mu\Big(\partial B\Big(\tilde{x}_i, \frac{\delta_1}{4} + \eta\Big)\Big) = \tilde{\mu}\Big(\partial B\Big(\tilde{x}_i, \frac{\delta_1}{4} + \eta\Big)\Big) = \mu_{n_l}\Big(\partial B\Big(\tilde{x}_i, \frac{\delta_1}{4} + \eta\Big)\Big) = 0 \tag{7.1.9}$$

and

$$\tilde{\mu}\left(B\Big(\tilde{x}_i, \frac{\delta_1}{4} + \eta\Big) \backslash B\Big(\tilde{x}_i, \frac{\delta_1}{4}\Big)\right) < \delta \tag{7.1.10}$$

where the centers are the ones from the construction in the proof of Proposition 6.3.1.

Such value of η exists since the set of values of $\eta > 0$ such that some of the expressions in (7.1.9) is positive for some $i \in \{1, \dots, S\}$ and some $l \geq 1$ is denumerable because the measures involved are probability measures. Moreover, we can also get (7.1.10) because $\tilde{\mu}$ is regular probability measure. Thus we may take $\eta > 0$ satisfying (7.1.9) and (7.1.10) arbitrarily close to zero.

We consider now the finite open cover $\tilde{\mathcal{C}} = \{B(\tilde{x}_i, \delta_1/4 + \eta) : i = 1, \dots, S\}$ as in (6.5.4) and we analogously construction a new partition $\tilde{\mathcal{P}}$ with the same number of elements of the original partition.

Moreover, for $\mathcal{P} = \{P_i;\ 1 \leq i \leq S\}$ and $\tilde{\mathcal{P}} = \{\tilde{P}_i;\ 1 \leq i \leq S\}$, we have by construction that $\tilde{\mu}(P_i \triangle \tilde{P}_i) < \delta$ for all $i = 1, \dots S$. Thus we have, the same properties (1-3) in the proof of Claim 6.5.1 replacing ξ by $\delta_1/4$ together with

(1) $\tilde{\mu}(\partial\tilde{\mathcal{P}}) = 0 = \mu_{n_l}(\partial\tilde{\mathcal{P}})$ for all $l \geq 1$;

(2) $\tilde{\mu}(P_i \triangle \tilde{P}_i) < \delta$ for all $i = 1, \dots S$; and

(3) $x_Q \in \tilde{Q} \cap Q$ for all $Q \in \mathcal{E}_j$ where $\tilde{Q} \in \tilde{\mathcal{E}}_j$ and $f^{h_j}(x_Q) = w \in \tilde{P} \cap P$.

Note that $H(\tilde{\mathcal{P}}^{n_l}, \nu_{n_l}) = H(\mathcal{P}^{n_l}, \nu_{n_l})$ by definition of the ν_{n_l} and by construction of the $\tilde{\mathcal{P}}$ as a perturbation of $\mathcal{P}$. Following the proof of Lemma 4.2.2 (see inequality (4.2.2)) there exists $l_0 \geq 0$ such that

$$\frac{1}{n_l} H(\mathcal{P}^{n_l}, \nu_{n_l}) = \frac{1}{n_l} H(\tilde{\mathcal{P}}^{n_l}, \nu_{n_l}) \leq h(\tilde{\mathcal{P}}, \tilde{\mu}) + \frac{\delta_2}{4}, \quad \forall l \geq l_0.$$

Since, $h(\tilde{\mathcal{P}}, \tilde{\mu}) \leq h(\mathcal{P}, \tilde{\mu}) + H_{\tilde{\mu}}(\tilde{\mathcal{P}}/\mathcal{P})$ we have that by Lemma 4.2.1 and item (2) above, that $H_{\tilde{\mu}}(\tilde{\mathcal{P}}/\mathcal{P}) < \frac{\delta_2}{4}$. Therefore,

$$\frac{1}{n_l} H(\mathcal{P}^{n_l}, \nu_{n_l}) \leq h(\mathcal{P}, \tilde{\mu}) + \frac{\delta_2}{2} \leq h(\mathcal{P}, \mu) + \frac{\delta_2}{2} + \tau, \quad \forall l \geq l_0,$$

where the last inequality is valid by the choice of δ_2 in (7.1.3). Thus,

$$\frac{1}{n_l} H(\mathcal{P}^{n_l}, \nu_{n_l}) \leq h(\mathcal{P}, \mu) + \frac{\delta_2}{2} + \tau \leq h_\mu(f) + \frac{3}{2}\tau, \quad \forall l \geq l_0. \tag{7.1.11}$$

and remember that $\delta_2 < \tau$.

Combining the assertions (7.1.5), (7.1.6), (7.1.11) and (7.1.8) we have that

$$\begin{aligned}
\mathrm{Leb}(A_{\varepsilon,n_l}(\mu)) &\leq \frac{n_l}{n_l-1}\exp\left[n_l\left(2\frac{\delta_2}{3}+\frac{1}{n_l}\log L(n_l)\right)\right] \\
&\leq \frac{n_l}{n_l-1}\exp\left[n_l\left(\frac{2}{3}\tau+\int\psi d\mu_{n_l}+\frac{1}{n_l}H(\mathcal{P}^{n_l},\nu_{n_l})\right)\right] \\
&\leq \frac{n_l}{n_l-1}\exp\left[n_l\left(3\tau+h_\mu(f)+\int\psi d\mu\right)\right]
\end{aligned}$$

Hence,

$$\frac{1}{n_l}\log\mathrm{Leb}(A_{\varepsilon,n_l}(\mu)) \quad\leq\quad \frac{1}{n_l}\log\left(\frac{n_l}{n_l-1}\right)+h_\mu(f)+\int\psi d\mu+3\tau, \quad \forall l\geq l_0.$$

Since $\mu\in\mathcal{W}_f^*$, we conclude that,

$$0=\limsup_{n\to+\infty}\frac{1}{n}\log\mathrm{Leb}(A_{\varepsilon,n}(\mu))=\lim_{l\to+\infty}\frac{1}{n_l}\log\mathrm{Leb}(A_{\varepsilon,n_l}(\mu))\leq h_\mu(f)+\int\psi d\mu+3\tau.$$

As $\tau>0$ is arbitrary, the proof of the Lemma is complete. □

Remark 7.1.8. *Proposition B is true replacing f by an open expanding and topologically transitive map T of a compact metric space X;* Leb *by a ϕ-conformal measure ν with $\mathcal{L}_\phi^*(\nu)=\lambda\nu$, for some $\lambda>0$ and for a continuous potential $\phi: X\to\mathbb{R}$, since we used that* Leb *is ψ-conformal together with a covering lemma that clearly holds for expanding maps, recall Remark 7.1.7. Thus, we have*

$$h_\mu(f)+\int\phi d\mu-\log\lambda\geq 0,\ \textit{for all}\ \mu\in\mathcal{W}_T^*(\nu).$$

In particular, this shows together with the Lemma 6.2.2 that $P_{top}(T,\phi)=\log\lambda$ and all ν-weak-SRB-like probability measures are ϕ-equilibrium states.

7.2 Proof of Theorem D

To prove Theorem D, first consider a weak-SRB-like measure μ. Since $\mu\in\mathcal{W}_f^*$ is an expanding probability measure, there exists $\sigma\in(0,1)$ such that $\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\|Df(f^j(x))^{-1}\|\leq\log\sigma<0$ for μ-a.e $x\in M$.

Thus, the Lyapunov exponents are non-negative. Hence the sum $\Sigma^+(x)$ of the positive Lyapunov exponents of a $\mu-$generic point x, counting multiplicities, is such that $\Sigma^+(x)=\lim_{n\to+\infty}\frac{1}{n}\log|\det Df^n(x)|$ (by the Multiplicative Ergodic Theorem) and $\int\Sigma^+d\mu=\int\log|\det Df|d\mu=-\int\psi d\mu$ by the standard Ergodic Theorem.

For C^1-systems, Ruelle's Inequality [36] states that for any f-invariant probability measure μ on the Borel σ-algebra of M, the corresponding measure-theoretic entropy $h_\mu(f)$ satisfies $h_\mu(f) \le \int \Sigma^+ d\mu$ and consequently $h_\mu(f) + \int \psi d\mu \le 0$.

By definition, Pesin's Entropy Formula holds if the latter difference is equal to zero. Since $\mu \in \mathcal{W}_f^*$, by Proposition B, we have that $h_\mu(f) + \int \psi d\mu = 0$ which proves the first statement of Theorem D.

The next corollary concludes the proof of Theorem D.

Corollary 7.2.1. *Let $f : M \to M$ be non-uniformly expanding. Then all the ergodic SRB-like probability measures are expanding probability measures.*

Proof. The assumptions on f ensure that there exists $\sigma \in (0,1)$ such that $\mathrm{Leb}(H(\sigma)) = 1$.

The proof uses a simple lemma.

Lemma 7.2.2. *If $f : M \to M$ is a C^1 local diffeomorphism such that* $\mathrm{Leb}(H(\sigma)) = 1$ *for some $\sigma \in (0,1)$, then each $\mu \in \mathcal{W}_f$ satisfies $\int \log \|(Df)^{-1}\| d\mu < \log \sqrt{\sigma}$.*

Proof. Fix $0 < \varepsilon < -\frac{1}{2}\log\sigma$ small enough. Since that $\varphi(x) := \log \|Df(x)^{-1}\|$ is a continuous potential, from the definition of the weak* topology in space $\mathcal{M}_1$ of probability measures, we deduce that there exists $0 < \varepsilon_1 < \varepsilon$ such that if $\mathrm{dist}(\mu,\nu) < \varepsilon_1$ then $|\int \varphi d\mu - \int \varphi d\nu| < \varepsilon$ for all $\mu, \nu \in \mathcal{M}_1$.

Let $\mu \in \mathcal{W}_f$, then $\mathrm{Leb}(A_{\varepsilon_1}(\mu)) > 0$, take $x \in A_{\varepsilon_1}(\mu) \cap H(\sigma)$ and consider $\nu_x \in p\omega(x)$ such that $\mathrm{dist}(\mu, \nu_x) < \varepsilon_1$. Thus,

$$\left|\int \log \|(Df)^{-1}\| d\mu - \int \log \|(Df)^{-1}\| d\nu_x\right| < \varepsilon,$$

and therefore there exists $n_k \nearrow \infty$ so that $\sigma_{n_k}(x) \xrightarrow{w^*} \nu_x$ and then

$$\begin{aligned}
\int \log \|(Df)^{-1}\| d\mu &\le \int \log \|(Df)^{-1}\| d\nu_x + \varepsilon \\
&= \lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \log \|Df(f^j(x))^{-1}\| + \varepsilon \\
&\le \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| + \varepsilon \\
&< \log\sigma + \varepsilon < \log \sqrt{\sigma}
\end{aligned}$$

as stated. □

Going back to the proof of the Corollary, since μ is f-invariant and ergodic, then by the previous lemma and by the Ergodic Theorem

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f^j(y))^{-1}\| = \int \log\|(Df)^{-1}\|d\mu < \log\sqrt{\sigma} \text{ for } \mu - \text{a.e } y \in M.$$

Therefore μ is an expanding measure. This finishes the proof of the corollary. □

This completes the proof of Theorem D.

Chapter 8

Ergodic weak-SRB-like measure

In this chapter, we prove Corollary E on existence of ergodic weak-SRB-like measures for non-uniformly expanding local diffeomorphisms $f : M \to M$.

8.1 Ergodic expanding invariant measures

Theorem 8.1.1. *Let $f : M \to M$ be a C^1 local diffeomorphism. If μ is an ergodic expanding f-invariant probability measure, then*

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to+\infty} \frac{\log(\mathrm{Leb}(A_{\varepsilon,n}(\mu)))}{n} \geq \int \psi d\mu + h_\mu(f). \tag{8.1.1}$$

Proof. Since μ is an ergodic probability measure, we have $\lim_{n\to+\infty} \sigma_n(x) = \mu$ for μ-a.e. $x \in M$. So for μ-a.e. $x \in M$, there exists $N(x) \geq 1$ such that $\mathrm{dist}(\sigma_n(x), \mu) < \varepsilon/4$ $\forall n \geq N(x)$.

Given $\varepsilon > 0$ and any natural value of $N \geq 1$, define the set

$$B_N := \{x \in M : \ \mathrm{dist}(\sigma_n(x), \mu) < \varepsilon/4 \ \ \forall n \geq N\}. \tag{8.1.2}$$

Consider $\delta_1 > 0$ such that for each σ-hyperbolic time $h \geq 1$ time for x, $f^h|_{B(x,h,\delta_1)}$ maps $B(x, h, \delta_1)$ diffeomorphically to the ball of radius δ_1 around $f^h(x)$.

Fix $\delta > 0$ such that Lemma 4.1.1 holds with $\varepsilon/8$ in the place of ε and fix $\xi > 0$ satisfying $|\psi(x) - \psi(y)| < \frac{\varepsilon}{4}$ if $d(x, y) < \xi$.

Consider $0 < \gamma_0 < \min\{\xi, \delta, \delta_1/2\}$ and let $N(n, \gamma, b)$ be the minimum number of points needed to (n, γ)-span a set of μ-measure b (see (2.1.2)). Choose $0 < \gamma_1 < \gamma_0$ such that

$$\liminf_{n\to+\infty} \frac{1}{n} \log N(n, 4\gamma, 1/2) \geq h_\mu(f) - \frac{\varepsilon}{2} \ \ \forall \gamma < \gamma_1, \tag{8.1.3}$$

and let $0 < \gamma_2 \le \gamma_1$ be such that

$$\mu\left(\left\{x \in M : \limsup_{n\to+\infty} -\frac{1}{n}\log \mathrm{Leb}(B(x,n,\gamma_2)) \le h_{\mathrm{Leb}}(f,\mu) + \frac{\varepsilon}{4}\right\}\right) > \frac{2}{3}.$$

This is possible by definition of $h_{\mathrm{Leb}}(f,\mu)$ (see (2.1.1)). We have implicitly assumed that $h_{\mathrm{Leb}}(f,\mu) < \infty$ here. If $h_{\mathrm{Leb}}(f,\mu) = \infty$, then there is nothing to prove since $h_\mu(f) < h_{\mathrm{top}}(f) < \infty$ because f is a C^1 local diffeomorphism. Denote

$$A = \left\{x \in M : \limsup_{n\to+\infty} -\frac{1}{n}\log \mathrm{Leb}(B(x,n,\gamma_2)) \le h_{\mathrm{Leb}}(f,\mu) + \frac{\varepsilon}{4}\right\}. \quad (8.1.4)$$

Note that $B_N \subset B_{N+1}$ and $\mu(\cup B_N) = 1$. So there exists $N \ge 1$ such that $\mu(B_N) \ge \frac{5}{6}$. If $C_N := A \cap B_N$ then $\mu(C_N) \ge \frac{1}{2}$ and for all $x \in C_N$ and $n \ge N(x)$ we have:

(1) $B(x,n,\gamma_2) \subset A_{\varepsilon,n}(\mu)$;

(2) $\mathrm{Leb}(B(x,n,\gamma_2)) \ge e^{-(h_{\mathrm{Leb}}(f,\mu)+\varepsilon/2)n}$.

Note that (2) immediately follows from (8.1.4). Moreover, (1) holds because, given $y \in B(x,n,\gamma)$ then $d(f^j(y), f^j(x)) < \gamma$ for all $j = 0,\dots,n-1$. By Lemma 4.1.1 we have $\mathrm{dist}(\sigma_n(y),\sigma_n(x)) < \frac{\varepsilon}{8}$. Since $x \in B_N$, by the triangular inequality

$$\mathrm{dist}(\sigma_n(y),\mu) \le \mathrm{dist}(\sigma_n(y),\sigma_n(x)) + \mathrm{dist}(\sigma_n(x),\mu) < \varepsilon.$$

Therefore, $y \in A_{\varepsilon,n}(\mu)$.

For each n, let $E_n = E_n(2\gamma_2)$ be a maximal $(n, 2\gamma_2)$-separated subset of points contained in C_N. Then $\bigcup_{x\in E_n} B(x,n,4\gamma_2) \supset C_N$ by maximality of E_n and so $\#E_n \ge N(n, 4\gamma_2, 1/2)$. Also, given $x, y \in E_n$, $x \ne y$ then $B(x,n,\gamma_2) \cap B(y,n,\gamma_2) = \emptyset$. Thus,

$$\begin{aligned}
\liminf_{n\to+\infty} \frac{1}{n}\log \mathrm{Leb}(A_{\varepsilon,n}(\mu)) &\ge \liminf_{n\to+\infty} \frac{1}{n}\log \sum_{x\in E_n} \mathrm{Leb}(B(x,n,\gamma_2)) \\
&\ge \liminf_{n\to+\infty} \frac{1}{n}\log\left(\#E_n \cdot e^{-(h_{\mathrm{Leb}}(f,\mu)+\varepsilon/2)n}\right) \\
&\ge \liminf_{n\to+\infty} \frac{1}{n}\log N(n,4\gamma_2,1/2) - h_{\mathrm{Leb}}(f,\mu) - \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, by (8.1.3) we have,

$$\liminf_{n\to+\infty} \frac{1}{n}\log \mathrm{Leb}(A_{\varepsilon,n}(\mu)) \ge h_\mu(f) - h_{\mathrm{Leb}}(f,\mu) - \varepsilon. \quad (8.1.5)$$

Moreover, since μ is expanding, then there exist $\theta > 0$ and $\sigma \in (0,1)$ such that μ-a.e. $x \in M$ has positive frequency $\geq \theta$ of σ-hyperbolic times of f. Let $\tilde{M} \subset M$ be such that for all $x \in \tilde{M}$, $\lim_{n\to+\infty} \sigma_n(x) = \mu$ and $f^h|_{B(x,h,\gamma_2)}$ maps $B(x,h,\gamma_2)$ diffeomorphically to the ball of radius γ_2 around $f^h(x)$, for h a σ-hyperbolic time for x.

Note that $|\psi(f^j(y)) - \psi(f^j(z))| < \frac{\varepsilon}{4}$ for all $z \in B(y,h,\gamma_2)$ since $d(f^jx, f^jy) \leq \gamma_2$ for all $j = 0,\dots,h-1$.

Hence, since Lebesgue measure gives weight uniformly bounded away from zero to balls of fixed radius and the by uniform continuity of ψ, we obtain

$$\begin{aligned} 0 < \alpha(\gamma_2) &\leq \operatorname{Leb}(B(f^hx,\gamma_2)) = \int_{B(x,h,\gamma_2)} |\det Df^h| \, d\operatorname{Leb} \\ &\leq \int_{B(x,h,\gamma_2)} e^{-S_h\psi} \, d\operatorname{Leb} \leq e^{-S_h\psi(x)+h\varepsilon/4} \cdot \operatorname{Leb}(B(x,h,\gamma_2)). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \liminf_{h\to+\infty} \frac{1}{h} \log \alpha(\gamma_2) \\ &\leq \liminf_{h\to+\infty} -\frac{1}{h} S_h\psi(x) + \frac{\varepsilon}{4} + \liminf_{h\to+\infty} \frac{1}{h} \log \operatorname{Leb}(B(x,h,\gamma_2)) \\ &\leq -\limsup_{h\to+\infty} \int \psi d\sigma_h(x) + \frac{\varepsilon}{4} - \limsup_{h\to+\infty} -\frac{1}{h} \log \operatorname{Leb}(B(x,h,\gamma_2)) \\ &= -\int \psi d\mu + \frac{\varepsilon}{4} - h_{\operatorname{Leb}}(f,x). \end{aligned}$$

Therefore, $h_{\operatorname{Leb}}(f,x) \leq -\int \psi d\mu + \frac{\varepsilon}{4}$ for μ-a.e $x \in M$, hence $h_{\operatorname{Leb}}(f,\mu) \leq -\int \psi d\mu + \frac{\varepsilon}{4}$ and by (8.1.5)

$$\lim_{\varepsilon\to0^+} \limsup_{n\to+\infty} \frac{1}{n} \log \operatorname{Leb}(A_{\varepsilon,n}(\mu)) \geq h_\mu(f) + \int \psi d\mu,$$

which gives the desired estimate. □

Corollary 8.1.2. *Let $f : M \to M$ be a C^1 local diffeomorphism. Every expanding ergodic f-invariant probability measure μ such that $\int \psi d\mu + h_\mu(f) \geq 0$ is a weak-SRB-like probability measure.*

Proof. Let $\mu \in \mathcal{M}_f$ be a expanding ergodic probability measure such that $\int \psi d\mu + h_\mu(f) \geq 0$. By Theorem 8.1.1, we have that

$$\lim_{\varepsilon\to0^+} \limsup_{n\to+\infty} \frac{\log(\operatorname{Leb}(A_{\varepsilon,n}(\mu)))}{n} \geq h_\mu(f) + \int \psi d\mu \geq 0.$$

Moreover, if $\varepsilon_1 < \varepsilon_2$ then $A_{\varepsilon_1,n}(\mu) \subset A_{\varepsilon_2,n}(\mu)$. So $\limsup_{n\to+\infty} \frac{\log(\mathrm{Leb}(A_{\varepsilon,n}(\mu)))}{n}$ is increasing with $\varepsilon > 0$. Thus $\limsup_{n\to+\infty} \frac{\log(\mathrm{Leb}(A_{\varepsilon,n}(\mu)))}{n} \geq 0$ for all $\varepsilon > 0$. But since Leb is a probability measure, we conclude that

$$\limsup_{n\to+\infty} \frac{\log(\mathrm{Leb}(A_{\varepsilon,n}(\mu)))}{n} = 0 \ \ \forall \varepsilon > 0.$$

Thus, we deduce that μ is weak-SRB-like measure. □

Remark 8.1.3. *Theorem 8.1.1 is true replacing f by an open expanding and topologically transitive map T of a compact metric space X;* Leb *by a ϕ-conformal measure ν, with $\mathcal{L}^*_\phi(\nu) = \lambda\nu$ for some $\lambda > 0$, and for a continuous potential $\phi : X \to \mathbb{R}$ with the extra assumption that $h_\nu(T, \mu)$ is finite. So we get*

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to+\infty} \frac{1}{n} \log \nu(A_{\varepsilon,n}(\mu)) \geq h_\mu(f) + \int \phi d\mu - \log \lambda,$$

since we used that Leb *is ψ-conformal together with general results from Ergodic Theory. Analogously for Corollary 8.1.2.*

Corollary 8.1.4. *Let $f : M \to M$ be a C^1 local diffeomorphism. Every expanding ergodic f-invariant probability measure that satisfies Pesin's Entropy Formula is a weak-SRB-like probability measure.*

Proof. Let $\mu \in \mathcal{M}_f$ be an expanding ergodic probability measure such that $h_\mu(f) = \int \sum^+ d\mu$, where $\sum^+(x)$ is the sum of the positive Lyapunov exponents of a μ-generic point x counting multiplicities.

We deduce by the Multiplicative Ergodic that $-\int \psi d\mu \leq \int \sum^+ d\mu = h_\mu(f)$. Therefore, $h_\mu(f) + \int \psi d\mu \geq 0$ and the corollary follows from Corollary 8.1.2. □

8.2 Ergodic expanding weak-SRB-like measures

Now we are ready to prove Corollary E.

Proof of Corollary E. Since $P_{\mathrm{top}}(f, \psi) = 0$ we conclude by Proposition B that all weak-SRB-like measures are equilibrium states for the potential ψ, that is, $h_\mu(f) + \int \psi d\mu = 0$ for all $\mu \in \mathcal{W}^*_f$.

We know that there exists $\sigma \in (0, 1)$ such that $\mathrm{Leb}(H(\sigma)) = 1$. Given $\mu \in \mathcal{W}_f$, by Lemma 7.2.2 we have that $\int \log \|(Df)^{-1}\| d\mu \leq \log \sqrt{\sigma}$.

By the Ergodic Decomposition Theorem, there exists $A \subset M$ such that $\mu(A) > 0$ and for all $y \in A$, $\int \log \|(Df)^{-1}\| d\mu_y \leq \log \sqrt{\sigma}$, where μ_y is an ergodic component of μ.

Fix $y_0 \in A$. By Birkhoff's Ergodic Theorem,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(y))^{-1}\| = \int \log \|(Df)^{-1}\| d\mu_{y_0} \leq \log \sqrt{\sigma},$$

for μ_{y_0}-a.e. $y \in M$. Therefore μ_{y_0} is an expanding probability measure.

Since $P_{\text{top}}(f, \psi) = 0$ and $h_\mu(f) + \int \psi d\mu = 0$ we conclude that $h_{\mu_y}(f) + \int \psi d\mu_y = 0$ for all μ-a.e $x \in M$. In particular, by Corollary 8.1.2 we conclude that there exist $y_0 \in A$ such that $\mu_{y_0} \in \mathcal{W}_f^*$ showing that there exist expanding ergodic weak-SRB-like probability measure such that satisfies Pesin's Entropy Formula ending the proof of Corollary E. □

Chapter 9

Weak-Expanding non-uniformly expanding maps

In this chapter we reformulate Pesin's Entropy Formula for a class of weak-expanding and non-uniformly expanding maps with C^1 regularity and prove Corollary F.

9.1 Weak-SRB-like, equilibrium and expanding measures

We divide the proof of Corollary F into the next two corollaries below.

Corollary 9.1.1. *Let $f : M \to M$ be weak-expanding and non-uniformly expanding. Then, all (necessarily existing) weak-SRB-like probability measures are ψ-equilibrium states and, in particular, satisfy Pesin's Entropy Formula. Moreover, there exists some ergodic weak-SRB-like probability measure.*

Proof. Let $f : M \to M$ be as in statement of Corollary F then for every $x \in M$ and all $v \in T_xM \backslash \{0\}$ we have $\liminf_{n\to+\infty} \frac{1}{n} \log \|Df^n(x) \cdot v\| \geq 0$. Thus, the Lyapunov exponents for any given $f-$invariant probability measure μ are non-negative. Hence $\Sigma^+(x) = \lim_{n\to+\infty} \frac{1}{n} \log |\det Df^n(x)|$ and $\int \Sigma^+ d\mu = \int \log |\det Df| d\mu = -\int \psi d\mu$.

For C^1-systems, Ruelle's Inequality [36] ensures $h_\mu(f) \leq \int \Sigma^+ d\mu$ then $h_\mu(f) + \int \psi d\mu \leq 0$. By Proposition B, we have that $P_{\text{top}}(f, \psi) = 0$ and $h_\mu(f) + \int \psi d\mu = 0$ for all $\mu \in \mathcal{W}_f^*$.

Therefore, all weak-SRB-like probability measures are ψ-equilibrium states and satisfy Pesin's Entropy Formula. Moreover, by Corollary E we conclude that there exist ergodic weak-SRB-like measures. □

Here we obtain a sufficient condition to guarantee that all ψ-equilibrium states are generalized convex combinations of weak-SRB-like measures.

Corollary 9.1.2. *Let $f : M \to M$ be weak-expanding and non-uniformly expanding. If $\psi < 0$ there is no atomic weak-SRB-like probability measure. Moreover, if $\mathcal{D} = \{x \in M; \|Df(x)^{-1}\| = 1\}$ is finite and $\psi < 0$ then almost all ergodic components of a ψ-equilibrium state are weak-SRB-like measures and all weak-SRB-like probability measures μ have ergodic components μ_x which are expanding weak-SRB-like probability measure for μ-a.e. $x \in M \backslash \mathcal{D}$.*

Proof. Note that if $\psi < 0$ then $\int \psi d\mu < 0$ for all $\mu \in \mathcal{M}_f$. On the other hand, if $\mu \in \mathcal{M}_f$ is an atomic invariant probability measure, then $h_\mu(f) = 0$. Therefore μ does not satisfy Pesin's Entropy Formula and by Corollary 9.1.1, we conclude that there is no atomic weak-SRB-like probability measure.

Since $\int \psi d\mu < 0$ and $h_\mu(f) \leq -\int \psi d\mu$ for all $\mu \in \mathcal{M}_f$, then a ψ-equilibrium state satisfies $h_\mu(f) + \int \psi d\mu = 0$ and so $h_{\mu_x}(f) + \int \psi d\mu_x = 0$, μ-a.e. x by the Ergodic Decomposition Theorem and $h_{\mu_x}(f) > 0$.

Hence μ_x is non-atomic and thus expanding because $\mathrm{supp}(\mu_x) \subsetneq \mathcal{D}$ so $\int \log \|(Df)^{-1}\| d\mu_x < 0$ and μ_x is ergodic, for μ-a.e x. Such μ_x also satisfies the Entropy Formula. Therefore Corollary 8.1.4 ensures that μ_x is weak-SRB-like for μ-a.e. $x \in M$.

Consider $\mu \in \mathcal{W}_f^*$, then $\int \log \|(Df)^{-1}\| d\mu < 0$, otherwise, we have $\mathrm{supp}(\mu) \subset \mathcal{D}$ and we see that this is not possible.

Because μ is a ψ-equilibrium state, then $h_\mu(f) + \int \psi d\mu = 0$. Moreover $0 > \int \log \|(Df)^{-1}\| d\mu = \int_{M \backslash \mathcal{D}} \log \|(Df)^{-1}\| d\mu$, thus by the Ergodic Decomposition Theorem we conclude that for μ-a.e $x \in M \backslash \mathcal{D}$ we have $h_{\mu_x}(f) + \int \psi d\mu_x = 0$ (remember that $P_{\mathrm{top}}(f, \psi) = 0$) and $\int \log \|(Df)^{-1}\| d\mu_x < 0$. Therefore, μ-a.e $x \in M \backslash \mathcal{D}$ have expanding ergodic components that are ψ-equilibrium states. By Corollary 8.1.2, we deduce that μ_x is an weak-SRB-like probability measure for μ-a.e $x \in M \backslash \mathcal{D}$ and finishes the proof.

□

Putting Corollaries 9.1.1 and 9.1.2 together we complete the proof of Corollary F.

9.2 Expanding Case

The Corollary H improves the main result of [19] and allows rewriting all the results from [19], which were only proved for C^1-expanding maps in circle. In this section we prove Corollary H.

Proof of Corollary H. The assumptions on f ensure that all f-invariant probability measures μ are expanding. Moreover, by Proposition B and Ruelle's Inequality we conclude that $P_{\text{top}}(f,\psi) = 0$ and every weak-SRB-like probability measures are ψ-equilibrium states and satisfy Entropy Formula.

Then, on the one hand, if $\mu \in \mathcal{M}_f$ satisfies Entropy Formula, then $h_\mu(f) + \int \psi d\mu = 0$. By Ergodic Decomposition Theorem we have that $h_{\mu_x}(f) + \int \psi d\mu_x = 0$ for all μ-a.e $x \in M$, because $P_{\text{top}}(f,\psi) = 0$. By Corollary 8.1.2 we have that μ_x is weak-SRB-like probability measure for μ-a.e $x \in M$.

On the other hand, if $\mu \in \mathcal{M}_f$ is such that its ergodic components μ_x are weak-SRB-like probability measures for μ-a.e. $x \in M$, then $h_{\mu_x}(f)+\int \psi d\mu_x = 0$ for μ-a.e $x \in M$. Thus, by the Ergodic Decomposition Theorem we have that

$$-\int \psi d\mu = -\int \left(\int \psi d\mu_x\right) d\mu(x) = \int \left(h_{\mu_x}(f)\right) d\mu(x) = h_\mu(f).$$

Assume now that μ is the unique weak-SRB-like probability measure. By item G of Theorem 1 in [17] we have that μ is SRB and $\text{Leb}(B(\mu)) = 1$.

See the proof of Corollary C for large deviation bound. This finishes the proof of Corollary H. □

As observed in [19], the SRB-like condition is a sufficient but not necessary condition for a measure μ to be an equilibrium state for the potential ψ, because it may exist a non-ergodic invariant measure $\mu << \text{Leb}$ that is neither SRB nor SRB-like (see [32]). In such a case μ satisfies Pesin's Entropy Formula, as stated the following lemma.

Corollary 9.2.1. *Let $f : M \to M$ be a C^1-expanding map. Let μ be a non-ergodic f-invariant probability such that $\mu << \text{Leb}$. Then μ satisfies Pesin's Entropy Formula.*

Proof. See Corollary 2.6 in [19]. □

9.3 Proof of Corollary I

Now we are ready to prove of Corollary I.

Proof of Corollary I. Let $f_n \xrightarrow[n\to+\infty]{} f$ in the C^1-topology, where $f_n, f : M \to M$ are C^1-expanding maps for all $n \geq 1$. For each $n \geq 1$ consider μ_n a weak-SRB-like measures associated to f_n and let μ be a weak* limit point: $\mu = \lim_{j\to+\infty} \mu_{n_j}$.

Fix $\mathcal{P}$ a generating partition for every f_{n_j} for all $j \geq 1$ and such that $\mu(\partial\mathcal{P}) = 0$.

This is possible, since f_{n_j} is C^1-expanding and $f_n \to f$ in C^1-topology and f is also C^1-expanding.

By Kolmogorov-Sinai Theorem this implies that $h_{\mu_n}(f_n) = h(\mathcal{P}, \mu_n)$ and $h_\mu(f) = h(\mathcal{P}, \mu)$, that is,

$$h_{\mu_{n_j}}(f_{n_j}) = \inf_{k\geq 1} \frac{1}{k} H(\mathcal{P}^k_{n_j}, \mu_{n_j}) \text{ and } h_\mu(f) = \inf_{k\geq 1} \frac{1}{k} H(\mathcal{P}^k, \mu)$$

Since μ gives zero measure to the boundary of $\mathcal{P}$ then $H(\mathcal{P}^k_{n_j}, \mu_{n_j})$ converge to $H(\mathcal{P}^k, \mu)$ as $j \to \infty$. Furthermore, for every $\varepsilon > 0$ there is $n_0 \geq 1$ such that

$$h_{\mu_{n_j}}(f_{n_j}) \leq \frac{1}{n_0} H(\mathcal{P}^{n_0}_{n_j}, \mu_{n_j}) \leq \frac{1}{n_0} H(\mathcal{P}^{n_0}, \mu) + \varepsilon \leq h_\mu(f) + 2\varepsilon.$$

By Corollary H, $h_{\mu_{n_j}}(f_{n_j}) + \int \psi_{n_j} d\mu_{n_j} = 0$ for all $j \geq 0$, since μ_{n_j} is weak-SRB-like probability measure and $\psi_{n_j} = -\log|\det Df_{n_j}|$.

Since $\psi_{n_j} \to \psi$ in the topology of uniform converge, we have that $\int \psi_{n_j} d\mu_{n_j} \to \int \psi d\mu$. By Ruelle's inequality, $h_\nu(f) + \int \psi d\nu \leq 0$ for any f-invariant probability measure ν on the Borel σ-algebra of M. Thus,

$$0 \geq h_\mu(f) + \int \psi d\mu \geq \limsup_{n\to+\infty} \left(h_{\mu_n}(f_n) + \int \psi_n d\mu_n \right) = 0.$$

This shows that μ satisfies Pesin's Entropy Formula, is a ψ-equilibrium state since $P_{\text{top}}(f, \psi) = 0$ and by Corollary H, its ergodic components μ_x are weak-SRB-like probability measures for μ-a.e $x \in M$. □

9.4 Proof of Corollary C

Finally we prove Corollary C.

Proof of Corollary C. From Remark 7.1.7 and 7.1.8 we can use Proposition B, Theorem 8.1.1 and Corollary 8.1.2 in the setting of Theorem B. Let $\mu \in \mathcal{M}_T$ such that

$$h_\mu(T) + \int \phi d\mu = P_{\text{top}}(T, \phi) = \int \left(h_{\mu_x}(T) + \int \phi d\mu_x \right) d\mu(x)$$

we also have $h_{\mu_x}(T) + \int \phi d\mu_x \leq P_{\text{top}}(T, \phi)$ and so $h_{\mu_x}(T) + \int \phi d\mu_x = P_{\text{top}}(T, \phi)$ for μ-a.e x. Now from Theorem 8.1.1 (see Remark 8.1.3) since $h_\nu(T, \mu_x) < \infty$

for μ-a.e. $x \in X$ we get

$$\lim_{\varepsilon\to 0^+} \limsup_{n\to+\infty} \frac{1}{n} \log \nu(A_{\varepsilon,n}(\mu_x)) \geq h_{\mu_x}(T) + \int \phi d\mu_x - \log \lambda = 0$$

and then we conclude that $\mu_x \in \mathcal{W}_T^*(\nu)$ following the same argument in the proof of Corollary 8.1.2.

Assume now that μ is the unique ϕ-equilibrium state. By Proposition 4.1.2 and Theorem B we conclude that there exist unique μ ν-SRB-like and by Theorem 1.6 in [18] follows that μ is ν-SRB and $\nu(B(\mu)) = 1$.

Let now $\mathcal{V}$ be a small neighborhood of μ in $\mathcal{M}_1$. Since $\{\mathcal{K}_r(\phi)\}_r$ is decreasing with r and $\{\mu\} = \mathcal{K}_0(\phi) = \cap_{r>0}\mathcal{K}_r(\phi)$ we have that there exists $r_0 > 0$ such that $\mathcal{K}_r(\phi) \subset \mathcal{V}$ for all $0 < r < r_0$. Since $\mathcal{K}_r(\phi)$ is weak* compact (by upper semicontinuity of the metric entropy) we have $\mathcal{K}_r(\phi) = \bigcap_{\varepsilon>0} \mathcal{K}_r^\varepsilon(\phi)$, where $\mathcal{K}_r^\varepsilon(\phi) = \left\{\mu \in \mathcal{M}_T;\ \mathrm{dist}(\mu, \mathcal{K}_r(\phi)) \leq \varepsilon\right\}$ with the weak* distance defined in (4.1.1). Consider $0 < \varepsilon < r_0$, such that $\mathcal{K}_r^\varepsilon(\phi) \subset \mathcal{V}$. By Proposition A, there exists $n_0 \geq 1$ and $\kappa = \kappa(\varepsilon, r) > 0$ such that

$$\begin{aligned} \nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{V}\}) &\leq \nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{K}_r^\varepsilon(\phi)\}) \\ &= \nu(\{x \in X;\ \mathrm{dist}(\sigma_n(x), \mathcal{K}_r(\phi)) \geq \varepsilon\}) < \kappa e^{n(\varepsilon - r)}, \end{aligned}$$

for all $n \geq n_0$. Thus $\limsup_{n\to+\infty} \frac{1}{n} \log \nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{V}\}) < \varepsilon - r$. As $\varepsilon > 0$ can be taken arbitrary small, we conclude that $\limsup_{n\to+\infty} \frac{1}{n} \log \nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{V}\}) < -r$ for all $0 < r < r_0$ and $r_0 = I(\mathcal{V}) := \sup\{r > 0;\ \mathcal{K}_r(\phi) \subset \mathcal{V}\}$. Therefore

$$\limsup_{n\to+\infty} \frac{1}{n} \log \nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{V}\}) \leq -I(\mathcal{V}).$$

This shows that the probability $\nu(\{x \in X :\ \sigma_n(x) \in \mathcal{M}_1 \backslash \mathcal{V}\})$ decreases exponentially fast with n at a rate that depends on the "size" of $\mathcal{V}$.

The assumptions on μ are the same as in Corollary H with $\nu = \mathrm{Leb}$ and $\phi = \psi$, so the upper large deviations statement of this corollary follows with the same proof. $\square$

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