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Instituto de Matemática - IM
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Tese de Doutorado

ON THE BEHAVIOUR OF THE SINGULAR VALUES OF EXPANDING LORENZ MAPS

Fabíola de Oliveira Pedreira

Salvador-Bahia
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Tese de Doutorado apresentada ao Colegiado da Pós-Graduação em Matemática da Universidade Federal da Bahia, como requisito parcial para obtenção do Título de Doutora em Matemática.

Orientador: Prof. Dr. Vilton Jeovan Viana Pinheiro.

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Banca examinadora:

Prof. Dr. Vilton Jeovan Viana Pinheiro (Orientador)
UFBA

Prof. Dr. Paulo César Rodrigues Pinto Varandas
UFBA

Prof. Dr. Manuel Stadlbauer
UFRJ

Prof^{a}. Dr^{a}. Maria José Pacífico
UFRJ

Prof. Dr. Alberto Adrego Pinto
FCUP

À meu filho e minha mãe
pelo carinho e apoio.

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"A persistência é o caminho do êxito."

Charles Chaplin.

Abstract

In this work we study one-dimensional expanding Lorenz maps f with the same singular point c. We show that if the orbits of singular values satisfy a condition of slow recurrence, then every ergodic invariant probability has slow recurrence to the singularity and it has finite Lyapunov exponent. Moreover, we show that generically the singular values do not belong to the basin of its SRB measure. Also, we show that singularity allows the existence of many ergodic invariant measures with full support, having positive entropy, fast recurrence to the singular region and infinite Lyapunov exponent. Furthermore, we consider a two-parameter standard family of these maps and prove that there is a cone in the parameter space, in which we find sets of points on the curves, which has positive Hausdorff dimension, so that the maps associated to these points have finite Lyapunov exponent for every ergodic invariant probability, and there is one and only one equilibrium state for a given Hölder potential.

Keywords: Expanding Lorenz, Lyapunov exponent, slow recurrence, two-parameter family, Hausdorff dimension.

Resumo

Neste trabalho, estudamos mapas de Lorenz expansor unidimensional f com mesmo ponto singular c. Mostramos que se as órbitas dos valores singulares satisfazem uma condição de recorrência lenta, então toda probabilidade invariante ergódica possui recorrência lenta à singularidade e expoente de Lyapunov finito. Além disso, mostramos que genericamente, os valores singulares de um mapa de Lorenz expansor, não pertencem à bacia de sua medida SRB. Mostramos também, que a singularidade permite a existência de muitas medidas invariantes ergódicas com suporte total, entropia positiva, recorrência rápida à região singular e expoente de Lyapunov infinito. Além disso, consideramos uma família a dois parâmetros destes mapas e provamos que existe um cone no espaço de parâmetros, no qual encontramos conjuntos de pontos em curvas, com dimensão de Hausdorff positiva, de modo que os mapas associados a estes pontos possuem expoente de Lyapunov finito para todas as probabilidades invariantes ergódicas, e existe um único estado de equilíbrio para cada potencial Hölder.

Palavras-chave: Lorenz Expansor, expoente de Lyapunov, recorrência lenta, família a dois-parâmetros, dimensão de Hausdorff.

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Introduction

One of the most impactful works in the area of Dynamic Systems were the studies of mathematician and meteorologist Edward Lorenz, published in the Journal of Atmospheric Sciences [L] in 1963. Motivated by an attempt to understand the fundamentals of weather forecasting, he obtained a model for the convection of thermal fluids, given by the system of differential equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy \end{aligned} \tag{1}$$

for parameters $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$.

The behaviour observed by him in system (1), originated what is now known as a strange attractor, whose shape is well known for being similar to a butterfly (see Figure 1).

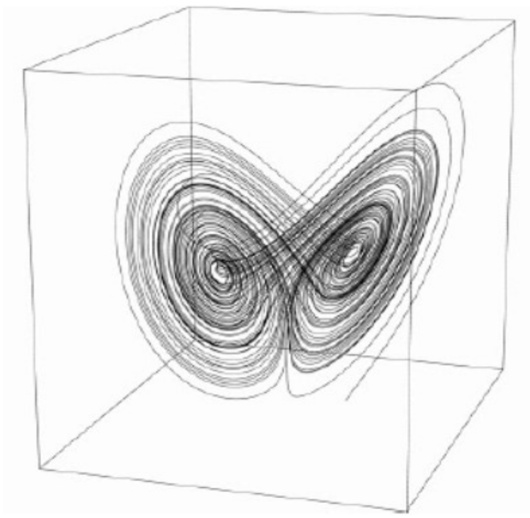


Figure 1: Strange attractor.

Lorenz and others observed, using numerical simulations, what in the open neighborhood of the parameters almost all the points in the phase space tend to that they called a strange attractor.

The difficulty in obtaining a rigorous analysis of the equations, caused many researchers to suggest a geometric model for the Lorenz attractor. Among them, Afraimovich, Bykov and Shil'nikov [ABS] in 1977, Guckenheimer and Williams [GW] in 1979, presented a construction of the model, dynamically similar to that of Lorenz, in a linearized neighborhood, whose origin is a singularity with eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ and with expanding condition $\lambda_1 + \lambda_3 > 0$.

Some authors such as Rovella [R] and Brandão [BR2], consider a vector field similar, but modifying the eigenvalues of the singularity to a contracting condition, taking $\lambda_1 + \lambda_3 < 0$ and then working with properties of the contracting Lorenz attractor.

The Lorenz attractor can be described using global cross-sections. The dynamical behaviour is then analyzed by taking the Poincaré return map to a section $\Sigma = \{|x| \leq 1/2; |y| \leq 1/2; z = 1\}$. From there, we obtain the first return map to Σ, $P : \Sigma^* \to \Sigma$ that has the form $P(x, y) = (f(x), g(x, y))$, where $\Sigma^* = \Sigma \setminus \{x = 0\}$ (see Figure 2). A more detailed study of the attractor and its properties can be seen in [AP].

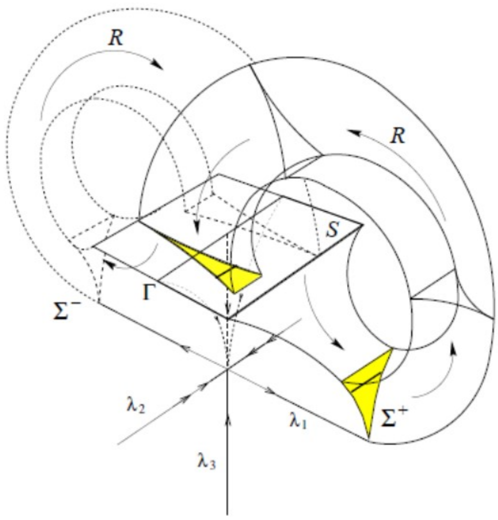


Figure 2: Lorenz geometric attractor. (Figure in [AP])

Due to the contraction of vertical leaves of Σ, points on the same leaf of the stable foliation have essentially the same behaviour in the future, so, to understand the dynamics of the P map, just observe the behaviour of a single point on each leaf, looking at the quotient dynamics f.

Thus, many are dedicated to studying the properties of map f, and the purpose here in this work is to study these Lorenz maps for the expanding case, observing the behaviour of the trajectory of their singular values, since these orbits play a fundamental role in describing topological and metrical properties of the dynamics. Here at work the maps will be considered with the domain range being $[0, 1]$ and we will denote by $\mathcal{L}$ the set of all these one-dimensional expanding Lorenz maps f in the interval $[0, 1]$, with the same singular point $c \in (0, 1)$. In Chapter 1, we will see about the map properties for the expanding case and in the Figure 3 below, we can see the graph for the two cases, on the left the expanding case and on the right the contracting case.

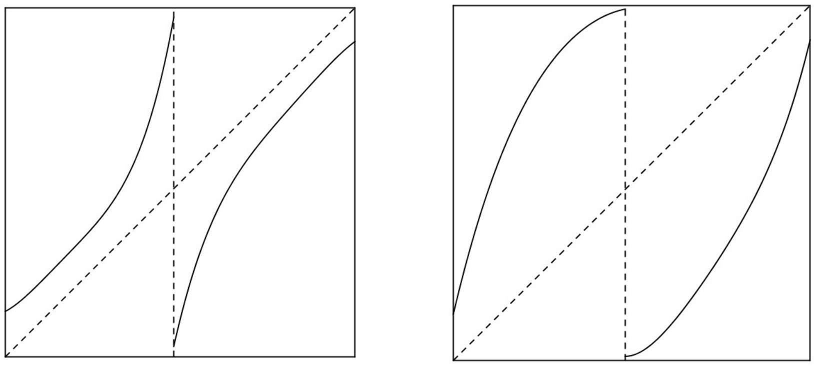


Figure 3: Expanding case for $\lambda_1 + \lambda_3 > 0$ and contracting case for $\lambda_1 + \lambda_3 < 0$.

One of the main goal in the study of Dynamical Systems is to describe the behaviour for the orbits of as many points as possible when time goes to infinity. The existence of invariant measures gives us information about this, they describe the behaviour of asymptotic time and provide a description of the measure of an attractor without necessarily knowing it.

The most natural measure to think that would be the Lebesgue measure m is not always invariant for the system, so we often try to find invariant measures that are comparable, in a certain sense, with the Lebesgue measure, which is the case of the absolutely continuous invariants probabilities (a.c.i.p.). Throughout the text, whenever we mention a.c.i.p. we will be referring to in relation to the Lebesgue measure.

Birkhoff's ergodic theorem establishes the importance of these measures, but says nothing about their existence.

Theorem (Birkhoff)**.** . *Let $f : X \to X$ preserve a probability measure μ. Given any $\varphi \in L^1(\mu)$ there exists $\varphi^* \in L^1(\mu)$ with $\varphi^* \circ f = \varphi^*$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) = \varphi^*(x) \tag{2}$$

for μ almost every $x \in X$. Moreover, if f is ergodic, then $\varphi^* = \int \varphi d\mu$ almost everywhere.

We know (see Viana [Vi97]) that each map $f \in \mathcal{L}$ admits a unique measure μ a.c.i.p. with respect to m, which guarantees μ ergodic and therefore μ almost every point satisfies (2) for $\varphi^* = \int \varphi d\mu$. When this occurs for any $\varphi : X \to \mathbb{R}$ continuous, it means that these points belong to the basin of measure μ, $B(\mu)$, see Definition 2.0.1. So, in the case of f we have that the $B(\mu)$ has full μ measure, which implies that the $B(\mu)$ has a positive Lebesgue measures, since μ is a.c.i.p. and therefore μ also is a SRB measures. However, this may not be the behaviour for any chosen point, or specifically for the singular values $f(c_\pm)$ of the map f.

Introduced in the decade of 1960 by Smale, the (uniformly) Hyperbolic Systems became a reference in the study of Chaotic dynamical systems. For such a system, there are three types of orbits, the contracting, the expanding and the saddle ones. For maps of the interval, hyperbolic orbits can be only expanding or contracting. If f is a C^1 interval map, a point p has a contracting orbit if $|(f^n)'(p)| \leq Ce^{-\lambda n}$ and an expanding one if $|(f^n)'(p)| \geq Ce^{\lambda n}$ for every $n \geq 1$, where C and $\lambda > 0$. The Lyapunov exponent of a point defined as

$$\lambda_f(x) := \lim_n \frac{1}{n} \log |(f^n)'(x)|$$

whenever this limit exists. Hence, an expanding point p has positive Lyapunov exponent and a contracting one has negative Lyapunov exponent. If μ is an ergodic f-invariant probability, it follows from Birkhoff's ergodic theorem that $\lambda_f(x) = \int_{x\in[0,1]} \log |f'(x)| d\mu$ for μ almost every point. Thus, the Lyapunov exponent of μ is defined as

$$\lambda_f(\mu) = \int_{x\in[0,1]} \log |f'(x)| d\mu.$$

The following is a consequence of Theorem B in [Pr]:

Theorem (Przytycki)**.** *Let $\mathcal{C}$ be a finite subset of $(0,1)$ and $f : [0,1] \setminus \mathcal{C} \to [0,1]$ be a C^{1+} local diffeomorphism with non-flat critical region $\mathcal{C}$ such that $\lim_{d(x,\mathcal{C})\to 0} f'(x) = 0$. If μ is an ergodic f-invariant probability and $\lambda_f(\mu) < 0$ then there exists $p \in [0,1]$ and $\ell \geq 1$ such that $\mu = \frac{1}{\ell}\sum_{j=0}^{\ell} \delta_{f^j(p_-)}$ or $\mu = \frac{1}{\ell}\sum_{j=0}^{\ell} \delta_{f^j(p_+)}$.*

Where for $p \in [0,1]$ and $j \geq 0$, define $f^j(p_\pm) = \lim_{0<\epsilon\to 0} f^j(p \pm \epsilon)$.

It follows from Przytycki's result that ergodic probabilities μ with $\lambda_f(\mu) = -\infty$ must be supported on a "periodic" critical orbit. In particular, those measures are very simples ones: finite support and zero entropy ($h_\mu(f) = 0$).

For an expanding Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$, we have that $\lambda_f(\mu) > 0$ for every ergodic invariant probability μ. If $f^\ell(c_\pm) = c$ for some $\ell \geq 1$, then $\lambda_f(\mu) = +\infty$ for $\mu = \frac{1}{n}\sum_{j=1}^{n} \delta_{f^j(c_\pm)}$. Of course, in this case, μ has a finite support and zero entropy. Nevertheless, in the expanding Lorenz's context, we do not have the Przytycki's result and so, we don't have a priori that μ has finite support and zero entropy when $\lambda_f(\mu) = +\infty$. The existence of "bad measures", that is, measures with infinite Lyapunov exponents, fast recurrence to the critical/singular region and positive entropy have been conjectured for many years. In the present thesis, we show that bad measures indeed exist on the expanding Lorenz context (Theorem A). We also give a condition on the singular values to assure that all invariant measures of a expanding Lorenz map have finite Lyapunov exponents (Theorem B). In particular, this condition is satisfied when the singular values belong to the basin of attraction of the SRB measure (Theorem 1). Nevertheless, we also show that the singular values for generic expanding Lorenz maps do not belong to the basin of attraction of the SRB measure (Theorem C).

In the study of non-uniform hyperbolicity for system with critical or singular region, the slow recurrence to the critical/singular region plays a crucial role. For such systems, it is common to assume a non-degenerated (non-flat) critical or singular region. For these cases, almost all points, with respect to invariant probability μ, have slow recurrence to the critical/singular region if and only if they have only finite Lyapunov exponents.

It is interesting to note that the contracting Lorenz maps are far more diverse than the expanding ones. In general, the study of the contracting Lorenz maps are more complicated and subtle. But, due to the "bad measures", this is not the case for the Thermodynamical Formalism.

A map is called Misiurewicz when the critical/singular region is not recurrent. For a Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$ this means that forward orbit of the critical/singular values stay away form the critical/singular point (i.e., $c \notin \overline{\mathcal{O}_f^+(f(c_\pm))}$). Misiurewicz maps always have interesting properties. In particular, they satisfy the hypothesis of Theorem B and so, Misiurewicz expanding Lorenz maps does not admit "bad measures". In Theorem D we study the presence of Misiurewicz maps on the parameter space the standard two-parameters families of expanding Lorenz maps. Although all results here, with respect to the parameter space, are about the expanding Lorenz maps, we believe that they can be adapted to the contracting case when the initial map has an absolutely continuous invariant probability.

Statement of the main results

Let $\mathcal{M}^1(f)$ be the set of all f-invariant probabilities and $\mathcal{M}^1_e(f)$ be the set of all ergodic f-invariant probabilities.

Theorem A. *Let $f : [0,1] \setminus \{c\} \to [0,1]$ be a non-flat C^2 expanding Lorenz map with singular point $c \in (0,1)$. If there exist $t \geq 1$ and $r > 0$ such that $f^t(c_+) = c$ and $\mathcal{O}_f^+(f(c_-)) \cap (c, c+r) = \emptyset$, then there exists an uncountable set $\mathcal{M}_c \subset \mathcal{M}^1(f)$ such that if $\mu \in \mathcal{M}_c$ then*

1. *μ is ergodic;*
2. $\operatorname{supp} \mu = [f(c_+), f(c_-)]$, *i.e., full support;*

3. $h_\mu(f) > 0$, *i.e., positive entropy;*

4. $\int_{x\in\mathbb{X}} |\log \operatorname{dist}(x,c)| d\mu = +\infty$, *i.e., fast recurrence to the singular region;*

5. $\lim_n \frac{1}{n}\log(f^n)'(x) = +\infty$ *for* μ *almost every* x, *i.e., infinite Lyapunov exponent.*

Furthermore, $\sup\{h_\mu(f)\,;\,\mu\in\mathcal{M}_c\} = \sup\{h_\mu(f)\,;\,\mu\in\mathcal{M}^1(f)\} =: h_{top}(f)$.

In Theorem B below, we show that when the orbits of singular values have a condition of slow approximation to the singular region, we obtain that all the ergodic invariant probabilities for the system have slow recurrence the singularity and have finite Lyapunov exponent.

Theorem B. *Let* $f : [0,1]\setminus\{c\} \to [0,1]$ *be a non-flat* C^2 *expanding Lorenz map with singular point* $c \in (0,1)$. *If*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} -\log|f^j(c_\pm)-c| < +\infty \tag{3}$$

then there is $\Upsilon > 0$ *such that* $\int_{x\in[0,1]} |\log|x-c||d\mu \le \Upsilon$ *for every* $\mu \in \mathcal{M}_e^1(f)$. *Moreover, if* (3) *holds then, every* $\mu\in\mathcal{M}_e^1(f)$ *has slow recurrence to the singularity and has finite Lyapunov exponent.*

In particular, if the singular values $f(c_\pm) \in B(\mu)$, for the SRB measure μ, then f satisfies the condition (3).

For the case of the quadratic family $f_a(x) = 1 - ax^2$ on $[-1,1]$, Benedicks and Carleson in 1985 [BC], proved that there is a set $\Delta_\infty \subset (1,2)$ of parameters a, having positive Lebesgue measure, so that for almost all $a \in \Delta_\infty$, f_a admits an a.c.i.p. and which the critical point is typical with respect to this a.c.i.p., i.e., it belongs to the basin of measure.

In this work we show, that with respect to the C^2 topology, generically the Lorenz maps in $\mathcal{L}$ do not have their singular values belonging to the basin of the measure μ.

Theorem C. *Generically, the singular values of an expanding Lorenz map do not belong to the statistical basin of attraction of its SRB measure. Furthermore, generically, expanding Lorenz maps do not satisfies the condition (3).*

Theorem C it follows from more precise characterization of recurrence for singular set in Chapter 2.

We seek to obtain in $\mathcal{L}$ "many" maps for which their singular values satisfy condition (3) of Theorem B, and with that purpose we study a two-parameter family of expanding Lorenz maps $\mathcal{F} = \{f_{t,s}\}$, defined in Section 4.4, with a same singular point c, and $(t,s)\in\Gamma=[1/2,1]\times[1/2,1]$. Observing the behaviour of the singular values of the maps and points of a Cantor set, in the phase space, with an arbitrarily large Hausdorff dimension, we obtained the following result.

Theorem D. *Let $\{f_{t,s}\}_{(t,s)\in\Gamma}$ be a two-parameter family of expanding Lorenz maps, where $\Gamma=[1/2,1]\times[1/2,1]$. Then there is a cone $\Gamma'\subset\Gamma$, so that for each $(t,s)\in\Gamma'$ on any smooth curve ψ_0 passing through (t,s) and tangent to the cone, there is a set of points Γ_0 with positive Hausdorff dimension in the curve ψ_0, such that, in each point $(t_i,s_i)\in\Gamma_0$ will pass a smooth curve ψ_i, crossing transversely ψ_0 (see Figure 4.14), and containing a set of points Γ_i with positive Hausdorff dimension in the curve ψ_i, such that for each $(t,s)\in\Gamma_i$, there is a constant $C_{t,s}>0$ such that the Lyapunov exponent of every ergodic $f_{t,s}$ invariant probability is bounded by $C_{t,s}$.*

The previous results give many Lorenz maps having no ergodic invariant probability with infinite Lyapunov exponent. Furthermore, we can guarantee for these maps, the existence of an unique measure of equilibrium state for Hölder potential and this equilibrium state must be a probability with finite Lyapunov exponent.

In systems with critical or singular points, measures of equilibrium states may not exist or they may not have finite Lyapunov exponents. Pinheiro e Varandas [PV], considering the set of all ergodic f-invariant zooming probability with exponential zooming contraction, $\mathcal{E}(f)$, for f non-uniformly expanding map (n.u.e.), show that there exists at most one f-equilibrium state $\mu\in\bigcup_{n\in\mathbb{N}}\mathcal{E}(f^n)$ for a given Hölder potential ϕ. Moreover, for the set $\overline{\bigcup_{n\in\mathbb{N}}\mathcal{E}(f^n)}$, the existence of such a measure is guaranteed, according to Theorem 5.

For each $f\in\mathcal{L}$, due to the existence of the measure a.c.i.p. μ mentioned, we know that f is n.u.e., and by Lemma B2 of [Pi20] each measure with finite Lyapunov exponent belongs to $\mathcal{E}(f)$. Thus, for each $f\in\mathcal{L}$ that satisfies condition (3) of Theorem B, we have that f has one and only one equilibrium states for any given Hölder potential φ which is Theorem 6.

From Theorem D and Theorem 6, we obtain the following Theorem.

Theorem E. *Let $\{f_{t,s}\}_{(t,s)\in\Gamma}$ be a two-parameter family of expanding Lorenz maps, $\Gamma=[1/2,1]\times[1/2,1]$. Then there is a cone $\Gamma'\subset\Gamma$, so that for each $(t,s)\in\Gamma'$ on the any smooth curve ψ_0 passing through (t,s) and tangent to the cone, there is a set of points Γ_0 with positive Hausdorff dimension in the curve ψ_0, such that, in each point $(t_i,s_i)\in\Gamma_0$ passes a smooth curve ψ_i, crossing transversely ψ_0, and containing a set of points Γ_i with positive Hausdorff dimension in the curve ψ_i, such that for each $(t,s)\in\Gamma_i$ there is one and only one $f_{t,s}$-equilibrium state for a given Hölder potential $\varphi:[0,1]\to\mathbb{R}$.*

The text is organized as follows. In Chapter 1, we present some definitions and results that will be important for the work, especially for the proof of Theorem C. We show how nice intervals containing the singularity c, can be found for a map $f \in \mathcal{L}$, in order to accumulate in c.

Chapter 2 is dedicated to the proof of Theorem C. In it we prove, using local perturbations at nice intervals, that there are dense subsets $\mathcal{D}_0$ and $\mathcal{D}_1$ of $\mathcal{L}$ such that each $f \in \mathcal{D}_0$ is preperiodic and each $f \in \mathcal{D}_1$ is periodic. From these subsets, we constructed two residual subsets $\mathcal{R}_0$ and $\mathcal{R}_1$ whose intersection will give us the desired residual subset in the statement of Theorem.

In Chapter 3 the proof of Theorem B is made. We use the notion of bound period, which gives a period of time in which a point when returning to a neighborhood of the singularity, has its orbit following the orbit of this singular value. An estimate is obtained for this period of time, according to the depth at which the point returns to that neighborhood and then we can obtain estimates of the recurrence of the points orbit using condition (3).

In Chapter 4 we show Theorem D. For this, we constructed Cantor sets using an induced Markov map in hyperbolic time in order to have distortion control and allow us to calculate the Hausdorff dimension of this set. We have seen that it is possible to obtain Cantor sets with arbitrarily large dimensions and being close enough to the singular values of Lorenz maps. Then we define a two-parameter family of expanding Lorenz maps and analyzing the behaviour of its singular values and the points of Cantor sets invariant for the dynamics, we show that by making small perturbations in an initial map, we obtain in the parameter space, sets of points contained in curves and having positive Hausdorff dimension, so that their singular values for the corresponding maps do not accumulate in c. To obtain an estimate of the Hausdorff dimension in the parameter space we use a bi-Hölder map that projects the points of the space into the parameter space.

In Chapter 5, the proof of Theorem E is made, which it follows from the Theorem 5 of Pinheiro and Varandas, applied to Theorem D. And finally in Chapter 6, we will see the proof of Theorem A, which guarantees that there are expanding Lorenz maps having many ergodic measures with infinite Lyapunov exponent whose entropy is positive and full support.

Chapter 1

Definitions and Preliminary Results

In this chapter we will present some definitions and results that will be important for all the work, but mainly for Chapter 2.

1.1 Expanding Lorenz Maps

The one-dimensional Lorenz maps are well studied in dynamical systems. In this work, we focus on expanding Lorenz maps with a single point of discontinuity on a closed interval, which is a singularity.

Let us begin by explaining what we mean by expanding Lorenz maps.

Definition 1.1.1. *(Expanding Lorenz maps). We say that a C^2 map $f : [0,1] \setminus \{c\} \to [0,1]$, $0 < c < 1$, is a expanding Lorenz map if $f(0) = 0, f(1) = 1, f'(x) \geq \lambda > 1, \forall x \in [0,1] \setminus \{c\}$.*

Furthermore, $f'(c_\pm) = \lim\limits_{x \to c_\pm} f'(x) = \infty$. We will denote by $\mathcal{L}$ the set of all Expanding Lorenz maps.

Definition 1.1.2. *(non-flat). A C^2 expanding Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$, is called non-flat if there exist constants $\alpha, \beta \in (0,1), d_0, d_1 \in [0,1]$ and C^2 diffeomorphisms preserving the orientation $\phi_0 : [0,c] \to [0, d_0^{1/\alpha}]$ and $\phi_1 : [c,1] \to [0, d_1^{1/\beta}]$ such that*

$$f(x) = \begin{cases} d_0 - (\phi_0(c-x))^\alpha & \text{if } x < c, \\ 1 - d_1 + (\phi_1(x))^\beta & \text{if } x > c. \end{cases}$$

where $f(c_-) = d_0$ and $f(c_+) = 1 - d_1$.

Example 1.1.3. *An example of such a function is describe in the Figure 1.1 below.*

Given $n \geq 1$, define $f^n(c_-) = \lim\limits_{x \uparrow c} f^n(x)$ and $f^n(c_+) = \lim\limits_{x \downarrow c} f^n(x)$. The singular values of f are $f(c_-)$ and $f(c_+)$ and the study of positive orbit of this points, $\mathcal{O}_f^+(f(c_-))$ and

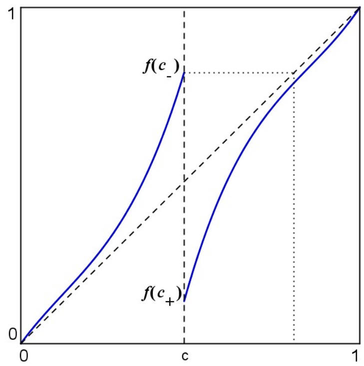


Figure 1.1: Expanding Lorenz Maps.

$\mathcal{O}_f^+(f(c_+))$, plays a fundamental role to describe topological and metric properties of the dynamics. The *pre-orbit* of a point $x \in [0,1]$ is the set $\mathcal{O}_f^-(x) := \bigcup_{n\geq 0} f^{-n}(x)$. For a point $x \in [0,1] \setminus \mathcal{O}_f^-(c)$, denote the positive orbit of x by $\mathcal{O}_f^+(x) = \{f^j(x); j \geq 0\}$. And, if $\exists \ell \geq 1$ such that $f^\ell(c_-) = c$ (we can assume that ℓ is the smallest positive integer with this property), we define $\mathcal{O}_f^+(c_-) = \{f^j(c_-); 1 \leq j \leq \ell\}$, similarly we defined $\mathcal{O}_f^+(c_+)$. A branch of f^n is a maximal closed interval I such that f^n is a diffeomorphism in the interior of I. This means that the points on the edge of I are either 0, or 1 or a point in the pre-orbit of c.

Sometimes we write f_0 or f_1 to specify that we are talking about the left or right branch of f, respectively. Since the function is not necessarily injective, some points have more than one pre-image, so we will denote the point images using the inverse branches by the functions f_0, f_1, written as f_0^{-1} and f_1^{-1}.

1.1.1 Nice Interval

A notion very used in the work is the nice interval that we will see below. In addition, we will see that they are easy to obtain and in the case of expanding Lorenz maps we can accumulate in c.

Definition 1.1.4. *(Nice Interval). An open interval $J = (a,b)$ containing the singular point c, is called a nice interval of f if $\mathcal{O}_f^+(\partial J) \cap J = \emptyset$. We denote by Λ_J the set of points whose future orbit avoids J, i.e., $\Lambda_J = \{x \in [0,1]; \mathcal{O}_f^+(x) \cap J = \emptyset\}$, and $\mathcal{P}_J$ the set of connected components of $[0,1] \setminus \Lambda_J$. An element of $\mathcal{P}_J$ is called a gap of Λ_J.*

Remark 1.1.5. *The usual definition of a nice interval does not require that it contain the point c, however here we will assume this is always the case.*

Definition 1.1.6. *(Misiurewicz map) We say that a map f with a singular point c is a Misiurewicz map if there is a neighborhood V of c such that $\mathcal{O}_f^+(c_\pm) \cap V = \emptyset$. This means* *that the singular point is not recurrent.*

Definition 1.1.7. *(Preperiodic and periodic Lorenz maps). We say that a C^2 Expanding Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$, is periodic if $c_\pm \in Per(f)$, that is, there exist j_+, j_- such that*

$$f^{j_+}(c_+) = c \quad \textit{and} \quad f^{j_-}(c_-) = c$$

And we say that f is preperiodic if $c_\pm \notin Per(f)$ and the points $f(c_+)$ and $f(c_-)$ are preperiodic, that is, there exist k_+, k_-, n_+, n_- such that:

$$\begin{aligned} f^{k_+ + n_+}(f(c_+)) &= f^{n_+}(f(c_+)) \\ f^{k_- + n_-}(f(c_-)) &= f^{n_-}(f(c_-)). \end{aligned}$$

In section 2.1 we prove that there are dense subsets $\mathcal{D}_0$ and $\mathcal{D}_1$ of $\mathcal{L}$ such that each $f \in \mathcal{D}_0$ is preperiodic and each $f \in \mathcal{D}_1$ is periodic. Note that every preperiodic map is a Misiurewicz map.

Lemma 1.1.8. *(see Lemma 6.1. [BR1]) If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map then $c \in \omega_f(x)$ for Lebesgue almost every x.*

Remember that $\omega_f(x)$ is the $\omega-$limit set of x, i.e., is the set of accumulation points of the positive orbit of x.

According to the previous Lemma we have that given any open interval I with nonempty interior in $[0,1]$, there is n such that $c \in f^n(I)$. We will define $S(I)$ as the smallest positive integer so that happens, that is

$$S(I) := \min\{n \in \mathbb{N} : c \in f^n(I)\}$$

Lemma 1.1.9. *Let J be a nice interval of an C^2 expanding Lorenz map $f : [0,1] \setminus \{c\} \to [0,1]$. Then*

(1) $S(I) = \min\{j \geq 0; f^j(I) \cap J \neq \emptyset\}, \forall I \in \mathcal{P}_J;$

(2) $f^{S(I)} |_I$ *is a diffeomorphism and* $f^{S(I)}(I) = J, \forall I \in \mathcal{P}_J;$

Proof. Given $I = (p,q) \in \mathcal{P}_J$ let $n = \min\{j \geq 0; f^j(I) \cap J \neq \emptyset\}$. Then by definition of $S(I)$, $n \leq S(I)$. As $p, q \in \Lambda_J$ and $f(\Lambda_J) \subset \Lambda_J$ we have $f^n(p), f^n(q) \notin J$, moreover as $f^n(I) \cap J \neq \emptyset$ it follows that $f^n(I) \supset J \ni c$, thus $n = S(I)$.

Now we have that, if $f^n(I) \supsetneq J$, $\exists\, a \in \partial J \cap f^n(I)$ which implies $\exists\, x = (f^n |_I)^{-1}(a) \in I$, and then $f^j(x) \notin J, \forall j \geq 0$ which is absurd, since $x \notin \Lambda_J$. Therefore $f^n(I) = J$ and how $c \notin f^j(I)$ to $j < S(I)$ we have $f^{S(I)} |_I$ is a diffeomorphism. $\square$

Lemma 1.1.10. *(see Lemma 4.5. [BR2]) If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map then*

$$\mathcal{O}_f^+(x) \cap (0,c) \neq \emptyset \neq \mathcal{O}_f^+(x) \cap (c,1), \quad \forall x \in (0,1) \setminus \mathcal{O}_f^-(c).$$

The results Lemma 1.1.10 and Lemma 1.1.13 it follows from [BR2], if we replace the hypothesis that the map is contracting with that is expanding, but the proofs remain the same.

1.1.2 First Return maps and construction of nice intervals

Definition 1.1.11. *Given an interval J, denote the first return map to J by $F_J : J^* \to J$, i.e., $F_J(x) = f^{R(x)}(x)$, where $J^* = \{x \in J; \mathcal{O}_f^+(f(x)) \cap J \neq \emptyset\}$ and $R(x) = min\{j \geq 1; f^j(x) \in J\}$ that is, called the first return time. We will denote by $\mathcal{C}_J$ be the collection of connected components of J^*.*

Definition 1.1.12. *(Left and right renormalizations). A Lorenz map f is said to be renormalizable by the left side with respect to the nice interval $J = (a,b)$, if $(a,c) \subset J^*$, i.e., $F_J \mid_{(a,c)} = f^n \mid_{(a,c)}$ for some $n > 1$. Analogously, f is said to be renormalizable by the right side with respect to $J = (a,b)$, if $(c,b) \subset J^*$. Moreover, f is renormalizable if $J^* = (a,c) \cup (c,b)$ for some interval nice $J = (a,b) \ni c$.*

The following Lemma 1.1.13 guarantees that if $c \notin \partial I$, for $I \in \mathcal{C}_J$, the restricted first return map $F_J \mid_I$ is a diffeomorphism of I over J, so the return of this interval provide full branch.

Lemma 1.1.13. *(see Lemma 4.1. and Corollary 4.2. [BR2]) Let $f : [0,1] \setminus \{c\} \to [0,1]$, $0 < c < 1$, be a C^2 expanding Lorenz map, $J = (a,b)$ be a nice interval, and $F_J : J^* \to J$ the first return map. The following statements are true:*

(i) $((p,q) \in \mathcal{C}_J$ *and* $p \neq c) \Rightarrow F_J((p,q)) = (a, f^{R|_{(p,q)}}(q))$;

(ii) $((p,q) \in \mathcal{C}_J$ *and* $q \neq c) \Rightarrow F_J((p,q)) = (f^{R|_{(p,q)}}(p), b)$;

(iii) $(c \notin \partial I, I \in \mathcal{C}_J) \Rightarrow F_J(I) = J$;

(iv) $a \in \partial I$*(or* $b \in \partial I$*), for some* $I \in \mathcal{C}_J \Leftrightarrow a \in Per(f)$*(or* $b \in Per(f)$*).*

As $f \in \mathcal{L}$ can not be $\infty-$renormalizable, since it is expanding (see [HS]), we can suppose without loss of generality that f is not renormalizable otherwise f is conjugated to a g that is not renormalizable.

Lemma 1.1.14. *(see Lemma 4.6. [BR2]) If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map then*

$$\overline{Per(f) \cap (0,c)} \ni c \in \overline{(c,1) \cap Per(f)}.$$

Proposition 1.1.15. *If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map and $p \in Per(f) \cap [f(c_+), f(c_-)]$, then*

$$\overline{\mathcal{O}_f^-(p) \cap (0,c)} \ni c \in \overline{(c,1) \cap \mathcal{O}_f^-(p)}.$$

(i.e., the pre-orbit of p accumulates on both sides of the c.)

Proof. By Lemma 1.1.10 we know that $\mathcal{O}_f^-(p) \cap (0,c) \neq \emptyset \neq (c,1) \cap \mathcal{O}_f^-(p)$. Suppose that the pre-orbit of p does not accumulate on the left side of the c (the proof for right side is analogous).

Let (a,b) be the connected component of $[0,1] \setminus \overline{\mathcal{O}_f^-(p)}$ containing c. So, $0 < a \leq c \leq b < 1$ and by the hypothesis we have $a < c$. Let ℓ be the smallest integer bigger than 0 such that $c \in f^\ell((a,c))$. Notice that $f^\ell(a) \leq a$ otherwise $a < f^\ell(a) < c$ gives us $\overline{\mathcal{O}_f^-(p)} \cap (a,c) \neq \emptyset$.

If $c = b$ we have $c \in \overline{(c,1) \cap \mathcal{O}_f^-(p)}$ and $f^\ell(b_-) = f^\ell(c_-) > c = b$, Figure 1.2. Let $p_0 \in \mathcal{O}_f^-(p) \cap [b,1] \cap f^\ell((a,c))$ and $q \in (a,c)$ such that $f^\ell(q) = p_0$. Thus, $q \in \mathcal{O}_f^-(p) \cap (a,c)$ and this contradicts the definition of (a,b).

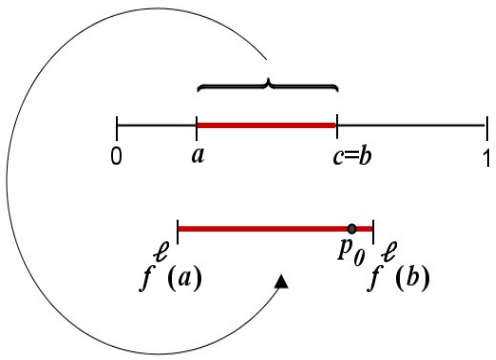


Figure 1.2: $f^\ell((a,b))$.

In the case where $c \neq b$, as (a,b) is not a renormalization interval, if $f^\ell(a) = a$ we have $f^\ell(c_-) > b$ and then the result it follows as before, there exists p_0 and there is $q \in \mathcal{O}_f^-(p) \cap (a,c)$. If $f^\ell(a) < a$ there exists $p_0 \in (f^\ell(a), a) \cap \mathcal{O}_f^-(p)$ and then there is $q \in (a,c)$ such that $f^\ell(q) = p_0$, thus $q \in \mathcal{O}_f^-(p) \subset \overline{\mathcal{O}_f^-(p)}$ and we get a contradiction again. Therefore $\overline{\mathcal{O}_f^-(p) \cap (0,c)} \ni c$. Analogously $c \in \overline{(c,1) \cap \mathcal{O}_f^-(p)}$.

$\square$

The Lemma 1.1.14 and Proposition 1.1.15 allow us to construct nice interval of arbitrary sizes using periodic points or periodic pre-orbit points, which will be very useful for obtaining dense sets $\mathcal{D}_0$ and $\mathcal{D}_1$ in Chapter 2.

Proposition 1.1.16. *If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map then there are sequences $a_n \nearrow c$ and $b_n \searrow c$ such that for each n, $J_n = (a_n, b_n)$ is a nice interval and*

$$J_n \supset J_{n+1} \supset \ldots \textit{ with } |J_n| \to 0.$$

Proof. Take $p_1, p_2 \in Per(f)$ and let $a_1 = \max\{\mathcal{O}_f^+(p_i) \cap (0,c)\}$ and $b_1 = \min\{\mathcal{O}_f^+(p_i) \cap (c,1)\}$. We have that $J_1 = (a_1, b_1)$ is a nice interval. Consider F_{J_1} the first return map to J_1 and $\mathcal{C}_{J_1}$ according to definition 1.1.11. We know from Lemma 1.1.8 that $c \in \omega_f(x)$ for Leb almost every x, and how $(c, b_1) \notin \mathcal{C}_{J_1}$ there exists $(p,q) \in \mathcal{C}_{J_1}$ such that $p \neq c$ or $q \neq b_1$.

If $q \neq b_1$, we have by Lemma 1.1.13, for $n = R\mid_{(p,q)}$, $f^n(q) = b_1$ and since n is the first return time, q does not return to J_1 therefore, for $b_2 = q < b_1$, $J_2 = (a_1, b_2)$ is a nice interval.

If $q = b_1$, we have $p \neq c$ and then $f^n(p) = a_1$, and J_2 can be obtained for $b_2 = p < b_1$.

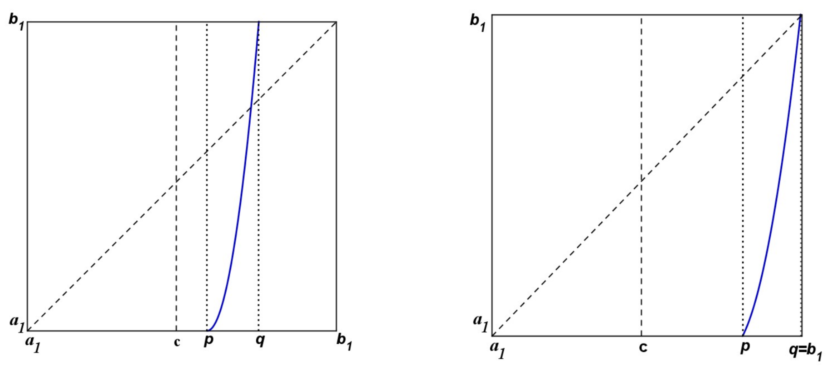


Figure 1.3: Return of interval (p,q) to J_1.

Consider now F_{J_2}, and using the same reasoning we find $c < b_3 < b_2 < b_1$, so that $J_3 = (a_1, b_3)$ is a nice interval. Recursively the result goes on finding $b_n \searrow c$ and, analogously we get $a_n \nearrow c$. By construction, we can conclude that it is possible to get J_n with $|J_n| \to 0$ for $n \to \infty$.

$\square$

Remark 1.1.17. *Note that points a_n and b_n of the boundary of J_n may belong to the same orbit. Moreover, by Brandão[BR1], Λ_{J_n} is a Cantor set if and only if $\mathcal{O}_f^+(a_n) \cap \mathcal{O}_f^+(b_n) = \emptyset$.*

Given a nice interval J we will denote by P_J^- and P_J^+, the connected component of $\mathcal{P}_J$ containing the singular values $f(c_-)$ and $f(c_+)$, respectively, whenever $f(c_\pm) \notin \Lambda_J$.

And in the case where J_n is a nice interval, we will denote by $P_n^{\pm}$, $\mathcal{P}_n$ and Λ_n to $P_{J_n}^{\pm}$, $\mathcal{P}_{J_n}$ and Λ_{J_n}.

Lemma 1.1.18. *Let $f : [0,1] \setminus \{c\} \to [0,1]$ be a C^2 expanding Lorenz map such that $c \in \omega_f(f(c_{\pm}))$ and f is not periodic, then there is nice interval sequence $J_1 \supset J_2 \supset \ldots$, so that for $P_1^{\pm} \supset P_2^{\pm} \supset \ldots \ni f(c^{\pm})$ we have $S_1 < S_2 < \ldots$ where $S_i = S(P_i)$.*

Proof. Suppose that $c \in \omega_f(f(c_-))$, and take a nice interval $J_1 = (a_1, b_1)$, the case $f(c_+)$ is analogue. As $f(c_-) \in P_1^-$ and f is not periodic, $f^{S_1}(f(c_-)) \in (a_1, c)$ or $f^{S_1}(f(c_-)) \in (c, b_1)$. Consider it to be the first case, and take $\epsilon_1 \leq |c - f^{S_1}(f(c_-))|$. By Proposition 1.1.16, we can find a nice interval $J_2 = (a_2, b_2)$ so that $J_2 \subset (c - \epsilon_1, b_1) \subset J_1$ and $f(c_-) \in P_2^- \subset P_1^-$. By definition of S_i and Lemma 1.1.9 we have $S_2 \geq S_1$. Furthermore, as $f^{S_1}(f(c_-)) \notin J_2$ it follows that $S_2 > S_1$. Repeating the argument recursively gives the result.

□

Lemma 1.1.19. *Let $f : [0,1] \setminus \{c\} \to [0,1]$ be a C^2 expanding Lorenz map such that $c \in \omega_f(f(c_{\pm}))$. If $J_1 = (a_1, b_1) \supset J_2 = (a_2, b_2)$ are nice intervals with $b_2 < b_1$ then $b_2^{\pm} < b_1^{\pm}$ and $a_1 < a_2$ then $a_1^{\pm} < a_2^{\pm}$ where $f(c_{\pm}) \in P_i^{\pm} = (a_i^{\pm}, b_i^{\pm}) \in \mathcal{P}_i$.*

Proof. Let S_1, S_2 such that $f^{S_1}(P_1^-) = J_1$ and $f^{S_2}(P_2^-) = J_2$. As $J_2 \subset J_1$, we have $P_2^- \subset P_1^-$ and $S_2 \geq S_1$.

Suppose $b_2 < b_1$ but $b_2^- = b_1^-$. So, $b_2 = f^{S_2}(b_2^-) = f^{S_1+r}(b_2^-) = f^r(f^{S_1}(b_1^-)) = f^r(b_1)$ for $r \geq 0$, but this is absurd, since $\mathcal{O}_f^+(b_1) \cap J_1 = \emptyset$ and $b_2 < b_1$. The other cases follow similarly.

□

By the previous results Proposition 1.1.16 and Lemma 1.1.19, it follows the statement.

Corollary 1.1.20. *If $f : [0,1] \setminus \{c\} \to [0,1]$ is a C^2 expanding Lorenz map such that $c \in \omega_f(f(c_{\pm}))$, then there is infinite sequence of nice intervals $J_n \supset J_{n+1}$, so that*

$$\operatorname{diam}(\mathcal{P}_n) := \max\{\operatorname{diam}(P) | P \in \mathcal{P}_n\} \to 0$$

when $n \to \infty$.

Chapter 2

Proof of Theorem C

This chapter is devoted to proof of the existence of C^2 residual subset $\mathcal{R} \subset \mathcal{L}$ which makes up the proof of Theorem C. For this we will initially recall some concepts necessary to understand the result.

Let $f : \mathbb{X} \to \mathbb{X}$ be a measurable function. We say that a measure μ is *invariant* by f (or f-invariant), if $\mu(f^{-1}(A)) = \mu(A)$ for every measurable set $A \subset \mathbb{X}$. If the measure μ satisfies $\mu(\mathbb{X}) = 1$, we say that μ is a *probability* measure. If μ and ν are finite measures, we say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$. And are said to be *equivalent* if both $\mu \ll \nu$ and $\nu \ll \mu$. An invariant measure μ is said to be *ergodic* if for every invariant measurable set A, i.e., $f^{-1}(A) = A$, implies either $\mu(A) = 0$ or $\mu(\mathbb{X} \setminus A) = 0$.

Definition 2.0.1. *We define $\mathcal{B}(\mu)$, the basin of μ , as the set of those points $x \in M$ for which*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu,$$

for every continuous function $\varphi : M \to \mathbb{R}$. This is equivalent to say that

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \xrightarrow{w^*} \mu \quad as \quad n \to \infty. \tag{2.1}$$

We say that x is a *μ-generic point* when (2.1) occurs. If the $\mathcal{B}(\mu)$ has positive Lebesgue measure, for μ probability measures f-invariant, we say that μ is an *Sinai-Ruelle-Bowen* (SRB) measures.

By Birkhoff's ergodic theorem, if μ is an ergodic probability measure then $\mathcal{B}(\mu)$ has full μ measure. Moreover, if μ is absolutely continuous with respect to the Lebesgue measure m, then the basin of μ has positive Lebesgue measure and therefore μ is an SRB measure.

Recall that the *variation* var φ of a function $\varphi : [0,1] \to \mathbb{R}$ is defined by

$$\text{var } \varphi = \sup \sum_{i=1}^{n} |\varphi(x_{i-1}) - \varphi(x_i)|$$

where the supremum is taken over all finite partitions $0 = x_0 < x_1 < \cdots < x_n = 1,\ n \geq 1$, of $[0,1]$, and says that φ has *bounded variation* if var $\varphi < \infty$.

We know that piecewise expanding Lorenz map f is topologically transitive, which implies that the closure of the maximal invariant set by f is the whole interval $[f(c_+), f(c_-)]$. In addition, $\dfrac{1}{f'}$ is BV, which according to Viana [Vi97] guarantees the existence of an SRB invariant measure.

Proposition 2.0.2. *([Vi97] Cor.3.4 and 3.5) The expanding Lorenz map f admits a unique invariant probability μ which is absolutely continuous with respect to Lebesgue measure m, it is ergodic and then a SRB measure for the map. Moreover, $d\mu/dm$ is a BV function.*

So $d\mu/dm$ is bounded, and it follows from Ledrappier-Young formula that the Lyapunov exponent is finite (see Definition 3.0.1) for μ almost every x.

However, although for every expanding Lorenz map Lebesgue almost every point belongs to the basin of this measure μ given by the Proposition 2.0.2, we show here that this does not happen residually for the singular values of these maps.

Definition 2.0.3. *A subset $\mathcal{R}$ of a topological space $\mathcal{X}$ is residual, if it contains an enumerable intersection of open and dense subsets. We say that a property P is generic in an open set $\mathcal{U}$ of $\mathcal{X}$, if there is a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that every $f \in \mathcal{R}$ satisfies property P.*

Remark 2.0.4. *An enumerable intersection of residual subsets is a residual set.*

For this purpose we will consider the metric induced by the norm C^2, i.e., given $f, g \in \mathcal{L}$

$$d(f,g) = \|f - g\|_2 = \sup_{x \in [0,1] \setminus \{c\}} \{|f(x) - g(x)|, |f'(x) - g'(x)|, |f''(x) - g''(x)|\}.$$

The following propositions 2.0.5 and 2.0.6 play a key role for the proof of Theorem C.

Proposition 2.0.5. *There exists a C^2 residual subset $\mathcal{R}_0$ of $\mathcal{L}$ such that for each $f \in \mathcal{R}_0$ we have*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |\log |f^j(c_\pm) - c|| < +\infty.$$

Proposition 2.0.6. *There exists a C^2 residual subset $\mathcal{R}_1$ of $\mathcal{L}$ such that for each $f \in \mathcal{R}_1$ we have*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |\log |f^j(c_\pm) - c|| = +\infty.$$

The proofs of Proposition 2.0.5 and Proposition 2.0.6 are given in the next section.

2.1 Construction of residual sets $\mathcal{R}_0$ e $\mathcal{R}_1$

In the Lemma 2.1.2 and in the Lemma 2.1.4 below, we will make small local perturbations in the maps in order to make them a preperiodic map or a periodic map.

Our goal is to perturb the map so that the singular values that are in gaps, of a nice interval, are pushed to the points of the boundary of the gaps, in the first case. And, in the second case, are pushed to pre-orbit points of the singular point.

For this, we will use the function φ defined in Remark 2.1.1 below.

Remark 2.1.1. *Let $v : \mathbb{R} \to \mathbb{R}$, be a C^∞ map defined by $v(x) = e^{\frac{1}{x(x+1)}}$, for $-1 < x < 0$, and $v(x) = 0$, for the other values (Figure 2.1).*

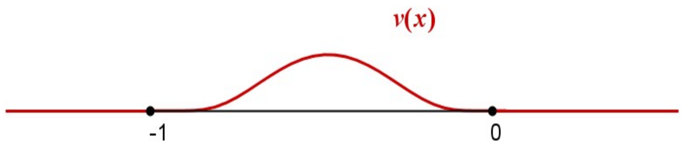


Figure 2.1: Map $v(x)$.

Now let $h : \mathbb{R} \to \mathbb{R}$, be the C^∞ map given by,

$$h(x) = \frac{\int_{-\infty}^{x} v(s)ds}{\int_{-\infty}^{\infty} v(s)ds} = \frac{\int_{-1}^{x} v(s)ds}{\int_{-1}^{0} v(s)ds}.$$

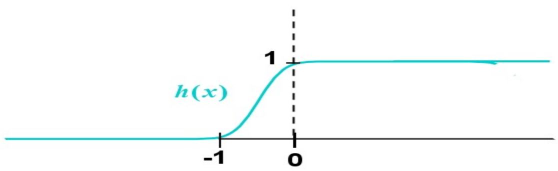


Figure 2.2: Map $h(x)$.

Finally, we define $\varphi : \mathbb{R} \to \mathbb{R}$, be the C^∞ map by (Figure 2.3)

$$\varphi(x) = h(-|x - 0.5| + 0.5).$$

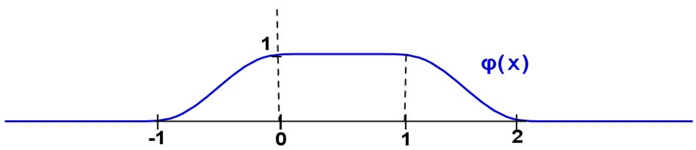


Figure 2.3: Bump Function $\varphi(x)$.

2.1.1 Construction of set $\mathcal{R}_0$

Lemma 2.1.2. *(Density of Misiurewicz maps) There exists a dense subset $\mathcal{D}_0$ of $\mathcal{L}$ such that each $f \in \mathcal{D}_0$ is a preperiodic map.*

Proof. Fix $f \in \mathcal{L}$, and suppose f is not Misiurewicz map, otherwise is finished. So, given any nice interval J we have the singular values $f(c_-)$ and $f(c_+)$ belong to connected component P_J^- and P_J^+ of $\mathcal{P}_J$, respectively.

Let $J_1 = (a, b)$ be a nice interval, which we can consider with the boundary points belonging to the orbit or pre-orbit of the periodic points of f. We will make a small perturbation of f in (a, c) pushing $f(c_-)$ to the right boundary of a gap which $f(c_-)$ belongs to, making it pre-periodic. For this we will use the function of Remark 2.1.1.

Fix $\epsilon > 0$ and let $M := \max_{x\in[-1,2]}\{|\varphi'(x)|, |\varphi''(x)|\}$ and $\gamma := \min\{\epsilon, \epsilon 8(c-a)^2/10M\}$. According to Proposition 1.1.16 and Lemma 1.1.19, we can find $b_2 \in (c, b)$ with $J_2 = (a, b_2)$ nice interval, such that $\delta := |b_2^- - f(c_-)| < \gamma$. We will define the perturbation of f (Figure 2.4)

$$\tilde{f}(x) = \begin{cases} f_0(x) + \varphi_\delta(x) & \text{if } x < c, \\ f_1(x) & \text{if } x > c. \end{cases}$$

Where

$$\varphi_\delta(x) = \delta\varphi\left(\frac{10(x-c)+(c-a)}{9(c-a)}\right) = \begin{cases} 0 & \text{if } x \le a, \\ y \in (0,\delta) & \text{if } a < x < c - |c-a|/10 \\ \delta & \text{if } c - |c-a|/10 \le x \le c. \end{cases}$$

Note that f and $\tilde{f}$ are equal outside the interval (a, c) and $\tilde{f}(c_-) = f(c_-) + \varphi_\delta(c) = f(c_-) + \delta = b_2^-$, therefore $c \notin \omega_{\tilde{f}}(\tilde{f}(c_-))$. In addition, $d(f, \tilde{f}) < \epsilon$. Indeed,

$$\begin{aligned} |f(x) - \tilde{f}(x)| &= |\varphi_\delta(x)| \le \delta < \gamma \le \epsilon \\ |f'(x) - \tilde{f}'(x)| &= |\varphi_\delta'(x)| = \delta|\varphi'(y(x))|\frac{10}{9(c-a)} < \gamma M \frac{10}{9(c-a)} < \epsilon \\ |f''(x) - \tilde{f}''(x)| &= |\varphi_\delta''(x)| = \delta|\varphi''(y(x))|\left(\frac{10}{9(c-a)}\right)^2 < \epsilon. \end{aligned}$$

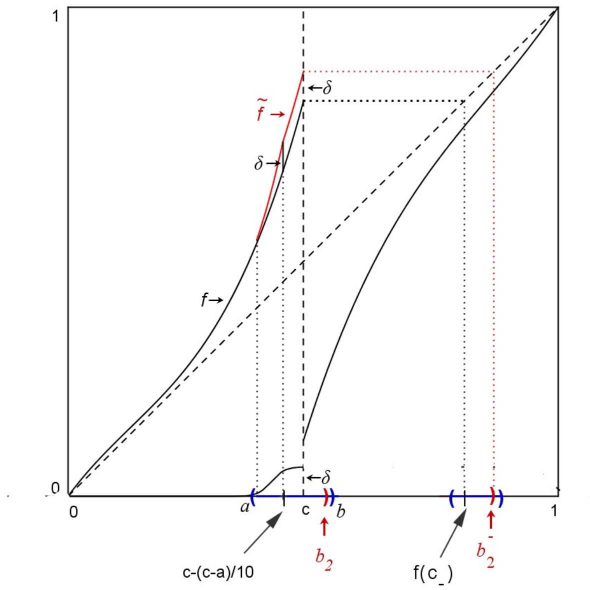


Figure 2.4: Nice Interval and pre image containing $f(c_-)$ and $\tilde{f}$ perturbation of f.

Now we make the same procedure for $f(c_+)$, setting the interval (c, b_2), getting $a_2 \in (a, c)$ with $J_3 = (a_2, b_2)$ nice interval, and the perturbation at branch f_1 pushing $f(c_+)$ to the left boundary of a gap which $f(c_+)$ belongs to.

□

Corollary 2.1.3. *For any $f \in \mathcal{D}_0$, there are $M_f > 0$ and $n_f \geq 1$ such that*

$$\frac{1}{n}\sum_{j=1}^{n} |\log|f^j(c_\pm) - c|| < M_f, \forall n \geq n_f.$$

Consequently,

$$\limsup_n \frac{1}{n}\sum_{j=1}^{n} |\log|f^j(c_\pm) - c|| < M_f$$

Moreover, $\mathcal{D}_0 \ni f \mapsto M_f$ is a continuous map.

Proof. Let $f \in \mathcal{D}_0$, then there exists $k_f > 1$ so that $f^j(c_-) \in \mathcal{O}_f^+(p_f^-)$ and $f^j(c_+) \in \mathcal{O}_f^+(p_f^+)$ for all $j \geq k_f$, for some $p_f^-, p_f^+ \in Per(f)$. Take $\alpha = \text{dist}(\mathcal{O}_f^+(p_f^\pm), c) > 0$ and $M_f = 2|\log \alpha| > 0$. So, for $n > k_f$,

$$\begin{aligned}
\sum_{j=1}^{n} |\log|f^j(c_\pm) - c|| &= \sum_{j=1}^{k_f-1} |\log|f^j(c_\pm) - c|| + \sum_{j=k_f}^{n} |\log|f^j(c_\pm) - c|| \\
&< \sum_{j=1}^{k_f-1} |\log|f^j(c_\pm) - c|| + (n - k_f + 1)|\log \alpha|
\end{aligned}$$

then choosing n_f the smallest integer bigger than k_f such that

$$\frac{1}{n_f}\sum_{j=1}^{k_f-1}|\log|f^j(c_\pm)-c||<|\log\alpha|,$$

the result follows.

□

Proof of Proposition 2.0.5.

Proof. For $f_0\in\mathcal{D}_0$ let M_{f_0} and n_{f_0} satisfying the Corollary 2.1.3. By continuity of the hyperbolic periodic point $p^{\pm}_{f_0}$ there is an open set $\mathcal{V}=\mathcal{V}_{f_0}\subset\mathcal{L}$ so that we can assume without loss of generality, that for all $f\in\mathcal{V}\cap\mathcal{D}_0$ we have

$$\limsup_n \frac{1}{n}\sum_{j=1}^{n}|\log|f^j(c_\pm)-c||<\gamma$$

where γ does not depend on f only of $\mathcal{V}$. We will construct a residual subset in $\mathcal{V}$ with the required property and so follows the result.

Fix $0<\delta<1$ and for each $f\in\mathcal{V}\cap\mathcal{D}_0$ and $n\in\mathbb{N}$ we defined the open sets $A_n(f)$ to be the set of maps $g\in\mathcal{V}$ so that, there is $m_f(n)\in\mathbb{N}$ with $\dfrac{n_f}{m_f(n)}<10^{-n}$ and $|f^j(c_\pm)-g^j(c_\pm)|<\delta|f^j(c_\pm)-c|$ for all $1\le j\le m_f(n)$. So, for $g\in A_n(f)$

$$\begin{aligned}|f^j(c_\pm)-c| &\le |f^j(c_\pm)-g^j(c_\pm)|+|g^j(c_\pm)-c|\Rightarrow\\ (1-\delta)|f^j(c_\pm)-c| &\le |g^j(c_\pm)-c|,\end{aligned}$$

for all $1\le j\le m_f(n)$, and then for each $n_f\le k\le m_f(n)$,

$$\begin{aligned}\sum_{j=1}^{k}|\log|g^j(c_\pm)-c|| &\le k\log(1-\delta)^{-1}+\sum_{j=1}^{k}|\log|f^j(c_\pm)-c||\\ &< k\log(1-\delta)^{-1}+k\gamma,\end{aligned}$$

therefore

$$\frac{1}{k}\sum_{j=1}^{k}|\log|g^j(c_\pm)-c|| \;<\; \log(1-\delta)^{-1}+\gamma=\Gamma. \tag{2.2}$$

Take $\mathcal{V}_n=\cup_{f\in(\mathcal{D}_0\cap\mathcal{V})}A_n(f)$ the open and dense subsets of $\mathcal{V}$. Therefore $\mathcal{R}_{f_0}=\cap_{n\in\mathbb{N}}\mathcal{V}_n$ is a residual subset of $\mathcal{V}=\mathcal{V}_{f_0}$ and for each $g\in\mathcal{R}_{f_0}$ we have

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|\log|g^j(c_\pm)-c||<\Gamma \tag{2.3}$$

The set $\mathcal{R}_0=\cup_{f_0\in\mathcal{D}_0}\mathcal{R}_{f_0}$ is a Baire generic subset and satisfies the Proposition. □

Note that the estimate in (2.3) cannot be guaranteed for lim sup, since as the neighborhoods of f depend on n_f, they may not be the same for any f in $\mathcal{D}_0$.

For example, if $g \in \mathcal{R}_0$, and for $n_2 > n_1$, $g \in A_{n_1}(f)$ and $g \in A_{n_2}(\overline{f})$ with $f \neq \overline{f}$, can happen from $m_f(n_1) \ll n_{\overline{f}}$, so, (2.2) is satisfied for $n_f \leq k \leq m_f(n_1)$ and $n_{\overline{f}} \leq k \leq m_{\overline{f}}(n_2)$, however we cannot guarantee the same for $m_f(n_1) \leq k \leq n_{\overline{f}}$.

2.1.2 Construction of set $\mathcal{R}_1$

Lemma 2.1.4. *(Density of periodic maps) There exists a dense subset $\mathcal{D}_1$ of $\mathcal{L}$ such that each $f \in \mathcal{D}_1$ is a periodic map.*

Proof. Let $f \in \mathcal{L}$. If $f^j(c_-) = c$ and $f^k(c_+) = c$ for some $j, k > 1$, we have nothing to do.

Suppose that $f^j(c_-) \neq c$, $\forall j \geq 1$. We can have two situations: either $c \in \omega_f(f(c_-))$ or $f(c_-) \in \Lambda_J$ for some nice interval J.

In the case $c \in \omega_f(f(c_-))$, given $\epsilon > 0$ arbitrary, we know by Lemma 1.1.18, that we can take a nice interval $J = (a, b)$ of arbitrarily small size to have $1/\lambda^{S(P_J^-)} < \epsilon$ and with $f(c_-) < q$, where q is the point of P_J^- such that $f^{S(P_J^-)}(q) = c$ (case $f(c_-) > q$, just fix b and take $a_n \nearrow c$ according to the Proposition 1.1.16 , which will cause q to be pushed up to reach $f(c_-)$ or transcend that point). We perturb the map f in the interval (a, c), pushing $f(c_-)$ to the point q, so that for the obtained map g, we have $g(c_-) = q$ and then $g^{S(P_J^-)}(g(c_-)) = g^{S(P_J^-)}(q) = f^{S(P_J^-)}(q) = c$, since f and g coincide outside the interval (a, c) and $S(P_J^-)$ is the first instant that the point q enters J by the map f.

Now, considering the situation $f(c_-) \in \Lambda_J$ for some nice interval J. As $c \in \omega_f(x)$ for *Leb*-almost every $x \in [0, 1]$ we have to given $\epsilon > 0$ arbitrary, there exists $q \in (f(c_-), f(c_-) + \epsilon)$ so that $f^n(q) = c$ for some $n \geq 1$ and then proceed as before. Note that this same reasoning could also be used in the previous case.

For the case where $f^k(c_+) \neq c, \forall k$, we proceed with the same reasoning perturb the map f in the interval (c, b).

□

Proof of the Proposition 2.0.6.

Proof. According to the previous lemma for each $f \in \mathcal{D}_1$, there are $\ell_f^{\pm}$ such that $f^{\ell_f^{\pm}}(c_{\pm}) = c$. Take $\ell_f = \max\{\ell_f^+, \ell_f^-\}$ and $m_f(n) = \max\{\ell_f, n\}$, set

$$B_n(f) = \{g \in \mathcal{L}; |g^j(c_{\pm}) - f^j(c_{\pm})| < e^{-n^2 \ell_f} \text{ for all } 1 \leq j \leq m_f(n)\}$$

so, if $g \in B_n(f)$ we obtain

$$\begin{aligned} |g^j(c_{\pm}) - c| &\leq |g^j(c_{\pm}) - f^j(c_{\pm})| + |f^j(c_{\pm}) - c| \\ &< e^{-n^2 \ell_f} + |f^j(c_{\pm}) - c| \text{ for all } 1 \leq j \leq m_f(n). \end{aligned}$$

Thus

$$|\log|g^j(c_\pm)-c|| \quad > \quad |\log(e^{-n^2\ell_f}+|f^j(c_\pm)-c|)|$$

for all $\quad 1 \leq j \leq m_f(n)$, and

$$\begin{aligned}
\sum_{j=1}^{m_f(n)} |\log|g^j(c_\pm)-c|| \quad &> \quad \sum_{j=1}^{m_f(n)} |\log(e^{-n^2\ell_f}+|f^j(c_\pm)-c|)| \\
&= \quad \sum_{\substack{1\leq j\leq m_f(n)\\ j\neq k\ell_f^\pm}} |\log(e^{-n^2\ell_f}+|f^j(c_\pm)-c|)| + \left[\frac{m_f(n)}{\ell_f^\pm}\right]|\log(e^{-n^2\ell_f})| \\
&> \quad |\log(e^{-n^2\ell_f})| = n^2\ell_f,
\end{aligned}$$

consequently

$$\frac{1}{m_f(n)}\sum_{j=1}^{m_f(n)} |\log|g^j(c_\pm)-c|| \quad > \quad \frac{1}{m_f(n)}n^2\ell_f > n.$$

As $B_n(f)$ is an open neighborhood of $f \in \mathcal{D}_1$, $\mathcal{U}_n = \cup_{f\in\mathcal{D}_1} B_n(f)$ is an open and dense subset of $\mathcal{L}$. Therefore $\mathcal{R}_1 = \cap_{n\in\mathbb{N}}\mathcal{U}_n$ is a residual subset of $\mathcal{L}$ and for each $g \in \mathcal{R}_1$ we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|\log|g^j(c_\pm)-c|| = +\infty.$$

$\square$

Proof of Theorem C. The proof of Theorem C follows from the intersection of the residuals $\mathcal{R}_0$ and $\mathcal{R}_1$ constructed in the Propositions 2.0.5 and 2.0.6.

Chapter 3

Proof of Theorem B

We will now work to obtain the proof of Theorem B. For this we need the following definitions.

Recall f is a C^2 local diffeomorphism in which for our main result we only need to consider $f' > 0$. Without loss of generality we will assume $c = 0$, and $f : [-1, 1] \setminus \{0\} \to [-1, 1]$, which will facilitate the computations, and then it would be enough to make a change of coordinates.

Definition 3.0.1. *(finite Lyapunov exponent) We say that a measure μ f-invariant ergodic probability has finite Lyapunov exponent if*

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \neq \pm\infty \tag{3.1}$$

for μ almost every x.

Definition 3.0.2. *(Slow recurrence) A set $\Lambda \subset [-1, 1]$ satisfies the slow recurrence condition to c if for each $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), c) \leq \epsilon \tag{3.2}$$

for every $x \in \Lambda$, where $\operatorname{dist}_\delta(x, c)$ denotes the δ-truncated distance from x to c defined as

$$\operatorname{dist}_\delta(x, c) = \begin{cases} \operatorname{dist}(x, c) & \textit{if} \ \ \operatorname{dist}(x, c) < \delta \\ 1 & \textit{if} \ \ \textit{otherwise.} \end{cases}$$

An ergodic f-invariant probability μ satisfies slow recurrent condition if there is a set Λ satisfying (3.2) *such that $\mu(\Lambda) = 1$.*

The fact that f is non-flat Lorenz map gives us, by Definition 1.1.2 and Remark 3.0.3 below, that there are C^1 diffeomorphisms $\varphi_0 : [-1, 0] \to [f'(-1), d_0]$ and $\varphi_1 : [0, 1] \to [d_1, f'(1)]$ such that

$$f'(x) = \begin{cases} \varphi_0(x)/|x|^{\alpha_0} & \text{if } x < 0 \\ \varphi_1(x)/|x|^{\alpha_1} & \text{if } x > 0 \end{cases} \tag{3.3}$$

where $\alpha_0, \alpha_1 \in (0,1)$ and $d_0 = \lim_{x\uparrow 0} f'(x)|x|^{\alpha_0}, d_1 = \lim_{x\downarrow 0} f'(x)|x|^{\alpha_1}$.

Remark 3.0.3. *As seen in Chapter 1, $f(x) = f(0_-) - (\phi_0(-x))^\alpha$ if $x < 0$, with $\alpha \in [0,1]$ and ϕ_0 being a C^2 diffeomorphism. Then*

$$|f'(x)| = \alpha \frac{|\phi_0'(-x)|}{|\phi_0(-x)|^{1-\alpha}} = \frac{1}{|x|^{1-\alpha}} \left(\alpha \frac{|\phi_0'(-x)||x|^{1-\alpha}}{|\phi_0(-x)|^{1-\alpha}} \right) = \frac{\varphi_0(x)}{|x|^{\alpha_0}},$$

where $\alpha_0 = 1-\alpha$ and φ_0 is C^1 diffeomorphism. For $x > 0$ it follows the same way.

Thus, from (3.3) we can take $a > 1$ and $0 < \alpha \le \beta < 1$ so that

$$\frac{1}{a}|x|^{-\alpha} \le f'(x) \le a|x|^{-\beta} \tag{3.4}$$

As $f' > 0$, we have $\inf \varphi_i > 0$, $i = 0, 1$. So, $\log|\varphi_i|$ is Lipschitz. Let $C > 0$ be so that

$$|\log|\varphi_i|(x) - \log|\varphi_i|(y)| \le C|x-y|, \ \forall i,$$

and so

$$\frac{|\varphi_i(x)|}{|\varphi_i(y)|} \le e^{C|x-y|} \tag{3.5}$$

Remark 3.0.4. *For all $x \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R}$ and $r > 0$, we have*

$$\begin{aligned} \left|\frac{y}{x}\right|^r &= \left|\frac{y}{x} - 1 + 1\right|^r = \left|\frac{y-x}{x} + 1\right|^r \le \left(\left|\frac{y-x}{x}\right| + 1\right)^r = \left(e^{\log\left(\left|\frac{y-x}{x}\right|+1\right)}\right)^r \\ &\le \left(e^{\frac{|y-x|}{|x|}}\right)^r = e^{r|y-x|/|x|} \end{aligned} \tag{3.6}$$

Remark 3.0.5. *If $xy > 0$ then*

$$\begin{aligned} |x-y| < \frac{|x|}{2} \ &\Leftrightarrow \ \left|1 - \frac{y}{x}\right| < \frac{1}{2} \Leftrightarrow \frac{1}{2} < \frac{y}{x} < \frac{3}{2} \\ &\Leftrightarrow \ \frac{1}{2} < \frac{|y|}{|x|} < \frac{3}{2} \Rightarrow \frac{1}{|y|} < \frac{2}{|x|}. \end{aligned} \tag{3.7}$$

From (3.5), (3.6), (3.7) we obtain a certain control of the derivatives of f in each branch of its domain. For this, consider $\alpha = \alpha_i, \varphi = \varphi_i$ e $d = d_i$ for each case when $x < 0$ or $x > 0$. Thus, $f'(x) = \varphi(x)/|x|^\alpha$ and $\varphi(0) = d > 0$.

Lemma 3.0.6. *There is $\gamma > 0$ such that for every $n \ge 1$ and every x, y so that $|f^j(x) - f^j(y)| \le \frac{|f^j(x)|}{2}$ and $f^j(x)f^j(y) > 0$ for every $0 \le j < n$, we have*

$$\frac{|(f^n)'(y)|}{|(f^n)'(x)|} \le e^{\gamma \sum_{j=0}^{n-1} \frac{|f^j(x) - f^j(y)|}{|f^j(x)|}} \tag{3.8}$$

Proof. Let $x, y \in [-1,1] \setminus \{0\}$ such that $xy > 0$ and $|x-y| \leq \frac{|x|}{2}$.Then

$$\begin{aligned}
\frac{|f'(y)|}{|f'(x)|} = \frac{|\varphi(y)|}{|\varphi(x)|}\frac{|x|^\alpha}{|y|^\alpha} &\underset{(3.5)}{\leq} e^{C|y-x|}\left|\frac{x}{y}\right|^\alpha \\
&\underset{|x|<1}{\leq} e^{C\frac{|y-x|}{|x|}}\left|\frac{x}{y}\right|^\alpha \\
&\underset{(3.6)}{\leq} e^{C\frac{|y-x|}{|x|}} e^{\alpha\frac{|x-y|}{|y|}} \\
&\underset{(3.7)}{\leq} e^{C\frac{|y-x|}{|x|}} e^{2\alpha\frac{|x-y|}{|x|}} \\
&= e^{(C+2\alpha)\frac{|x-y|}{|x|}}
\end{aligned}$$

Thus, taking $\gamma = C + 2\max\{\alpha_0, \alpha_1\}$, for every $x, y \in [-1,1]$ such that $xy > 0$ and $|x-y| \leq \frac{|x|}{2}$, we get

$$\frac{|f'(y)|}{|f'(x)|} \leq e^{\gamma\frac{|x-y|}{|x|}}. \tag{3.9}$$

Consequently, using (3.9), for every $n \geq 1$ and every x, y such that $|f^j(x) - f^j(y)| \leq \frac{|f^j(x)|}{2}$ and $f^j(x)f^j(y) > 0$ for every $0 \leq j < n$, we have

$$\frac{|(f^n)'(y)|}{|(f^n)'(x)|} = \prod_{j=0}^{n-1}\frac{|f'(f^j(y))|}{|f'(f^j(x))|} \leq \prod_{j=0}^{n-1} e^{\gamma\frac{|f^j(x)-f^j(y)|}{|f^j(x)|}} = e^{\gamma\sum_{j=0}^{n-1}\frac{|f^j(x)-f^j(y)|}{|f^j(x)|}}.$$

□

Now for a point $p \neq 0$ we will define what we will call the *bound period* of the orbit of p with the orbit of $c = 0$.

Definition 3.0.7. *Fix $0 < \delta \leq \frac{1}{2}$ and for any $p \neq 0$, we will call δ-bound period (or simply bound period) of p with 0 the number*

$$m(p) := -1 + \min\{j > 0; |f^j(p) - f^j(0_p)| \geq \delta|f^j(0_p)|\},$$

where

$$0_p = \begin{cases} 0_- & \textit{if} \;\; p < 0 \\ 0_+ & \textit{if} \;\; p > 0. \end{cases}$$

This means that for every $1 \leq j \leq m(p)$, we have, either $f^j(p)$ and $f^j(0_p) < 0$ or $f^j(p)$ and $f^j(0_p) > 0$, i.e.,

$$\frac{|f^j(p) - f^j(0_p)|}{|f^j(0_p)|} < \delta \;\; \text{and} \;\; f^j(p)f^j(0_p) > 0.$$

Consequently,

$$\sum_{i=1}^{j}\frac{|f^i(p)-f^i(0_p)|}{|f^i(0_p)|}<\delta j \text{ for every } 1\leq j\leq m(p),$$

and then of (3.8) for $x=f(0_p)$ and $y=f(p)$ we have

$$\frac{|(f^j)'(f(p))|}{|(f^j)'(f(0_p))|}\leq e^{\gamma\sum_{i=0}^{j-1}\frac{|f^i(f(p))-f^i(f(0_p))|}{|f^i(f(0_p))|}}<e^{\gamma\delta j} \tag{3.10}$$

for every $1\leq j\leq m(p)$.

Also, as $f^{m(p)+1}\mid_{(0,p)}$ if $p>0$ or $f^{m(p)+1}\mid_{(p,0)}$ if $p<0$ is diffeomorphism, for every $y\in\{tp+(1-t)(0_p);t\in[0,1]\}$ and $1\leq j\leq m(p)$ we have

$$|(f^j)'(f(y))|<e^{\gamma\delta j}|(f^j)'(f(0_p))|. \tag{3.11}$$

Lemma 3.0.8. *If there is $M>0$ such that*

$$M=\sup_n\frac{1}{n}\sum_{j=1}^{n}|\log|f^j(0_\pm)|| \tag{3.12}$$

then for each n, we have:

(*i*) $\prod_{j=1}^{n}|f^j(0_\pm)|\geq e^{-Mn}$ *in particular* $|f^j(0_\pm)|\geq e^{-Mn},\ 1\leq j\leq n$

(*ii*) $|(f^n)'(f(0_\pm))|\leq e^{bn}$ *where* $b:=\log a+\beta M$

and

$$|(f^j)'(f(0_\pm))|\leq e^{bn},\ 1\leq j\leq n.$$

Proof. In fact, as $M=\sup_n\frac{1}{n}\sum_{j=1}^{n}|\log|f^j(0_\pm)||$, we have for each n,

$$\begin{aligned} M &\geq \frac{1}{n}\sum_{j=1}^{n}|\log|f^j(0_\pm)|| \\ M &\geq -\frac{1}{n}\log\prod_{j=1}^{n}|f^j(0_\pm)|| \\ \Rightarrow & \quad \prod_{j=1}^{n}|f^j(0_\pm)|\geq e^{-Mn} \end{aligned}$$

and as, $|f^j(0_\pm)|<1$ for all j, in particular we obtain, $|f^j(0_\pm)|\geq e^{-Mn},\ 1\leq j\leq n$ establishing the item (i). Thus, using the condition (3.4) of f non-flat,

$$\begin{aligned} |(f^n)'(f(0_\pm))| &= \prod_{j=1}^{n}|f'(f^j(0_\pm))|\leq\prod_{j=1}^{n}a|f^j(0_\pm)|^{-\beta} \\ &\leq a^n\prod_{j=1}^{n}|f^j(0_\pm)|^{-\beta}\underset{(i)}{\leq}a^ne^{\beta Mn} \\ &= e^{bn} \end{aligned}$$

where $b = \log a + \beta M$. So, we also have $|(f^j)'(f(0_\pm))| \leq e^{bn},\ 1 \leq j \leq n$. $\square$

The next result provides us with a relation between the *depth* at which a point is and the *bound period* in which its orbit is linked to the singular orbit. Intuitively, it means that when a point is close to the singularity, we have an estimate for the bound period with the orbit of the singular point. Thus, supposing that there is $M > 0$ so that (3.12) is true, we have the following consequences.

Corollary 3.0.9. *Let $0 < A \leq \delta\frac{1-\beta}{a^2}$ and $B \geq 2\gamma\delta + 2M + b$. If $|p|^{1-\beta} \leq Ae^{-Bn}$, with $n \geq 1$, then $m(p) \geq n$.*

Proof. Initially, note that

$$|f(p) - f(0_p)| = \left|\int_{0_p}^{p} f'(x)dx\right| \leq \int_{0_p}^{p} |f'(x)|dx \leq a\int_{0}^{p} |x|^{-\beta}dx = \frac{a}{1-\beta}|p|^{1-\beta}$$

and by the mean value theorem and the definition of $m(p)$

$$|f^j(p) - f^j(0_p)| = |(f^{j-1})'(y)||f(p) - f(0_p)|,\ y \in (f(0_p), f(p))$$

for each $1 < j \leq m(p) + 1$.

Then, it follows from (3.11) that

$$\begin{aligned}
|f^j(p) - f^j(0_p)| &\leq e^{\gamma\delta(j-1)} \prod_{i=1}^{j-1} |f'(f^i(0_p))|\ |f(p) - f(0_p)| \\
&\leq e^{\gamma\delta(j-1)} \prod_{i=1}^{j-1} |f'(f^i(0_p))|\ \frac{a}{1-\beta}|p|^{1-\beta} \\
&\leq e^{\gamma\delta(j-1)} \prod_{i=1}^{j-1} |f'(f^i(0_p))|\ \frac{a}{1-\beta}Ae^{-Bn} \\
&\leq e^{\gamma\delta(j-1)} \prod_{i=1}^{j-1} |f'(f^i(0_p))|\ \frac{\delta}{a}e^{-Bn},
\end{aligned}$$

for every $j \leq m(p) + 1$.

Suppose that $m(p) < n$. Thus, using item (ii) of the Lemma 3.0.8 we have

$$|f^{m(p)}(p) - f^{m(p)}(0_p)| \leq \frac{\delta}{a}e^{\gamma\delta(m(p))}e^{-Bn}e^{bn} < \frac{\delta}{a}e^{(\gamma\delta - B + b)n}. \tag{3.13}$$

As $|f'(y)|/|f'(f^{m(p)}(0_p))| < e^{\gamma\delta}$ for each $y \in \{tf^{m(p)}(p) + (1-t)f^{m(p)}(0_p); t \in [0,1]\}$, by (3.11) we obtain that

$$\begin{aligned}
\frac{|f^{m(p)+1}(p) - f^{m(p)+1}(0_p)|}{|f^{m(p)+1}(0_p)|} &= \left| \int_{f^{m(p)}(0_p)}^{f^{m(p)}(p)} \frac{|f'(y)|}{|f'(f^{m(p)}(0_p))|} dy \right| \frac{|f'(f^{m(p)}(0_p))|}{|f^{m(p)+1}(0_p)|} \\
&\leq e^{\gamma\delta} |f^{m(p)}(p) - f^{m(p)}(0_p)| \frac{|f'(f^{m(p)}(0_p))|}{|f^{m(p)+1}(0_p)|} \\
&\underset{(3.13)}{<} e^{\gamma\delta} \frac{\delta}{a} e^{(\gamma\delta - B + b)n} a \frac{|f^{m(p)}(0_p)|^{-\beta}}{|f^{m(p)+1}(0_p)|} \\
&\underset{Lemma\ 3.0.8(i)}{\leq} \delta e^{\gamma\delta} e^{(\gamma\delta - B + b)n} e^{\beta M n} e^{M n} \\
&< \delta e^{(2\gamma\delta + b + 2M - B)n} < \delta.
\end{aligned}$$

But this contradicts the definition of $m(p)$. Therefore $m(p) \geq n$. □

Corollary 3.0.10. *If $n \geq 1$ is such that $Ae^{-B(n+1)} < |p|^{1-\beta} \leq Ae^{-Bn}$ then*

$$\sum_{j=0}^{m(p)} |\log |f^j(p)|| < \Gamma m(p) \tag{3.14}$$

where

$$\Gamma = \frac{|\log A|}{1-\beta} + 2\frac{B}{1-\beta} + \log\left(\frac{1}{1-\delta}\right) + M.$$

Proof. As

$$|f^j(p)| > |f^j(0_p)| - |f^j(p) - f^j(0_p)| > (1-\delta)|f^j(0_p)|, \ \forall 1 \leq j \leq m(p),$$

it follows that

$$|\log |f^j(p)|| < \log\left(\frac{1}{1-\delta}\right) + |\log |f^j(0_p)||, \ \forall 1 \leq j \leq m(p). \tag{3.15}$$

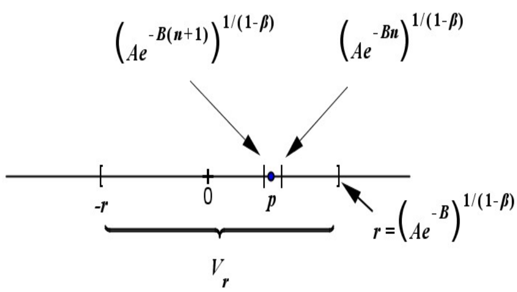


Figure 3.1: Neighborhood V_r of 0.

We know from the previous Corollary that $m(p) \geq n$. Now from $Ae^{-B(n+1)} < |p|^{1-\beta}$, we have

$$\begin{aligned}
\log A + (-B)(n+1) < (1-\beta)\log|p| \;&\Rightarrow\; \left|\frac{-\log A}{(1-\beta)} + \frac{B(n+1)}{(1-\beta)}\right| > |-\log|p|| \\
&\Rightarrow\; \frac{|\log A|}{(1-\beta)} + \frac{B(n+1)}{(1-\beta)} > |\log|p||. \qquad (3.16)
\end{aligned}$$

As a consequence of (3.15) and (3.16) it follows that

$$\begin{aligned}
\sum_{j=0}^{m(p)} |\log|f^j(p)|| &\leq |\log|p|| + \sum_{j=1}^{m(p)} |\log|f^j(p)|| \\
&< |\log|p|| + \sum_{j=1}^{m(p)} \left[\log\left(\frac{1}{1-\delta}\right) + |\log|f^j(0_p)||\right] \\
&< \frac{|\log A|}{(1-\beta)} + \frac{B(n+1)}{(1-\beta)} + m(p)\log\left(\frac{1}{1-\delta}\right) + \sum_{j=1}^{m(p)} |\log|f^j(0_p)|| \\
&\underset{(3.12)}{<} \frac{|\log A|}{(1-\beta)} + (n+1)\frac{B}{(1-\beta)} + m(p).\log\left(\frac{1}{1-\delta}\right) + Mm(p) \\
&< \frac{|\log A|}{(1-\beta)} + 2m(p)\frac{B}{(1-\beta)} + m(p)\log\left(\frac{1}{1-\delta}\right) + Mm(p) \\
&< \left[\frac{|\log A|}{(1-\beta)} + 2\frac{B}{(1-\beta)} + \log\left(\frac{1}{1-\delta}\right) + M\right] m(p).
\end{aligned}$$

Therefore,

$$\sum_{j=0}^{m(p)} |\log|f^j(p)|| < \Gamma m(p).$$

□

The goal here is to obtain a control for the recurrence of points to the singularity. Note that by this last Corollary, whenever a point is in the neighborhood of the singularity $c = 0$ we obtain a control to its orbit until a certain time, the bound period. And so, we are going to take a V_r neighborhood of 0 for that purpose, as in the Fig 3.1.

Proof of Theorem B.

Remember that we are considering $c = 0$.

Suppose that $\limsup_n \frac{1}{n}\sum_{j=1}^{n} -\log|f^j(0_\pm)| < +\infty$ then it means that there is $M > 0$ so that (3.12) happens, that is,

$$M = \sup_n \frac{1}{n}\sum_{j=1}^{n} |\log|f^j(0_\pm)||$$

and then the results Lemma 3.0.8, Corollary 3.0.9 and Corollary 3.0.10 can be used.

Let $r = (Ae^{-B})^{\frac{1}{1-\beta}}$ and $U = \{x \in [-1,1] \setminus \mathcal{O}_f^-(0); 0 \in \omega_f(x)\}$ and let $R : U \to \{0, 1, 2, \cdots\}$ be the map that gives the first entry time to $V_r = [-r, r]$, i.e., $R(x) = \min\{0 \leq j; |f^j(x)| \leq r\}$.

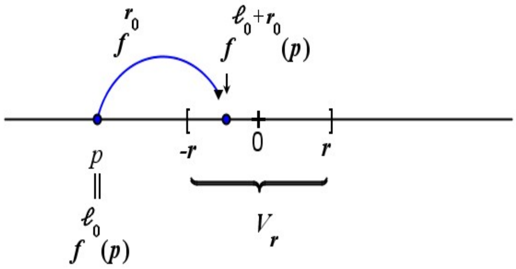


Figure 3.2: First entry of p in neighborhood V_r.

Given a point $p \in U$, let $\ell_0 = 0$, $r_0 = R(f^{\ell_0}(p))$ and $n_0 = m(f^{\ell_0+r_0}(p))$.

Note that even before the entry time in V_r, we have the following estimate for the orbit of the point p

$$\sum_{j=0}^{r_0-1} |\log|f^j(p)|| \leq r_0 \log \frac{1}{r}$$

and for $p_0 = f^{r_0}(p) = f^{\ell_0+r_0}(p)$ we know from (3.14), that until the bound period n_0 we have

$$\sum_{j=0}^{n_0} |\log|f^j(p_0)|| = \sum_{j=r_0}^{r_0+n_0} |\log|f^j(p)|| < \Gamma n_0.$$

Then

$$\begin{aligned}
\sum_{j=0}^{\ell_0+r_0+n_0} |\log|f^j(p)|| &= \sum_{j=0}^{r_0-1} |\log|f^j(p)|| + \sum_{j=r_0}^{\ell_0+r_0+n_0} |\log|f^j(p)|| \\
&\leq r_0 \log(1/r) + \Gamma n_0 \\
&\leq (\ell_0 + r_0 + n_0)\left(\log(1/r) + \Gamma\right) \\
&= (\ell_1)\left(\log(1/r) + \Gamma\right).
\end{aligned}$$

Inductively, let $\ell_{j+1} = \ell_j + r_j + n_j$, $r_{j+1} = R(f^{\ell_{j+1}}(p))$ and $n_{j+1} = m(f^{\ell_{j+1}+r_{j+1}}(p))$. Besides that, we have

$$\begin{aligned}
\sum_{j=0}^{\ell_k} |\log|f^j(p)|| &= \sum_{j=0}^{k}\sum_{i=\ell_j}^{\ell_{j+1}-1} |\log|f^i(p)|| \\
&= \sum_{j=0}^{k}\left(\sum_{i=0}^{r_j-1} |\log|f^i(f^{\ell_j}(p)|| + \sum_{i=0}^{n_j} |\log|f^i(f^{\ell_j+r_j}(p)||\right) \\
&\leq \sum_{j=0}^{k}\left(\sum_{i=0}^{r_j-1} \log(1/r) + n_j\Gamma\right) \\
&= \sum_{j=0}^{k} \left(r_j \log(1/r) + n_j\Gamma\right) \\
&\leq (\ell_k+1)\left(\log(1/r)+\Gamma\right).
\end{aligned}$$

Therefore,

$$\liminf \frac{1}{n}\sum_{j=0}^{n-1} |\log|f^j(p)|| < \log(1/r) + \Gamma = \Upsilon, \tag{3.17}$$

for every $p \in [-1,1] \setminus \mathcal{O}_f^-(0)$ (It is clear that the equation (3.17) also applies to points p such that $0 \notin \omega_f(p)$). Thus, by ergodic theorem of Birkhoff, for every ergodic f-invariant probability μ, we have

$$\int_{x\in[-1,1]} |\log|x||d\mu < \Upsilon. \tag{3.18}$$

This means that the logarithm of the distance to the singular point $c = 0$ is μ integrable. Consequently,

$$\int \log \operatorname{dist}_{\epsilon^{-n}}(x,c)d\mu \underset{\delta-trunc}{=} \int_{[x;\log\operatorname{dist}(x,0)<-n]} \log|x|d\mu \to 0$$

when $n \to \infty$. And so, by Birkhoff, μ satisfies the condition of slow recurrence.

Now, as f is non-flat, i.e., $\frac{1}{a}|x|^{-\alpha} \leq f'(x) \leq a|x|^{-\beta}$, we get

$$\log\left(\frac{1}{a}\right) + \alpha|\log|x|| \leq \log f'(x) \leq \log a + \beta|\log|x|| \tag{3.19}$$

and then for every $\mu \in \mathcal{M}^1(f)$, we have

$$\log\left(\frac{1}{a}\right) + \alpha\int_{x\in[-1,1]} |\log|x||d\mu \leq \int_{x\in[-1,1]} \log f' d\mu \leq \log a + \beta\int_{x\in[-1,1]} |\log|x||d\mu$$

Using (3.18), it follows

$$0 < \int \log f' d\mu < \log a + \beta\Upsilon.$$

Therefore, again by Birkhoff's theorem, for every ergodic f-invariant probability μ, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| = \int \log f' d\mu \neq \pm\infty,$$

that is, μ has finite Lyapunov exponent.

Here we obtain a kind of converse of Theorem B.

Theorem 1. *Let f be a C^2 expanding Lorenz map and μ the SRB measure. If the singular values $f(0_\pm) \in B(\mu)$, then f satisfies the condition* (3)*.*

Proof. In fact, as $\mu \ll m$, it follows from the Ledrappier-Young entropy formula and the variational principle (see Theorem 4)

$$\int \log f' d\mu = h_\mu(f) \leq h_{top}(f) < \infty.$$

And as f is non-flat, we have by inequality (3.19) that $\int_{x\in[-1,1]} |\log |x|| d\mu < \infty$.
So, if $f(0_\pm) \in B(\mu)$ we have

$$\lim \frac{1}{n} \sum_{j=0}^{n-1} |\log |f^j(f(0_\pm))|| = \int_{x\in[-1,1]} |\log |x|| d\mu < \infty$$

and therefore f satisfies the condition (3). $\square$

Chapter 4

Proof of Theorem D

In this chapter we will prove Theorem D. To do this, we will start presenting some concepts that will be important to achieve our goal. The intention is construct a Cantor set, dynamically defined, using an Induced Markov map in hyperbolic times, which are also zooming times. This will allow us to have a distortion control that will be fundamental to calculate the Hausdorff dimension of this set.

Let M be a compact Riemannian manifold of dimension $d \geq 1$ and $f : M \to M$ a map defined on M. We say that f is a *non-flat* map if it is a local C^{1+} diffeomorphism in the whole manifold except in a *non-degenerate critical/singular set* $\mathcal{C} \subset M$. Which means that there are constants $\beta > 0$ and $B > 1$ such that the following two conditions hold.

$$\frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta} \leq \frac{\|Df(x)v\|}{\|v\|} \leq B\operatorname{dist}(x,\mathcal{C})^{-\beta} \text{ for all } v \in T_xM. \tag{C.1}$$

For every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ we have

$$|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|| \leq \frac{B}{\operatorname{dist}(x,\mathcal{C})^{\beta}}\operatorname{dist}(x,y). \tag{C.2}$$

The condition $(C.1)$ says that f behaves like a power of the distance to $\mathcal{C}$ and the condition $(C.2)$ that $\log\|Df^{-1}\|$ is locally Lipschitz at points $x \in M \setminus \mathcal{C}$, with Lipschitz constant depending on $\operatorname{dist}(x,\mathcal{C})$.

In general $\mathcal{C}$ is a set of points where the derivative fails to be invertible or simply does not exist. In the case we are working on is the singularity c, and for $dim(M) = 1$ the definition non-flatness seen in Chapter 1 also satisfies the definition given above, as we saw in Chapter 3.

4.1 Markov Map and Nested set

Definition 4.1.1. *We say that a point $x \in M$ has positive Lyapunov exponent if*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| > 0 \tag{4.1}$$

and say that μ has all of its positive Lyapunov exponents if (4.1) holds for μ-almost every point $x \in M$.

Definition 4.1.2. *A positively invariant set $\mathcal{H} \subset M$, i.e., $f(\mathcal{H}) \subset \mathcal{H}$, is called λ- expanding, $\lambda \geq 0$ if*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|(Df(f^i(x)))^{-1}\|^{-1} > \lambda \tag{4.2}$$

for every $x \in \mathcal{H}$, and $\mathcal{H}$ satisfies the slow approximation condition, i.e., for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{C}) \leq \epsilon \tag{4.3}$$

for every $x \in \mathcal{H}$ where $\operatorname{dist}_\delta(x, \mathcal{C})$ denotes the δ-truncated distance from x to $\mathcal{C}$ defined as

$$\operatorname{dist}_\delta(x, \mathcal{C}) = \begin{cases} \operatorname{dist}(x, \mathcal{C}) & \textit{if} \quad \operatorname{dist}(x, \mathcal{C}) < \delta \\ 1 & \textit{if} \quad \textit{otherwise.} \end{cases}$$

Remark 4.1.3. *Notice that in the one-dimensional case the condition (4.2) is equivalent to the Lyapunov exponent of f on x to be bigger than λ and it is clear that for the expanding Lorenz maps all points have positive Lyapunov exponent. Condition (4.3) is a generalization of the concept in Definition 3.0.2.*

The measure μ given by the Proposition 2.0.2 is absolutely continuous with respect to Lebesgue. Moreover, μ has finite Lyapunov exponent, according to Definition 3.0.1. Then, we have the following result that can be found in Pinheiro [Pi20].

Lemma 4.1.4. *([Pi20] Lemma B.2) Let M be a Riemannian manifold and $f : M \to M$ a C^{1+} non-flat map with singular set $\mathcal{C} \subset M$. If μ is a f-invariant ergodic probability with all of its Lyapunov exponents finite, then $x \mapsto \log \operatorname{dist}(x, \mathcal{C})$ and $x \mapsto \log \|(Df(x))^{-1}\|$ are μ-integrable. In particular, μ satisfies the slow approximation condition.*

According to Remark 4.1.3 and Lemma above we have then that μ satisfies conditions (4.2) and (4.3) of the Definition 4.1.2. So, Lebesgue almost every point in $[0, 1]$ has non-uniform expansion and slow recurrence.

4.1.1 Hyperbolic Times

Now we will see the idea of hyperbolic times that was introduced by Alves et al.[ABV], and that play a fundamental role in the study of the statistical properties of many classes of dynamical systems. Hyperbolic times for a point are iterations at which the backward contraction holds, implying uniformly bounded distortion on some small neighborhood of that point which will be very useful for us to calculate the Hausdorff dimension of a given set in Section 4.2.

The definitions and results presented here can be found in Pinheiro [Pi11], Alves [A] and Alves et al. [ABV], we will seek to state them succinctly just so that there is an understanding of the tools used.

Let $B > 1$ and $\beta > 0$ as in the definition of non-flat. Let us fix $0 < b = \dfrac{1}{3}\min\{1, 1/\beta\} < \dfrac{1}{2}\min\{1, 1/\beta\}$.

Definition 4.1.5. *Given $0 < \sigma < 1$ and $\epsilon > 0$, we say that n is a (σ, ϵ)-hyperbolic time for a point $x \in M$ if for all $1 \le k \le n$,*

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \text{ and } \operatorname{dist}_\epsilon(f^{n-k}(x), \mathcal{C}) \ge \sigma^{bk}. \tag{4.4}$$

We denote the set of points of M such that $n \in \mathbb{N}$ is a (σ, ϵ)-hyperbolic time by $H_n(\sigma, \epsilon, f)$.

As we have seen that Lebesgue almost all points satisfy the Definition 4.1.2 of a expanding set, we have by the proposition below that Lebesgue almost all points in $[0, 1]$ has infinitely many moments with positive frequency of hyperbolic times. In particular, they have infinitely many hyperbolic times.

Proposition 4.1.6. *([A] Proposition 2.12) If $f : M \to M$ is non-uniformly expanding on $\mathcal{H} \subset M$, given $\lambda > 0$ there exist $\theta > 0$ and $\epsilon_0 > 0$ such that, for every $x \in \mathcal{H}$ and $\epsilon \in (0, \epsilon_0]$,*

$$\#\{1 \le j \le n; x \in H_j(\sigma, \epsilon, f)\} \ge \theta n,$$

whenever $\frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\|^{-1} \ge \lambda$ and $\frac{1}{n}\sum_{i=0}^{n-1} - \log \operatorname{dist}_\epsilon(f^j(x), \mathcal{C}) \le \dfrac{\lambda}{16\beta}$, where $\sigma = e^{-\lambda/4}$.

In addition, one of the main characteristics of hyperbolic times, is that they are also zooming times, definition given by Pinheiro (see [Pi11]), which in general terms gives a geometric property for hyperbolic times. As can be seen in the proposition below.

Proposition 4.1.7. *([A] Proposition 2.3) Given $0 < \sigma < 1$ and $\epsilon > 0$ there exists $\delta > 0$ such that if n is a (σ, ϵ)-hyperbolic time for x, then there exists a neighborhood $V_n(x)$ of x such that:*

(1) *f^n maps $\overline{V_n(x)}$ diffeomorphically onto the ball $\overline{B_\delta(f^n(x))}$;*

(2) $\operatorname{dist}(f^{n-j}(p), f^{n-j}(q)) \leq \sigma^{j/2} \operatorname{dist}(f^n(p), f^n(q))$ *for all $1 \leq j < n$ and $p, q \in V_n(x)$.*

The sets $V_n(x)$ are called *hyperbolic pre-balls* and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, *hyperbolic balls.*

As stated above, our main objective of using hyperbolic times is to obtain distortion control. And as a consequence of the previous proposition we have uniformly bounded distortion on hyperbolic pre-balls, given by the next result.

Proposition 4.1.8. *(Bounded Distortion) There exists $C_0 > 0$ such that for every hyperbolic pre-ball V_n and every $p, q \in V_n$*

$$\log \frac{|\det(f^n)'(p)|}{|\det(f^n)'(q)|} \leq C_0 \operatorname{dist}(f^n(p), f^n(q)).$$

Corollary 4.1.9. *There exists $C_1 > 0$ such that for every hyperbolic pre-ball V_n and every $p, q \in V_n$*

$$\frac{1}{C_1} \leq \frac{|\det(f^n)'(p)|}{|\det(f^n)'(q)|} \leq C_1$$

The Bounded Distortion will be important, to ensure in Section 4.2 that the Hausdorff measure of a given set is greater than 0.

The following Lemma gives us that the concatenation of hyperbolic times will still be a hyperbolic time, which happens as a straightforward consequence of the definition.

Lemma 4.1.10. *The Hyperbolic times have the following property.*

If $p \in H_j(\sigma, \epsilon, f)$ and $f^j(p) \in H_l(\sigma, \epsilon, f)$ then $p \in H_{j+l}(\sigma, \epsilon, f)$.

Notation 4.1.11. *For each $x \in \mathcal{H}$ denote by $h(x) = \{f^n(x), x \in H_n(\sigma, \epsilon, f)\}$ the set of all hyperbolic images of x and by $h = (h(x))_{x \in \mathcal{H}}$ the collection of hyperbolic images. Denote by $\mathcal{E}_{\mathcal{H}} = (\mathcal{E}_{\mathcal{H},n})_n$ the collection of all (σ, δ)-hyperbolic pre-balls, where $\mathcal{E}_{\mathcal{H},n} = \{V_n(x); x \in H_n(\sigma, \epsilon, f)\}$ is the collection of all (σ, δ)- hyperbolic pre-balls of order n.*

Remark 4.1.12. *Thus, if μ is the unique a.c.i.p. for the Lorenz map f, which is ergodic, Pinheiro (see Lemma 3.9 and Theorem 2 in [Pi11]) ensures the existence of a fat attractor A (i.e., $\mu(A) > 0$) such that $\omega_f(x) = A$ for μ-a.e.p. $x \in M = [0, 1]$, a compact set $A_h \subset A$ such that $\omega_{f,h}(x) = A_h$ for μ-a.e.p. $x \in M$, called hyperbolic ergodic attractor, and a compact set $A_{+,h} \subset A_h$ such that $\omega_{+,f,h}(x) = A_{+,h}$ for μ-a.e.p. $x \in M$, called statistical hyperbolic ergodic attractor.*

Here, the set $\omega_{f,h}(x)$ is the set of accumulation points of $h(x)$, that is, the set of points $p \in M$ such that there is a sequence $n_j \to +\infty$ satisfying $h(x) \ni f^{n_j}(x) \to p$. And $\omega_{+,f,h}(x)$ is the set of h-frequently visited points of x orbit, as the set of points $p \in M$ such that $\limsup \frac{1}{n}\#\{1 \leq j \leq n; f^j(x) \in h(x) \cap V\} > 0$ for every neighborhood V of p.

This deep knowledge about the measure, and these attractors will be useful to guarantee the construction of the Markov partition in a nested set.

Remark 4.1.13. *By Lemma 4.1.10, we have that the collection of all (σ, δ)-hyperbolic pre-balls, $\mathcal{E}_{\mathcal{H}} = (\mathcal{E}_{\mathcal{H},n})_n$, is a dynamically closed family of regular pre-images, i.e., if $f^l(E) \in \mathcal{E}_{n-l}, \forall E \in \mathcal{E}_n$ and $\forall\ 0 \leq l \leq n$.*

4.1.2 Nested set and Induced Markov map

The $\mathcal{E}_{\mathcal{H}}$ family mentioned at Remark 4.1.13 will be used to obtain the Markov partition on a nested set, as we will see later. The idea of nested sets introduced by Pinheiro in [Pi11], generalizes the concept of nice interval introduced by Martens in [Mar], and that we saw in Chapter 1.

We have that two sets I_1 and I_2 are *linked* if $I_1 \setminus I_2$, $I_2 \setminus I_1$ and $I_1 \cap I_2$ are not empty sets.

Definition 4.1.14. *($\mathcal{E}$-nested set). Let $\mathcal{E} = (\mathcal{E}_n)$ be a dynamically closed family of regular pre-images. A set V is called $\mathcal{E}$-nested if it is open and it is not linked with any $\mathcal{E}$-pre-image of itself.*

The fundamental property of a nested set is that any $\mathcal{E}$-pre-images I_1 and I_2 of it are not linked (see [Pi11]).

Pinheiro shows that nested sets are abundant in the presence of some expansion, see Corollary 2.9 and Lemma 5.12 in [Pi11]. For this, let λ be given by the Proposition 4.1.6 and let δ be given by the Proposition 4.1.7, for some $0 < r < \delta/4$ we have that $\alpha_n(r) = e^{-\frac{\lambda}{8}n}r$ is a zooming contraction (according to definition in the Section **??**) and $\sum_{n\geq 1} \alpha_n(r) < \frac{r}{4}$. And so, the existence of a nested ball I is ensured by Lemma 5.12 in [Pi11].

Definition 4.1.15. *(Induced Markov partition) Let $f : U \to U$ a mensurable map defined on a Borel set $U \subset M$. A countable collection $\mathcal{P} = \{P_1, P_2, ...\}$ of Borel subsets of U is called a induced Markov partition if it satisfies the following conditions*

(1) *$int(P_i) \cap int(P_j) = \emptyset$ if $i \neq j$;*

(2) *for each $P_i \in \mathcal{P}$ there is a $R_i \geq 1$ such that*

 (2.1) *if $n < R_i$ and $int(f^n(P_i)) \cap int(P_j) \neq \emptyset$ then $int(f^n(P_i)) \subset int(P_j)$ or $int(f^n(P_i)) \supset int(P_j)$;*

 (2.2) *if $int(f^{R_i}(P_i)) \cap int(P_j) \neq \emptyset$ then $f^{R_i}(P_i) \supset int(P_j)$.*

(3) *$\#\{f(P_i); i \in \mathbb{N}\} < \infty$;*

(4) $f\mid_{P_i}$ *is a homeomorphism and it can be extended to a homeomorphism sending* $\overline{P_i}$ *onto* $\overline{f(P_i)}$*;*

(5) $\lim_n \operatorname{diam}(\mathcal{P}_n(x)) = 0$*,* $\forall x \in \bigcap_{n\geq 0} f^{-n}(\bigcup_i P_i)$*,*

where $\mathcal{P}_n(x) = \{y; \mathcal{P}(f^j(y)) = \mathcal{P}(f^j(x)), \forall 0 \leq j \leq n\}$ *and* $\mathcal{P}(x)$ *denotes the element of* $\mathcal{P}$ *that contains* x*.*

Definition 4.1.16. *(Induced Markov map) The pair* $(F, \mathcal{P})$*, where* $\mathcal{P}$ *is a induced Markov partition of* $F : U \to U$*, is called a induced Markov map for* f *on* U *if there is a function* $R : U \to \mathbb{N} = \{0, 1, 2, ...\}$ *(called inducing time) such that*

(1) $\{R \geq 1\} = \bigcup_{P\in\mathcal{P}} P$*,*

(2) $R\mid_P$ *is constant,* $\forall P \in \mathcal{P}$

(3) $F(x) = f^{R(x)}(x), \forall x \in U$*.*

In addition, if $F(P) = U, \forall P \in \mathcal{P}$*,* $(F, \mathcal{P})$ *is called a full induced Markov map and so, if* U *is an open set then* P *is an open set* $\forall P \in \mathcal{P}$*.*

The nested ball I can be taken in a way that $I \cap A_{+,h} \neq \emptyset$, and thus it follows from Corollary 6.6 and Lemma 6.7 in [Pi11], that there exist the induced Markov partition $\mathcal{P} = \{I_1, I_2, I_3, ...\}$ of I and an $f-$induced full Markov map F in I, with $\{R \geq 1\} = \bigcup_{I_i\in\mathcal{P}} I_i$, such that R is the *first hyperbolic return time* to I.

4.2 Hausdorff Dimension

In this section we will show that f has invariant Cantor sets of points with arbitrarily large Hausdorff dimension, that avoid the singular region. We will construct Cantor set Λ in the nested ball I defined from the previous section, such that it has Hausdorff dimension arbitrarily close to one and then we will get f-invariant set $\tilde{\Lambda}$, containing Λ, with Hausdorff dimension greater than or equal to that of Λ. For this, some notions about the definition of Hausdorff measure and dimension will be presented.

Suppose that Λ is a subset of $\mathbb{R}$ and $\mathcal{U} = \{U_i\}$ is a countable (or finite) δ-cover of Λ by open intervals in $\mathbb{R}$, i.e., $\Lambda \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \leq \delta$ for each i. Recall that $|U| = \sup\{|x - y| : x, y \in U\}$ for any U non-empty subset of $\mathbb{R}$.

Define for $\alpha > 0$

$$H_\alpha(\mathcal{U}) = \sum_{i=1}^{\infty} |U_i|^\alpha$$

and then the *Hausdorff α-measure* of Λ is

$$m_\alpha(\Lambda) = \lim_{\delta\to 0} \left(\inf_{\mathcal{U}\ \delta-covers\ \Lambda} H_\alpha(\mathcal{U}) \right)$$

It can be shown that there is a unique number, the *Hausdorff dimension* of Λ, denoted by $HD(\Lambda)$, such that for $\alpha < HD(\Lambda)$, $m_\alpha(\Lambda) = \infty$ and for $\alpha > HD(\Lambda)$, $m_\alpha(\Lambda) = 0$.

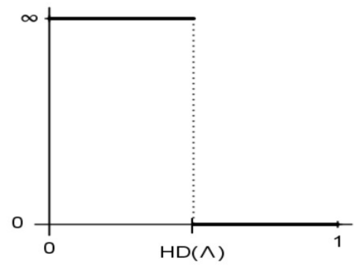


Figure 4.1: Hausdorff dimension of Λ.

Moreover, if Λ is contained in $\tilde{\Lambda}$ then $m_\alpha(\Lambda) \leq m_\alpha(\tilde{\Lambda})$ and thus $HD(\Lambda) \leq HD(\tilde{\Lambda})$.

The next results are very important properties related to the Hausdorff dimension, and can be found in Falconer.

Proposition 4.2.1. *(see Proposition 2.3 [Fa]) Let $\Lambda \subset \mathbb{R}^n$ and suppose that $f : \Lambda \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq C|x - y|^\alpha \qquad (x, y \in \Lambda)$$

for constants $C > 0$ and $\alpha > 0$. Then $HD(f(\Lambda)) \leq (1/\alpha)HD(\Lambda)$.

And so, as a consequence it follows that the Hausdorff dimension is invariant about bi-Lipschitz continuous maps.

Proposition 4.2.2. *(see Corollary 2.4 [Fa])*

(a) If $f : \Lambda \to \mathbb{R}^m$ is a Lipschitz transformation then $HD(f(\Lambda)) \leq HD(\Lambda)$.

(b) If $f : \Lambda \to \mathbb{R}^m$ is a bi-Lipschitz transformation, i.e.

$$C_1|x - y| \leq |f(x) - f(y)| \leq C_2|x - y| \qquad (x, y \in \Lambda)$$

where $0 < C_1 \leq C_2 < \infty$, then $HD(f(\Lambda)) = HD(\Lambda)$.

This property will be useful for us to guarantee that close to the singular values of the Lorenz map, we will obtain a set of points with Hausdorff dimension also arbitrarily close to one and that avoid the singular region.

Definition 4.2.3. *Let $f : X \to X$ defined in a set X, $I \subset X$, and $F : I^* \to I$ the f-induced map $F(x) = f^{R(x)}(x)$ defined in section 4.1.14 with $R : I^* \to \mathbb{N} := \{1, 2, ...\}$ the first return zomming time. Given a set $\Lambda \subset I$, we define the (f, R)-**spreading of** Λ as*

$$\tilde{\Lambda} = \bigcup_{x \in \Lambda} \bigcup_{j=0}^{R(x)-1} f^j(x) = \bigcup_{n \geq 1} \bigcup_{j=0}^{n-1} f^j(\Lambda \cap \{R = n\}) = \bigcup_{j \geq 0} f^j(\Lambda \cap \{R > j\})$$

Lemma 4.2.4. *(see Lemma 2.1[Pi20]) If $F(\Lambda) \subset \Lambda \subset I$ then $f(\tilde{\Lambda}) \subset \tilde{\Lambda}$. Also, if $F(\Lambda) = \Lambda \subset I$ then $f(\tilde{\Lambda}) = \tilde{\Lambda}$.*

To show that a set Λ has arbitrarily large Hausdorff dimension $HD(\Lambda)$, we will see that for every $\alpha < 1$ chosen arbitrarily close of 1 the Hausdorff measure, $m_\alpha(\Lambda) > 0$.

Lemma 4.2.5. *Let $f : [0,1]\backslash\{c\} \to [0,1]$ be a C^2 expanding Lorenz map and let $J = (a,b)$ be a nice interval of f so that $\Lambda_J = \{x \in [0,1] \setminus \{c\};\ \mathcal{O}_f^+(x) \cap J = \emptyset\}$ is a Cantor set. Then $HD(\Lambda_J) \to 1$ as $|J| \to 0$.*

Proof. Let I be a nested ball, $\mathcal{P}(I) = \{I_1, I_2, ..., I_n, ...\}$ an enumerable partition of I in increasing order of induction time, that is, if $i < j$ then $R(I_i) \leq R(I_j)$, and F the f-induced map of the Subsection 4.1.2. For simplicity of notation assume that $|I| = 1$ and then $\sum_{i=1}^{\infty} |I_i| = 1$, (otherwise we could normalize by obtaining $\sum_{i=1}^{\infty} |I_i|/|I| = 1$). Fixed $k > 1$, define

$$\begin{aligned}
\Lambda_k^{(1)} &= \overline{I_1} \cup \overline{I_2} \cup \cdots \cup \overline{I_k} \\
\Lambda_k^{(2)} &= \Lambda_k^{(1)} \cap F^{-1}(\Lambda_k^{(1)}) \\
&\vdots \\
\Lambda_k^{(n)} &= \cap_{j=0}^{n-1} F^{-j}(\Lambda_k^{(1)})
\end{aligned}$$

and so we get a dynamically defined Cantor set

$$\Lambda_k = \lim_{n\to\infty} \Lambda_k^{(n)}.$$

Note that the way it was constructed, Λ_k is F-invariant, it does not contain the point c and $\Lambda_k \subset \Lambda_{k+1}, \forall k$. Furthermore, if for all n, we consider the interval

$$I_{i_1 \dots i_n} = (F\mid_{I_{i_1}})^{-1}(I_{i_2 \dots i_n}) = \dots = (F\mid_{I_{i_1}})^{-1} \circ \dots \circ (F\mid_{I_{i_n}})^{-1}(I),$$

i.e., $F^n(I_{i_1 \dots i_n}) = I$, by Corollary 4.1.9 we will have that there is $C > 1$ so that

$$(1/C)|I_{i_1}||I_{i_2 \dots i_n}| \leq |I_{i_1 \dots i_n}| \leq C|I_{i_1}||I_{i_2 \dots i_n}|. \tag{4.5}$$

Now fix $0 < \alpha < 1$ and let $C > 1$ be given by the Corollary 4.1.9. The nested interval I can be considered so that $\sum_{i=1}^{\infty} |I_i|^\alpha > C$ (it is enough that we take I in order to have $|I_i|/|I| < \lambda^{-n_0}$, for all $i \geq 1$ and some n_0 large enough, i.e., $R(I_i) \geq n_0$ for all $i \geq 1$). Then, there is $k_1 > 1$ such that $\sum_{i=1}^{k_1} |I_i|^\alpha > C$. We will see that $HD(\Lambda_{k_1}) \geq \alpha$.

In fact, initially note that

$$\begin{aligned}\sum_{i_1,i_2=1}^{k_1} |I_{i_1 i_2}|^\alpha &\geq \sum_{i_1=1}^{k_1}\sum_{i_2=1}^{k_1} (1/C|I_{i_1}||I_{i_2}|)^\alpha \\ &\geq (1/C)^\alpha \sum_{i_1=1}^{k_1} |I_{i_1}|^\alpha \sum_{i_2=1}^{k_1} |I_{i_2}|^\alpha \\ &> (1/C)^\alpha C \sum_{i_2=1}^{k_1} |I_{i_2}|^\alpha \\ &> \sum_{i=1}^{k_1} |I_i|^\alpha > C > 0.\end{aligned}$$

And so, it follows that

$$\begin{aligned}\sum_{i_1,\dots,i_n=1}^{k_1} |I_{i_1\dots i_n}|^\alpha &\geq (1/C)^\alpha \left(\sum_{i_1=1}^{k_1} |I_{i_1}|^\alpha\right)\left(\sum_{i_2,\dots,i_n=1}^{k_1} |I_{i_2\dots i_n}|^\alpha\right) \\ &> \sum_{i_2,\dots,i_n=1}^{k_1} |I_{i_2\dots i_n}|^\alpha > C > 0. \end{aligned} \tag{4.6}$$

Furthermore, since Λ_{k_1} is compact, we need only consider coverings that are finite collection of intervals. Given $\delta > 0$ note that if $\mathcal{U} = \{U_j\}$ is a finite δ-cover of Λ_{k_1}, there is n such that $\mathcal{I} = \{|I_{i_1\dots i_n}|, 1 \leq i_1, ..., i_n \leq k_1\}$ is δ-cover of Λ_{k_1}, $\max\{|I_{i_1\dots i_n}|, 1 \leq i_1, ..., i_n \leq k_1\} < |U_j|, \forall U_j \in \mathcal{U}$, and for any $I_{i_1\dots i_n} \in \mathcal{I}$, $I_{i_1\dots i_n} \subset U_j$ for some j. Thus, it follows from (4.6) that

$$n \to \infty \ (\Leftrightarrow \delta \to 0) \Rightarrow m_\alpha(\Lambda_{k_1}) > 0$$

So, for each given $\alpha < 1$, $\exists k_1 > 1$ such that $m_\alpha(\Lambda_{k_1}) > 0$ and therefore $HD(\Lambda_{k_1}) \geq \alpha$.

However, we saw that Λ_{k_1} is invariant by map F, but it is not by map f, so we take the spreading of Λ_{k_1} and denote by $\tilde{\Lambda}_{k_1}$ which is f-invariant compact[1] set with $HD(\tilde{\Lambda}_{k_1}) \geq \alpha$, since $\Lambda_{k_1} \subset \tilde{\Lambda}_{k_1}$.

Also, note that c does not belong to $\tilde{\Lambda}_{k_1}$, since every point in Λ_{k_1} has infinite hyperbolic times, and as we saw that $\tilde{\Lambda}_{k_1}$ is compact we have $d = \text{dist}(c, \tilde{\Lambda}_{k_1}) > 0$. By Proposition 1.1.16 of Chapter 1, there is nice interval $J_1 = (a_1, b_1) \subset (c - d, c + d)$ with Λ_{J_1} being a Cantor set, and $\Lambda_{J_1} \supset \tilde{\Lambda}_{k_1}$, so, $HD(\Lambda_{J_1}) \geq \alpha$.

Note that, for any other nice interval J_2 contained in J_1, we have $\Lambda_{J_1} \subset \Lambda_{J_2}$ and then $HD(\Lambda_{J_2}) \geq HD(\Lambda_{J_1}) \geq \alpha > 0$. Moreover, as Lebesgue almost every point accumulates on both sides of the c, one can take $k_2 > k_1$, and with the same construction for $\tilde{\Lambda}_{k_2}$, get $J_2 \subsetneq J_1$ so that we have $\Lambda_{J_2} \cap J_1 \neq \emptyset$ on both sides of the c and $HD(\Lambda_{J_2}) \geq \alpha > 0$.

[1] $\tilde{\Lambda}_{k_1}$ is a finite union of iterates by f, at most $R(I_{k_1})$, of Λ_{k_1} which is a compact set.

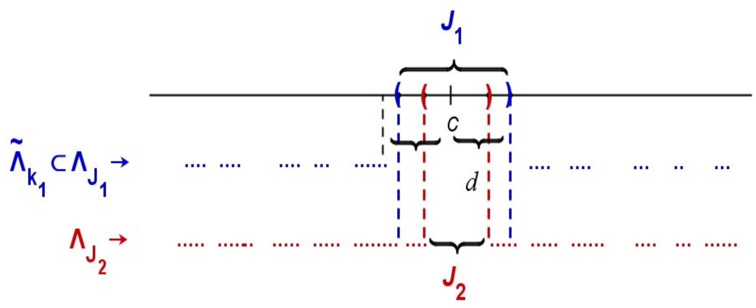


Figure 4.2: Cantor set Λ with arbitrarily large Hausdorff dimension.

$\square$

We will denote by Λ or Λ_f whenever we need to emphasize what the map, the set obtained by the previous lemma. And so, we can take Λ arbitrarily close to c on both sides, which gives us the consequence of Corollary 4.2.6.

As we deal with proximity between compact subsets, a natural way to measure the distance between two compact sets is to use Hausdorff metric. The *Hausdorff metric* of two compact sets A and B is defined as

$$d(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\}$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Intuitively means that it is the minimum distance by which two compact sets can shadow each other. For example, if $A = \{f(c_-)\}$ and $B \subset \Lambda$, $d(A, B)$ can be seen in the Figure 4.3 below.

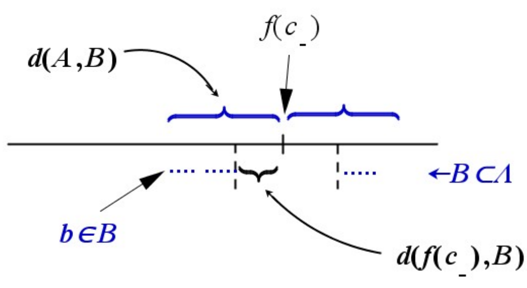


Figure 4.3: Hausdorff metric $d(f(c_-), B)$.

Corollary 4.2.6. *Let $f : [0, 1] \setminus \{c\} \to [0, 1]$ be a C^2 expanding Lorenz map such that $c \in \omega_f(f(c_\pm))$. Then there are subsets of a Cantor set Λ, with arbitrarily large Hausdorff dimension, and being arbitrarily close to $f(c_\pm)$, on both sides.*

Proof. Let J be a nice interval sufficiently small. By the previous lemma we saw that it is possible to obtain nice interval $\overline{J} \subsetneq J$, so that $\Lambda = \Lambda_{\overline{J}}$ has subsets with arbitrarily large Hausdorff dimension, that intersects J on the right side and on the left side.

Take P the connected component of $\mathcal{P}_J$ containing the singular value $f(c_-)$, as in the Chapter 1 (the case $f(c_+)$ is analogue). Thus, there is S such that $f^S(P) = J$ and since $f^S\mid_P$ is diffeomorphism, it follows by Proposition 4.2.2 that arbitrarily close to $f(c_-)$ we will also have subsets of Λ with the same Hausdorff dimension.

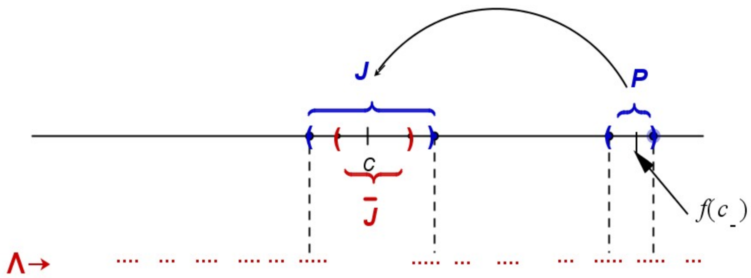


Figure 4.4: Subsets of Λ with large Hausdorff dimension close to $f(c_-)$.

□

4.3 Symbolic Dynamics

Let Σ_2^+ denote the set of sequences of $0's$ and $1's$, i.e., $\alpha : \mathbb{N} \to \{0,1\}$ endowed with the topology given by the metric

$$d(\alpha,\beta) = \sum_{i=0}^{\infty} \frac{|\alpha_i - \beta_i|}{2^i}$$

where $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ and $\beta = \beta_0\beta_1\beta_2\cdots$. The metric d allows us to decide how the strings are close to each other.

Let $\sigma : \Sigma_2^+ \to \Sigma_2^+$ be the shift map so that $\sigma(\alpha_0\alpha_1\alpha_2\cdots) = \alpha_1\alpha_2\cdots$, we have (Σ_2^+, d) is a compact metric space and σ is continuous (see [De]). Furthermore, consider in Σ_2^+ the lexicographical order: $\alpha < \beta$ if there is $n \in \mathbb{N}$ such that $\alpha_i = \beta_i$ for $i = 0, 1, \cdots, n-1$ and $\alpha_n = 0$ and $\beta_n = 1$.

We can relate the shift map σ to our map of the interval f. For this purpose be $I_0 = [0, c]$ and $I_1 = [c, 1]$ and following definition.

Definition 4.3.1. *Let $x \in [0,1] \setminus \bigcup_{j=0}^{\infty} f^{-j}(c)$. The itinerary of x is a sequence $\mathcal{I}(x) = \alpha_0\alpha_1\alpha_2\cdots$ where*

$$\alpha_j = 0 \text{ if } f^j(x) \in I_0 \text{ and } \alpha_j = 1 \text{ if } f^j(x) \in I_1.$$

For $x = c$, we have the sequences given by the itineraries

$$\mathcal{I}(c_+) = \lim_{x \downarrow c} \mathcal{I}(x) \text{ and } \mathcal{I}(c_-) = \lim_{x \uparrow c} \mathcal{I}(x)$$

which are called kneading invariants and denoted respectively by $K_+(f)$ and $K_-(f)$.

Remark 4.3.2. *The sequences kneading play a very important role, since the combinatorics of all possible words are determined by it.*

According to Melo and Martens [MM] the first n symbols of $K_-(f)$ are the symbols of the branch of f^{n+1} adjacent to c that is contained in I_0 and of $K_+(f)$ are the of the branch adjacent to c that is contained in I_1.

The idea is to observe the behaviour of the orbit of the points that belong to the set Λ and for that we use the itinerary of the point. Note that for two different points x, y in $[0,1]$, there is n so that $f^n(x)$ and $f^n(y)$ are on opposite sides of c, then if $x \neq y$, it follows that $\mathcal{I}(x) \neq \mathcal{I}(y)$, thus, the intersection of all branches that contains a given point is a point.

To each branch I of f^n we can associate a word of length n, $w = w_0 w_1 \cdots w_{n-1}$, where $w_i \in \{0,1\},\ 0 \leq i \leq n-1$ and there is at most one branch of f^n associated with it, which is called a cylinder, defined as

$$\mathcal{C}_+(w_0 w_1 \cdots w_{n-1}) = \{\alpha = \alpha_0 \alpha_1, \alpha_2 \cdots \in \Sigma_2^+;\ \alpha_i = w_i,\ 0 \leq i \leq n-1\}.$$

4.4 Two-parameter family

In this section we will study a family of maps with two-parameters, in order to observe the behaviour of their singular values and the points of Λ when we make small perturbations.

Given a expanding Lorenz map, as in the Chapter 1, $f : [0,1] \setminus \{c\} \to [0,1]$, $0 < c < 1$,

$$f(x) = \begin{cases} f_0(x), & x < c, \\ f_1(x), & x > c. \end{cases}$$

With full branch, i.e., $f_0(0) = 0, f_0(c_-) = 1, f_1(1) = 1, f_1(c_+) = 0$, and $f'(x) \geq \lambda > 1, \forall x \in [0,1] \setminus \{c\}$.

We consider the family associated to f as the two-parameter family $\mathcal{F} = \{f_{t,s}\}$ given by

$$f_{t,s}(x) = \begin{cases} t f_0(x), & x < c \\ 1 - s(1 - f_1(x)), & x > c \end{cases}$$

where $(t,s) \in \Gamma = (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]$. Note that $f_{1,1}(x) = f(x)$, $f_{t,s}(c_-) = t$ and $f_{t,s}(c_+) = 1 - s$. In addition, $f_{t,s}$ has the same regularity as f and the same singular point c.

Let $g_0 = f_0^{-1} = (f\mid_{[0,c)})^{-1}$ and $g_1 = f_1^{-1} = (f\mid_{(c,1]})^{-1}$. There exists $\gamma < 1$ so that

$$0 < g_i'(x) \leq \gamma < 1,\ \forall x \in [0,1],\ i = 0,1 \tag{4.7}$$

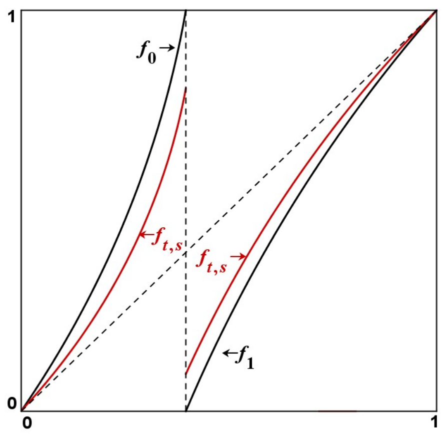


Figure 4.5: Two-parameters family $\mathcal{F} = \{f_{t,s}\}$.

and then the inverse branches of $f_{t,s}$ will be given by

$$h_{\alpha_k}(t,s,x) = \begin{cases} h_0(t,s,x) = g_0\left(\dfrac{x}{t}\right) & \text{if } \alpha_k = 0 \\ h_1(t,s,x) = g_1\left(\dfrac{x-(1-s)}{s}\right) & \text{if } \alpha_k = 1 \end{cases}$$

thus,

$$\begin{cases} h_0(t,s,x) = (g_0 \circ u)(t,s,x) \\ h_1(t,s,x) = (g_1 \circ v)(t,s,x) \end{cases}$$

where

$$u(t,s,x) = \frac{x}{t} \qquad \text{and} \qquad v(t,s,x) = \frac{x-(1-s)}{s}.$$

In what follows each α_k will be either 0 or 1. And to simplify the notation of $h_{\alpha_k}(t,s,x)$ we will omit t,s in some moments writing simply $h_{\alpha_k}(x)$, for the compositions $h_{\alpha_0\alpha_1\cdots\alpha_n}(x) = h_{\alpha_0}(t,s,h_{\alpha_1\cdots\alpha_n}(x))$ and $x_i = h_{\alpha_i\cdots\alpha_n}(x)$ for $i \geq 1$. Observe that

$$\begin{cases} \partial_x u(t,s,x) = \frac{1}{t} \\ \partial_x v(t,s,x) = \frac{1}{s} \end{cases} \qquad \begin{cases} \partial_t u(t,s,x) = -\frac{x}{t^2} \\ \partial_t v(t,s,x) = 0 \end{cases} \qquad \begin{cases} \partial_s u(t,s,x) = 0 \\ \partial_s v(t,s,x) = \frac{(1-x)}{s^2} \end{cases}$$

$$\partial_t h_{\alpha_k}(x) = \begin{cases} g_0'(u(t,s,x))\partial_t u(t,s,x) \geq -\gamma\frac{x}{t^2} & \text{if } \alpha_k = 0 \\ 0 & \text{if } \alpha_k = 1 \end{cases} \tag{4.8}$$

$$\partial_s h_{\alpha_k}(x) = \begin{cases} 0 & \text{if } \alpha_k = 0 \\ g_1'(v(t,s,x))\partial_s v(t,s,x) \leq \gamma\frac{1-x}{s^2} & \text{if } \alpha_k = 1 \end{cases} \tag{4.9}$$

Moreover, we also have,

$$\partial_t h_{\alpha_k}(x) < 0 \text{ if } \alpha_k = 0 \text{ and } \partial_s h_{\alpha_k}(x) > 0 \text{ if } \alpha_k = 1 \tag{4.10}$$

in cases where $x \neq 0$ and $x \neq 1$ respectively.

$$0 < \partial_x h_{\alpha_k}(x) = \begin{cases} g_0'(u(t,s,x))\partial_x u(t,s,x) \leq \frac{\gamma}{t} & \text{if } \alpha_k = 0 \\ g_1'(v(t,s,x))\partial_x v(t,s,x) \leq \frac{\gamma}{s} & \text{if } \alpha_k = 1 \end{cases} \tag{4.11}$$

Then we get by induction and using the Chain rule,

$$\partial_\ell h_{\alpha_0\alpha_1\cdots\alpha_n}(x) = \partial_\ell h_{\alpha_0}(x_1) + \sum_{k=1}^{n} \partial_\ell h_{\alpha_k}(x_{k+1}). \prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) \tag{4.12}$$

where $\ell = t, s$.

Remark 4.4.1. *Note that if x is close enough to 1, for $\alpha_k = 1$, the derivative $\partial_s h_{\alpha_k}(x)$ will be close enough to 0. And, if x is close enough to 0, for $\alpha_k = 0$, $\partial_t h_{\alpha_k}(x)$ will be close enough to 0.*

So that we can analyze the behaviour of the points of the Cantor set $\Lambda(t,s)$, each point $p(t,s) \in \Lambda(t,s)$ will be approximated by the inverse branches of the map $f_{t,s}$.

According to Remark 4.3.2, for each (t,s), we will have a subset $\Sigma_{t,s} \subset \Sigma_2^+$ given by the kneading sequence of $f_{t,s}$, which will give all possible combinatorics associated with the map $f_{t,s}$.

Finally, for each $(t,s) \in \Gamma$, $\alpha = \alpha_0\alpha_1\alpha_2 \cdots \in \Sigma_{t,s}$, and x_0 chosen in $[0,1]$, we define[2] the coding map $\phi_{t,s} : \Sigma_{t,s} \to [0,1]$ by

$$\phi_{t,s}(\alpha) := \lim_{n\to\infty} h_{\alpha_0} \circ h_{\alpha_1} \circ \cdots \circ h_{\alpha_n}(x_0) \tag{4.13}$$

Note that for each (t,s) the map $\phi_{t,s}(\alpha) = \phi(\alpha,t,s)$ is well defined because, as mentioned in Section 4.2, two different points have different itineraries.

Lemma 4.4.2. *The partial derivatives of ϕ satisfy the following estimates:*

$$0 > \partial_t \phi(\alpha,t,s) \geq -\frac{\gamma}{t^2} \sum_{k\geq 0} (1-\alpha_k) \left(\frac{\gamma}{t}\right)^{n_k} \left(\frac{\gamma}{s}\right)^{k-n_k} \tag{4.14}$$

$$0 < \partial_s \phi(\alpha,t,s) \leq \frac{\gamma}{s^2} \sum_{k\geq 0} \alpha_k \left(\frac{\gamma}{t}\right)^{n_k} \left(\frac{\gamma}{s}\right)^{k-n_k} \tag{4.15}$$

where $n_k = \#\{\alpha_j = 0; 0 \leq j < k\}$

Proof. Initially we see that for every k, by the equation (4.11), we have

$$\prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) \leq \left(\frac{\gamma}{t}\right)^{n_k} \left(\frac{\gamma}{s}\right)^{k-n_k}$$

[2] The definition of ϕ does not depend on the choice of x_0.

with $n_k = \#\{\alpha_j = 0; 0 \le j < k\}$. And using the equation (4.8), for $\ell = t$ we will have each term of the expression (4.12) will be

$$\partial_t h_{\alpha_k}(x_{k+1}).\prod_{i=0}^{k-1}\partial_x h_{\alpha_i}(x_{i+1}) \begin{cases} \ge -\dfrac{\gamma}{t^2}\left(\dfrac{\gamma}{t}\right)^{n_k}\left(\dfrac{\gamma}{s}\right)^{k-n_k} & \text{if } \alpha_k = 0 \\ = 0 & \text{if } \alpha_k = 1 \end{cases}$$

In particular

$$0 \ge \partial_t h_{\alpha_k}(x_{k+1}).\prod_{i=0}^{k-1}\partial_x h_{\alpha_i}(x_{i+1}) \ge -(1-\alpha_k)\frac{\gamma}{t^2}\left(\frac{\gamma}{t}\right)^{n_k}\left(\frac{\gamma}{s}\right)^{k-n_k}$$

and so equation (4.14) it follows from expression (4.12) when n goes to infinity. The equation (4.15)it follows in a similar way.

□

We will be interested to see how the points of the set Λ_f and the singular values of f vary in relation to the parameters, but precisely we will see that when making small perturbations in the branches of the function f, the singular values will go through a subset of Λ_f with Hausdorff dimension arbitrarily close to one.

In addition, we have that the two-parameter family $\{f_{t,s}\}$ of functions, is continuous in each parameter t, s, which means given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t_1 - t_0| < \delta \Rightarrow |f_{t_1,s}(x) - f_{t_0,s}(x)| < \epsilon$$

$$|s_1 - s_0| < \delta \Rightarrow |f_{t,s_1}(x) - f_{t,s_0}(x)| < \epsilon$$

for all $x \in [0, 1]$.

The next Proposition gives us an estimate of the partial derivative of ϕ with respect to parameters t, s and points close to 1 or close to 0.

Let $\gamma < 1$ be the contraction rate of inverse branches seen in (4.7) an let $1_{n_0} \subset \Sigma_2^+$ denote the cylinder $\mathcal{C}_+(\underbrace{1, \cdots, 1}_{n_0})$, that is, whose itinerary is formed by the first n_0 elements equal to 1 (similarly to $0_{n_0} \subset \Sigma_2^+$).

Furthermore, given $\alpha \in \Sigma_{t,s}$ and $\vec{v}$ a vector in the plane we will denote the directional derivative of $\phi_\alpha(t, s) = \phi(\alpha, t, s)$ in the direction of $\vec{v}$ by $D_{\vec{v}}\phi_\alpha(t, s)$.

Proposition 4.4.3. *Given arbitrarily small $\sigma < 1$, there are $a_0 = a_0(\gamma) \in (\frac{1}{2}, 1)$ and $n_0 > 1$ such that for each $t_0, s_0 \in [a_0, 1)$ with $1_{n_0}, 0_{n_0} \in \Sigma_{t_0,s_0}$ we have:*

If $\alpha \in 1_{n_0}$,

(i) $0 < |\partial_t\phi(\alpha, t_0, s_0)| < \dfrac{\sigma}{2}$,

(ii) $0 < \eta \le |\partial_s\phi(\alpha, t_0, s_0)| < \eta + \dfrac{\sigma}{2}$ *for some $\eta > 0$ depending on n_0,*

(iii) $0 < |D_{\vec{v}}\phi_\alpha(t,s)\,|_{(t_0,s_0)}\,| < \sigma$ *for* $\vec{v} = (\theta, 1), \forall \theta > 0$.

Moreover, if $\beta \in 0_{n_0}$,

(iv) $0 < |\partial_s \phi(\beta, t_0, s_0)| < \dfrac{\sigma}{2}$,

(v) $0 < \eta \leq |\partial_t \phi(\beta, t_0, s_0)| < \eta + \dfrac{\sigma}{2}$ *for some* $\eta > 0$ *depending on* n_0,

(vi) $0 < |D_{\vec{v}}\phi_\beta(t,s)\,|_{(t_0,s_0)}\,| < \sigma$ *for* $\vec{v} = (\theta, 1), \forall \theta > 0$.

Proof. Let $\gamma < 1$ be the contraction rate of inverse branches and let $0 < \sigma < 1$.

Take a_0 such that $\gamma < a_0 < 1$. So $\dfrac{\gamma}{a_0} < 1$ and then the series

$$\mathbb{S}_{a_0} = \frac{\gamma}{a_0^2} \sum_{k \geq 0} \left(\frac{\gamma}{a_0}\right)^k$$

is convergent. Thus, there is $n_{a_0} > 1$ so that

$$\mathbb{S}_{a_0}(k \geq n_{a_0}) = \frac{\gamma}{a_0^2} \sum_{k \geq n_{a_0}} \left(\frac{\gamma}{a_0}\right)^k < \frac{\sigma}{2}. \tag{4.16}$$

Let $n_0 \geq n_{a_0}$. Note that, for $t, s \geq a_0$, $\dfrac{\gamma}{t}, \dfrac{\gamma}{s} \leq \dfrac{\gamma}{a_0}$, so for all $t_0, s_0 \in [a_0, 1)$ and every α in the cylinder 1_{n_0}, it follows by (4.14)

$$\begin{aligned} 0 < |\partial_t \phi(\alpha, t_0, s_0)| &\leq \frac{\gamma}{t_0^2} \sum_{k \geq n_0} (1 - \alpha_k) \left(\frac{\gamma}{t_0}\right)^{n_k} \left(\frac{\gamma}{s_0}\right)^{k - n_k} \\ &< \mathbb{S}_{a_0}(k \geq n_0) < \frac{\sigma}{2}. \end{aligned}$$

For item *(ii)*, as $\alpha_k = 1$ for $0 \leq k < n_0$, we have

$$\partial_s h_{\alpha_k}(x_{k+1}) > 0, \ \forall 0 \leq k < n_0.$$

Thus, in opposition to the case of the partial derivative in relation to t we have

$$\sum_{k=0}^{n_0 - 1} \partial_s h_{\alpha_k}(x_{k+1}) . \prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) > 0. \tag{4.17}$$

Therefore $\partial_s \phi(\alpha, t_0, s_0)$ is strictly positive and greater than or equal to $\mathbb{S}_{s_0}(k < n_0)$. Moreover, for the upper bound we have

$$\begin{aligned} \partial_s \phi(\alpha, t_0, s_0) &= \sum_{k=0}^{n_0 - 1} \partial_s h_{\alpha_k}(x_{k+1}) . \prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) + \sum_{k \geq n_0} \partial_s h_{\alpha_k}(x_{k+1}) . \prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) \\ &< \mathbb{S}_{s_0}(k < n_0) + \frac{\sigma}{2} = \eta + \frac{\sigma}{2}. \end{aligned}$$

where

$$\mathbb{S}_{s_0}(k < n_0) = \frac{\gamma}{s_0^2} \sum_{k<n_0} \left(\frac{\gamma}{s_0}\right)^k = \eta. \tag{4.18}$$

Note that we are not necessarily requiring that η be less than $\frac{\sigma}{2}$, i.e., it is not necessary that $\partial_s \phi(\alpha, t_0, s_0) < \sigma$. However, for item (iii), we will see that it is still possible to obtain $\eta < \frac{\sigma}{2}$.

In fact, take $\epsilon < \sigma/2\mathbb{S}_{a_0}$, we know by (4.16) that for $n_0 \geq n_{a_0}$

$$\mathbb{S}_{a_0}(k \geq n_0) < \frac{\sigma}{2}.$$

Furthermore, n_0 can be taken large enough, i.e., t_0 and s_0 close enough to 1, so that for every point $p \in 1_{n_0}$, we have $f^j_{t_0,s_0}(p)$ close enough to 1 for $0 \leq j < n_{a_0}$, that is, $|1 - f^j_{t_0,s_0}(p)| < \epsilon$ for $0 \leq j < n_{a_0}$.

So, we will have according to the Remark 4.4.1, for the sum (4.17) up to n_{a_0}

$$\begin{aligned}\sum_{k=0}^{n_{a_0}-1} \partial_s h_{\alpha_k}(x_{k+1}).\prod_{i=0}^{k-1} \partial_x h_{\alpha_i}(x_{i+1}) &< \epsilon\frac{\gamma}{s_0^2}\left[1 + \frac{\gamma}{s_0} + \cdots + \left(\frac{\gamma}{s_0}\right)^{n_{a_0}-1}\right] \\ &< \epsilon\ \mathbb{S}_{a_0} < \sigma/2\end{aligned}$$

Now, for all $t_0, s_0 \in (a_0, 1)$ fixed, and every $\alpha \in 1_{n_0}$, analyzing the variation of $\phi_\alpha(t,s)$ along the line through point (t_0, s_0) in the direction of the vector $\vec{v} = (\theta, 1)$, we have

$$\begin{aligned}|D_{\vec{v}}\phi_\alpha(t,s)\,|_{(t_0,s_0)}\,| &= \frac{1}{\|\vec{v}\|}|\partial_t\phi_\alpha(t_0,s_0).\theta + \partial_s\phi_\alpha(t_0,s_0).1| \\ &\leq \frac{1}{\|\vec{v}\|}\left[\ |\partial_t\phi_\alpha(t_0,s_0)|.\theta + |\partial_s\phi_\alpha(t_0,s_0)|\ \right] \\ &\underset{(\alpha\in 1_{n_0})}{\leq} \frac{1}{\|\vec{v}\|}\left[\frac{\gamma}{t_0^2}\sum_{k\geq n_0}(1-\alpha_k)\left(\frac{\gamma}{t_0}\right)^{n_k}\left(\frac{\gamma}{s_0}\right)^{k-n_k}\theta + \right. \\ &\qquad \left. + \frac{\gamma}{s_0^2}\sum_{k\geq 0}\alpha_k\left(\frac{\gamma}{t_0}\right)^{n_k}\left(\frac{\gamma}{s_0}\right)^{k-n_k}\right] \\ &\underset{(t_0,s_0\geq a_0)}{\leq} \frac{1}{\|\vec{v}\|}\left[\frac{\gamma}{a_0^2}\sum_{k\geq n_0}(1-\alpha_k)\left(\frac{\gamma}{a_0}\right)^k\theta + \frac{\gamma}{a_0^2}\sum_{k\geq n_0}\alpha_k\left(\frac{\gamma}{a_0}\right)^k + \eta\right].\end{aligned}$$

So, if $\theta \leq 1$

$$\begin{aligned}|D_{\vec{v}}\phi_\alpha(t,s)\,|_{(t_0,s_0)}\,| &\leq \frac{1}{\|\vec{v}\|}\left[\frac{\gamma}{a_0^2}\sum_{k\geq n_0}\left(\frac{\gamma}{a_0}\right)^k + \eta\right] \\ &< \frac{1}{\|\vec{v}\|}\left[\frac{\sigma}{2} + \frac{\sigma}{2}\right] < \sigma\end{aligned}$$

since $1 < \|\vec{v}\| = \sqrt{\theta^2+1}$.

And if $\theta > 1$,

$$
\begin{aligned}
|D_{\vec{v}}\phi_\alpha(t,s)\,|_{(t_0,s_0)}\,| &\leq \frac{1}{\|\vec{v}\|}\left[\frac{\gamma}{a_0^2}\sum_{k\geq n_0}\left(\frac{\gamma}{a_0}\right)^k\theta+\eta\right] \\
&< \frac{1}{\|\vec{v}\|}\left[\frac{\sigma}{2}\theta+\frac{\sigma}{2}\right] < \frac{\theta}{\|\vec{v}\|}\sigma < \sigma
\end{aligned}
$$

since $\theta/\|\vec{v}\| < 1$.

And so item (iii) is established.

For the case $\beta \in 0_{n_0}$, just note that we will have the case analogous to the previous case with inversion in the variables t and s, and then by the expressions obtained in Lemma 4.4.2, we will have

$$
\begin{aligned}
0 < \partial_s\phi(\beta,t_0,s_0) &\leq \frac{\gamma}{s_0^2}\sum_{k\geq n_0}\beta_k\left(\frac{\gamma}{t_0}\right)^{n_k}\left(\frac{\gamma}{s_0}\right)^{k-n_k} \\
&< \mathbb{S}_{a_0}(k\geq n_0) < \frac{\sigma}{2}.
\end{aligned}
$$

for the derivative in t

$$
\begin{aligned}
0 < |\partial_t\phi(\beta,t_0,s_0)| &\leq \mathbb{S}_{t_0}(k<n_0)+\frac{\gamma}{t_0^2}\sum_{k\geq n_0}(1-\beta_k)\left(\frac{\gamma}{t_0}\right)^{n_k}\left(\frac{\gamma}{s_0}\right)^{k-n_k} \\
&< \eta+\mathbb{S}_{a_0}(k\geq n_0) < \eta+\frac{\sigma}{2}.
\end{aligned}
$$

and, for the directional derivative in $\vec{v}$, it also follows in a similar way to the previous one, observing only that, in this case, we will use the reasoning for points $p \in 0_{n_0}$ with $f^j_{t_0,s_0}(p)$ close enough to 0 for $0 \leq j < n_{a_0}$, that is, $|f^j_{t_0,s_0}(p)| < \epsilon$ for $0 \leq j < n_{a_0}$, using Remark 4.4.1 again.

$\square$

Remark 4.4.4. *So, for any smooth curve*

$$
\begin{aligned}
\psi : (-\epsilon,\epsilon) &\rightarrow [a_0,1]\times[a_0,1] \\
\lambda &\mapsto \psi(\lambda) = (\psi_1(\lambda),\psi_2(\lambda))
\end{aligned}
$$

with $\psi(0) = (t_0,s_0)$*, we will have then for* $\alpha \in 1_{n_0}$ *(analogue for* $\beta \in 0_{n_0}$*)*

$$
|(\phi_\alpha\circ\psi)'(\lambda)\,|_{\lambda=0}\,| = |\,\nabla\,\phi_\alpha(t_0,s_0)\cdot\psi'(0)| \tag{4.19}
$$

where $\psi'(0) = (\theta, 1)$ *for some* $\theta > 0$ *depending on the curve* ψ *(Figure 4.6). So, from what we saw in item* (iii) *(analogue item* (vi) *for case* β*), we have the rate of change that* ϕ *varies in relation to* λ *along curve* ψ*, at point* (t_0, s_0)*, has the same estimate*

$$|(\phi_\alpha \circ \psi)'(\lambda)\ |_{\lambda=0}\ | < \sigma. \tag{4.20}$$

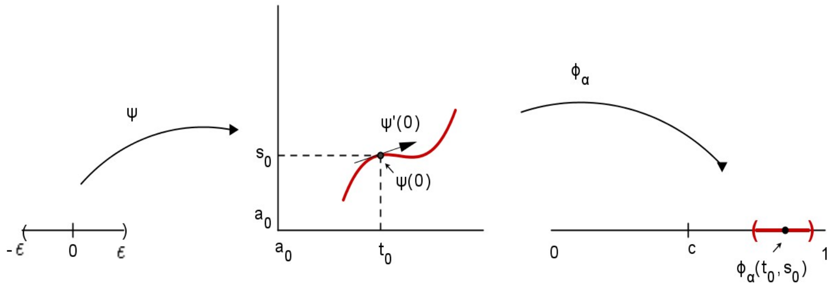


Figure 4.6: Variation of ϕ_α along curve ψ.

Remark 4.4.5. *For the singular values* $f_{t,s}(c_-) = f(t, s, c_-) = t$ *and* $f_{t,s}(c_+) = f(t, s, c_+) = 1 - s$ *of a map* $f_{t,s}$*, we can see that the rate of change at a point* (t_0, s_0) *in the direction of the vector* $\vec{v} = (\theta, 1)$*, for* $\theta > 0$*, is given by:*

$$\begin{aligned} D_{\vec{v}} f(t, s, c_-)\ |_{(t_0,s_0)} &= \frac{1}{\|\vec{v}\|}\left[\partial_t(f)(t_0, s_0, c_-).\theta + \partial_s(f)(t_0, s_0, c_-).1\right] \\ &= \frac{\theta}{\sqrt{\theta^2 + 1}} \end{aligned}$$

and

$$\begin{aligned} D_{\vec{v}} f(t, s, c_+)\ |_{(t_0,s_0)} &= \frac{1}{\|\vec{v}\|}\left[\partial_t(f)(t_0, s_0, c_+).\theta + \partial_s(f)(t_0, s_0, c_+).1\right] \\ &= \frac{-1}{\sqrt{\theta^2 + 1}} \end{aligned}$$

Note that for very large θ*, we will have* $D_{\vec{v}} f(t_0, s_0, c_+)$ *close to* 0 *and* $D_{\vec{v}} f(t_0, s_0, c_-)$ *close to* 1*,i.e., the vector* $\vec{v}$ *would be close to the direction* $(1, 0)$*, in which the left branch of* $f_{t,s}$ *is perturbed, while the right branch is not perturbed. And for very small* θ*,* $D_{\vec{v}} f(t_0, s_0, c_+)$ *close to* -1 *and* $D_{\vec{v}} f(t_0, s_0, c_-)$ *close to* 0*, in this case the vector* $\vec{v}$ *has a direction close to* $(0, 1)$*.*

These situations will not be interesting for the next result, as we do not want either of the two rates of change to be close to 0*. So, we will find an estimate for* θ *close to* 1*, so that the rates are greater than the value of* σ *taken.*

We have for $\theta = 1$,

$$D_{\vec{v}}f(t_0, s_0, c_+) = -1/\sqrt{2} \text{ and } D_{\vec{v}}f(t_0, s_0, c_-) = 1/\sqrt{2}$$

if $0 < \theta < 1$

$$\begin{cases} D_{\vec{v}}f(t_0, s_0, c_-) < 1/\sqrt{2} \\ D_{\vec{v}}f(t_0, s_0, c_+) < -1/\sqrt{2} \end{cases} \Rightarrow \begin{cases} |D_{\vec{v}}f(t_0, s_0, c_-)| < 1/\sqrt{2} \\ |D_{\vec{v}}f(t_0, s_0, c_+)| > 1/\sqrt{2} \end{cases} \tag{4.21}$$

and if $\theta > 1$

$$\begin{cases} D_{\vec{v}}f(t_0, s_0, c_-) > 1/\sqrt{2} \\ D_{\vec{v}}f(t_0, s_0, c_+) > -1/\sqrt{2} \end{cases} \Rightarrow \begin{cases} |D_{\vec{v}}f(t_0, s_0, c_-)| > 1/\sqrt{2} \\ |D_{\vec{v}}f(t_0, s_0, c_+)| < 1/\sqrt{2} \end{cases} \tag{4.22}$$

Thus, fixed $\sigma < 1/\sqrt{2}$*, in order to obtain* $D_{\vec{v}}f(t_0, s_0, c_-) > \sigma$*, we have*

$$\frac{\theta}{\sqrt{\theta^2+1}} > \sigma \quad \Rightarrow \quad \theta > \frac{\sigma}{\sqrt{1-\sigma^2}},$$

and for $D_{\vec{v}}f(t_0, s_0, c_+) < -\sigma$*, we have*

$$\frac{-1}{\sqrt{\theta^2+1}} < -\sigma \quad \Rightarrow \quad \theta < \frac{\sqrt{1-\sigma^2}}{\sigma}.$$

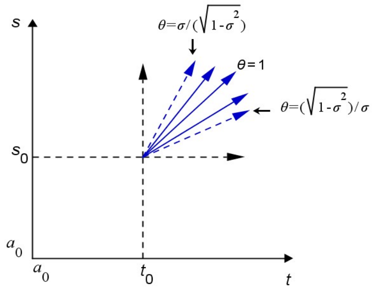


Figure 4.7: Variation of θ.

Therefore,

$$\frac{\sigma}{\sqrt{1-\sigma^2}} < \theta < \frac{\sqrt{1-\sigma^2}}{\sigma} \quad \Rightarrow \quad |D_{\vec{v}}f(t_0, s_0, c_\pm)| > \sigma, \tag{4.23}$$

for example, for $\sigma = 1/2$*, we can take* $1/\sqrt{3} < \theta < \sqrt{3}$.

The next result shows that, given a map $f_{t,s}$, the sets obtained $J_{t,s}$ and $\Lambda_{t,s}$ in Lemma 4.2.5 of the Section 4.2, are not destroyed under small perturbations. Intuitively this means that the boundary points of $J_{t,s}$ will not reach c, when we make variations in the parameters.

For the next results, we will be considering $\sigma < 1/\sqrt{2}$ and let a_0 and n_0 be as in the Proposition 4.4.3.. In addition, we will consider curves $\psi(\lambda) = (t(\lambda), s(\lambda))$, so that for every $(t, s) \in \psi$ the tangent vector to the curve at this point has direction satisfying the relation (4.23).

Lemma 4.4.6. *Assume that $f_{t_0,s_0} \in \mathcal{F}$ for some $t_0, s_0 \in (a_0, 1)$ with $t_0 \in 1_{n_0}$ and $1 - s_0 \in 0_{n_0}$, that $J_{t_0,s_0} = (a(t_0, s_0), b(t_0, s_0))$ is a nice interval of f_{t_0,s_0}, with $a(t_0, s_0) \in 01_{n_0}$ and $b(t_0, s_0) \in 10_{n_0}$ and $\Lambda_{t_0,s_0} = \Lambda_{J(t_0,s_0)}$, and that $\psi(\lambda) = (t(\lambda), s(\lambda))$ is a smooth curve passing through (t_0, s_0), with tangent vector at each point satisfying* (4.23).

Then, varying (t, s) along the curve ψ, with t, s increasing, the nice interval $J_{t,s}$ and $\Lambda_{t,s}$ continue to exist for the perturbed map $f_{t,s}$.

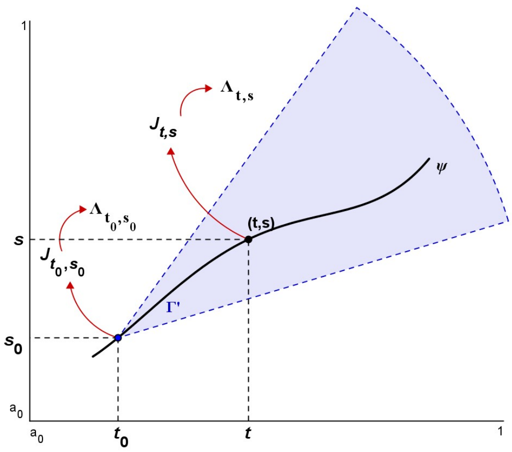


Figure 4.8: $J_{t,s}$ and $\Lambda_{t,s}$ persist in the cone.

Proof. Fix f_{t_0,s_0} and let J_{t_0,s_0} and Λ_{t_0,s_0} be as in the Lemma statement. Note that, as $a(t_0, s_0) \in 01_{n_0}$ and $b(t_0, s_0) \in 10_{n_0}$, we have $f_{t_0,s_0}(a(t_0, s_0)) \in 1_{n_0}$ and $f_{t_0,s_0}(b(t_0, s_0)) \in 0_{n_0}$.

We will fix the itineraries of $f_{t_0,s_0}(a(t_0, s_0))$ and $f_{t_0,s_0}(b(t_0, s_0))$, provided by the map f_{t_0,s_0}, and denote by α_0 and β_0, respectively. So, $\phi_{\alpha_0}(t_0, s_0) = f_{t_0,s_0}(a(t_0, s_0))$ and $\phi_{\beta_0}(t_0, s_0) = f_{t_0,s_0}(b(t_0, s_0))$.

Furthermore, note that (see Figure 4.9)

$$b(t_0, s_0) < f_{t_0,s_0}(a(t_0, s_0)) < f_{t_0,s_0}(c_-) \text{ and } f_{t_0,s_0}(c_+) < f_{t_0,s_0}(b(t_0, s_0)) < a(t_0, s_0).$$

We can see that, with these itineraries fixed, according to Remark 4.4.4, when varying (t, s) along a smooth curve ψ, passing through (t_0, s_0), with tangent vector to the curve at point (t_0, s_0) having the direction $\psi'(0) = (\theta, 1)$, we have the following estimates for the variation of the points $f_{t_0,s_0}(a(t_0, s_0))$ and $f_{t_0,s_0}(b(t_0, s_0))$,

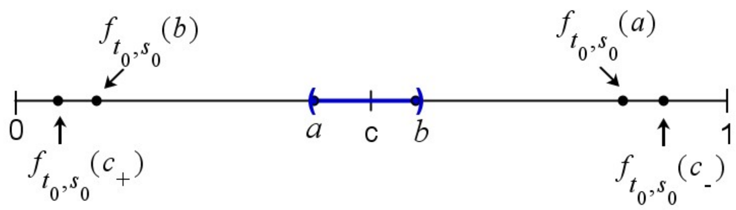


Figure 4.9: Nice interval and images of the boundary points.

$$|(\phi_{\alpha_0} \circ \psi)'(\lambda) \mid_{\lambda=0} | < \sigma \quad \text{and} \quad |(\phi_{\beta_0} \circ \psi)'(\lambda) \mid_{\lambda=0} | < \sigma. \tag{4.24}$$

Moreover, since θ satisfies relation (4.23), we have

$$|\frac{d}{d\lambda} f(t(\lambda), s(\lambda), c_{\pm}) \mid_{\lambda=0} | > \sigma \tag{4.25}$$

Thus, by increasing t, s, even though the points $f_{t,s}(a(t,s))$ and $f_{t,s}(c_-)$ move in the same direction we can see, by their positions and the rates of change given by equations (4.24) and (4.25), that $f_{t,s}(a(t,s))$ will not reach $f_{t,s}(c_-)$. If the directions are opposite, we will have the point $f_{t,s}(a(t,s))$ moving to the left, while $f_{t,s}(c_-)$ moves to the right and, clearly, they will not collide.

For points $f_{t,s}(b(t,s))$ and $f_{t,s}(c_+)$, the argument is the same, and also they will not collide.

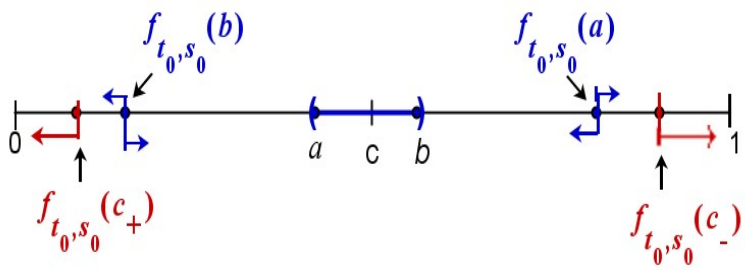


Figure 4.10: Point behaviour when varying (t, s) along the smooth curve.

In the Figure 4.10, the arrows indicate the direction of variation of the points, when we make t and s increase.

Therefore, the nice interval $J_{t,s}$, for $f_{t,s}$, having the edge points with the same itineraries as the interval J_{t_0,s_0}, for f_{t_0,s_0}, will continue to exist under perturbations. In fact, if this doesn't happen, that is, if $J_{t,s}$ are destroyed, then the point $a(t,s)$ or $b(t,s)$ will reach c. However, this occurs if and only if $f_{t,s}(a(t,s))$ reachs $f_{t,s}(c_-)$ (or $f_{t,s}(b(t,s))$ reachs $f_{t,s}(c_+)$), what we have just shown it does not happen.

Thus, all the itineraries that exist for the points of the Cantor set Λ_{t_0,s_0} will also exist for the Cantor set $\Lambda_{t,s}$. $\square$

Lemma 4.4.7. *Let $f_{t_0,s_0} \in \mathcal{F}$ for some $t_0, s_0 \in (a_0, 1)$ with $t_0 \in 1_{n_0}$, $1 - s_0 \in 0_{n_0}$, and let $J_{t_0,s_0} = (a(t_0, s_0), b(t_0, s_0))$ be a nice interval of f_{t_0,s_0}, with $a(t_0, s_0) \in 01_{n_0}$ and $b(t_0, s_0) \in 10_{n_0}$, such that Λ_{t_0,s_0} having subsets $\Lambda^-_{t_0,s_0}$ arbitrarily close to $f_{t_0,s_0}(c_-)$ and $\Lambda^+_{t_0,s_0}$ to $f_{t_0,s_0}(c_+)$ with $HD(\Lambda^\pm_{t_0,s_0}) > 1 - \delta$, for some δ small enough. Let ψ be a smooth curve passing through (t_0, s_0), with tangent vector at each point satisfying (4.23).*

Then, varying (t, s) along the curve ψ, with t, s increasing, the singular values $f_{t,s}(c_-)$ and $f_{t,s}(c_+)$ will cross the continuation of $\Lambda^-_{t_0,s_0}$ and of $\Lambda^+_{t_0,s_0}$, respectively, for map $f_{t,s}$.

We will see in the Section 4.5 that this will happen for a set of parameters with a positive Hausdorff dimension on the curve ψ.

Proof. Let f_{t_0,s_0} and ψ as in the Lemma statement. Given $\delta > 0$ small enough, let $J_{t_0,s_0} = (a(t_0, s_0), b(t_0, s_0))$ be an nice interval of f_{t_0,s_0}, with $a(t_0, s_0) \in 01_{n_0}$ and $b(t_0, s_0) \in 10_{n_0}$, and let Λ_{t_0,s_0} obtained by Lemma 4.2.5, so that the $HD(\Lambda_{t_0,s_0}) > 1 - \delta$.

According to the Corollary 4.2.6, there are subsets of points of Λ_{t_0,s_0} with Hausdorff dimension also greater than $1 - \delta$, arbitrarily close to $f_{t_0,s_0}(c_-)$, on both sides, and the same for $f_{t_0,s_0}(c_+)$. Denoted by $\Lambda^-_{t_0,s_0}$ and $\Lambda^+_{t_0,s_0}$, respectively. So that we can assume $\Lambda^-_{t_0,s_0} \subset 1_{n_0}$ and $\Lambda^+_{t_0,s_0} \subset 0_{n_0}$.

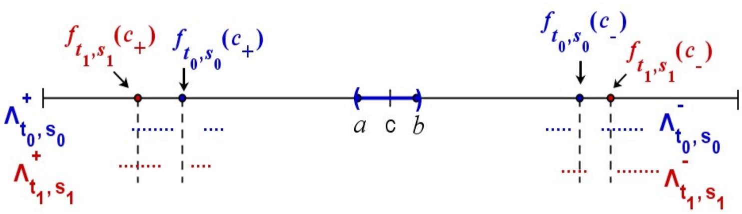


Figure 4.11: Variation of sets $\Lambda^\pm_{t_1,s_1}$ and points $f_{t_1,s_1}(c_-)$ and $f_{t_1,s_1}(c_+)$, for $t_1 > t_0$ and $s_1 > s_0$.

Thus, for each point $p(t_0, s_0) \in \Lambda^-_{t_0,s_0}$ we will have $p(t_0, s_0)$ belongs to the cylinder 1_{n_0}. And for each point $q(t_0, s_0) \in \Lambda^+_{t_0,s_0}$ we will have $q(t_0, s_0)$ belongs to the cylinder 0_{n_0}.

Now making perturbations of f_{t_0,s_0}, with (t, s) varying along the curve ψ, we know, by the Lemma 4.4.6, that the sets $J_{t,s}$, $\Lambda^-_{t,s}$ and $\Lambda^+_{t,s}$ persist. Moreover, according to Remark 4.4.4 and relation 4.23, analyzing the rates of change of the singular values and the points of $\Lambda^-_{t_0,s_0}$ and $\Lambda^+_{t_0,s_0}$, we have

$$|\frac{d}{d\lambda} f(t(\lambda), s(\lambda), c_-)\,|_{\lambda=0}\,| > \sigma > |(\phi_\alpha \circ \psi)'(\lambda)\,|_{\lambda=0}\,|$$

for $\alpha \in 1_{n_0}$, and

$$|\frac{d}{d\lambda} f(t(\lambda), s(\lambda), c_+)\,|_{\lambda=0}\,| > \sigma > |(\phi_\beta \circ \psi)'(\lambda)\,|_{\lambda=0}\,|$$

for $\beta \in 0_{n_0}$.

Then as $\Lambda^-_{t_0,s_0}$ and $\Lambda^+_{t_0,s_0}$ are arbitrarily close to $f_{t_0,s_0}(c_-)$ and $f_{t_0,s_0}(c_+)$, respectively, the result follows.

Notice that every time $f_{t,s}(c_-)$ reaches some points $p(t,s) \in \Lambda^-_{t,s}$, we have $f_{t,s}(c_-)$ do not accumulate in c for map $f_{t,s}$. Similarly for $f_{t,s}(c_+)$ reaches some points $q(t,s) \in \Lambda^+_{t,s}$. $\square$

The main goal is that we have both singular values, at the same time, not accumulating in c, when we make small perturbations on the map f_{t_0,s_0}. This will mean being to be Misiurewicz maps. We will see, in the Theorem below, how to obtain this.

Remark 4.4.8. *As in Definition 1.1.6 we have here that a point (t,s) in the parameter space is said to be a Misiurewicz point if the respective map $f_{t,s}$ is a Misiurewicz map .*

Theorem 2. *Let $f_{t_0,s_0} \in \mathcal{F}$, $\Lambda^\pm_{t_0,s_0}$, and ψ_0 be any smooth curve satisfying the conditions of the Lemma 4.4.7.*

Then, there is a set of points Γ_0 in ψ_0 , so that, for each point $(t_i,s_i) \in \Gamma_0$, there will be a smooth curve ψ_i passing through (t_i,s_i), transversely the curve ψ_0 (see Figure 4.14), and containing a set of points Γ_i, such that for each $(t,s) \in \Gamma_i$ the map $f_{t,s}$ associated is a Misiurewicz map, i.e., we will have $c \notin \omega_{f_{t,s}}(f_{t,s}(c_\pm))$.

In the next section, we will show that both the set Γ_0, and the sets Γ_i have a positive Hausdorff dimension in the curves ψ_0 and ψ_i, respectively, so we consider the Hausdorff dimensions of the Cantor sets $\Lambda^\pm_{t_0,s_0}$ in the Lemma 4.4.7 and Theorem 2.

Proof. Let $\sigma < 1/\sqrt{2}, a_0, n_0$ as already mentioned, satisfying the Proposition 4.4.3. Let $t_0, s_0 \in (a_0, 1)$ with $t_0 \in 1_{n_0}$ and $1 - s_0 \in 0_{n_0}$, and let J_{t_0,s_0} be an nice interval of f_{t_0,s_0}, and $\Lambda^\pm_{t_0,s_0}$ with $HD(\Lambda^\pm_{t_0,s_0}) > 1 - \delta$, for some δ small enough, as in Lemma 4.4.7.

Take any smooth curve ψ_0 passing through (t_0,s_0) so that for all $(t,s) \in \psi_0$ the tangent vector to the curve at this point having the direction satisfying the relation (4.23). Due to Lemma 4.4.7, we have that under the curve ψ_0 there is a set of points Γ_0 (Figure 4.12), arbitrarily close to (t_0,s_0), such that for each $(t,s) \in \Gamma_0$, the singular value $f_{t,s}(c_-)$, for map $f_{t,s}$, do not accumulate in c. That is, we are considering the points (t,s) in Γ_0 which correspond to the moments when $f_{t,s}(c_-)$ reaches some point of $\Lambda^-_{t,s}$ continuation of $\Lambda^-_{t_0,s_0}$.

We will denote the points of Γ_0, by (t_i,s_i). Note that for each (t_i,s_i) we have the nice interval $J_{t_i,s_i} = J(a(t_i,s_i), b(t_i,s_i))$ which is the continuation of $J_{t_0,s_0} = J(a(t_0,s_0), b(t_0,s_0))$ according to the Lemma 4.4.6. And, so the itinerary of $a(t_i,s_i)$ is the same for all i, as well as that of $b(t_i,s_i)$.

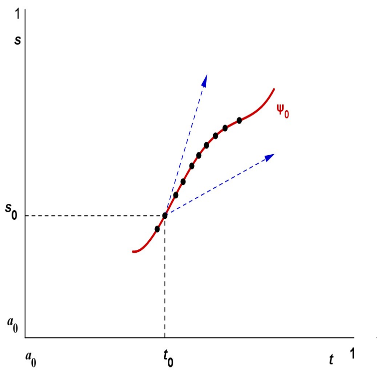


Figure 4.12: Points $(t,s) \in \Gamma_0$ on the curve ψ_0 with $c \notin \omega_{f_{t,s}}(f_{t,s}(c_-))$.

Furthermore, for each $(t_i, s_i) \in \Gamma_0$ we get $f_{t_i,s_i}(c_-) = t_i$ do not accumulate in c, i.e., $\phi_\alpha(t_i, s_i) = p(t_i, s_i) \in \Lambda^-_{t_i,s_i}$, where $\alpha \in 1_{n_0}$ and $f_{t_i,s_i}(c_-) = p(t_i, s_i) = t_i$.

For each point (t_i, s_i) we will fix the itinerary of t_i, provided by the map f_{t_i,s_i}, and denote by α_i. So, we know, by lexicographical order, seen in the Section 4.3, that for the nice intervals $J_{t,s}$ continuation of J_{t_0,s_0}, the points that have this itinerary α_i its future orbit avoid $J_{t,s}$.

We will find a smooth curve ψ_i passing through (t_i, s_i), so that all points $(t,s) \in \psi_i$ will have the itinerary of t, provided by the map $f_{t,s}$, being the same of t_i.

In fact, let

$$\zeta_i(t,s) = \phi_{\alpha_i}(t,s) - t.$$

Then, we have

$$\zeta_i(t_i, s_i) = \phi_{\alpha_i}(t_i, s_i) - t_i = 0,$$

and as, $\alpha_i \in 1_{n_0}$, and $t_i, s_i \in (a_0, 1)$, because they are close enough to t_0, s_0, respectively, we have by the item (ii) of the Proposition 4.4.3, that

$$\partial_s \zeta_i(t_i, s_i) = \partial_s \phi_{\alpha_i}(t_i, s_i) \neq 0.$$

Therefore, by the Implicit Function Theorem, we have that for each i, there is $t_i \in B_i$ such that we will have $s = g_i(t)$, for $t \in B_i$ (for simplicity we write only $g(t)$) and

$$\phi_{\alpha_i}(t, g(t)) = t \text{ with } g(t_i) = s_i.$$

Thus, we obtain the curve ψ_i passing through (t_i, s_i) as the graph of the function g in domain B_i.

Note that the tangent vector to the curve ψ_i at point (t_i, s_i) may not satisfy condition (4.23)(see Figure 4.13). In fact, we will have

$$\partial_t \phi_{\alpha_i}(t, g(t)) + \partial_s \phi_{\alpha_i}(t, g(t)).g'(t) = 1$$

thus,

$$g'(t_i) = \frac{1 - \partial_t \phi_{\alpha_i}(t_i, s_i)}{\partial_s \phi_{\alpha_i}(t_i, s_i)} \tag{4.26}$$

and, as by items (i) and (ii) of the Proposition 4.4.3, we have

$$-\sigma < \partial_t \phi_{\alpha_i}(t_i, s_i) < 0 \text{ and } 0 < \partial_s \phi_{\alpha_i}(t_i, s_i) < \sigma$$

follow that

$$g'(t_i) > \frac{1}{\sigma} > \frac{\sigma}{\sqrt{1-\sigma^2}}. \tag{4.27}$$

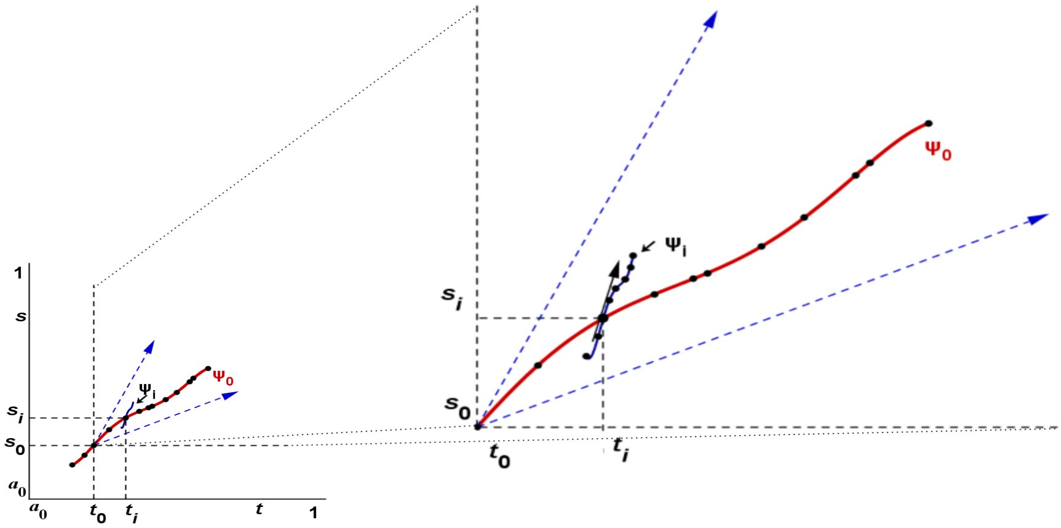


Figure 4.13: Curve ψ_i and points $(t, s) \in \Gamma_i$.

However, as we now want the singular values of the left $f(t, s(t), c_+) = f_{t,s(t)}(c_+) = 1 - s(t)$ to reach the points at $\Lambda^+_{t,s(t)}$, we will use the estimate (4.21) at the point (t_i, s_i).

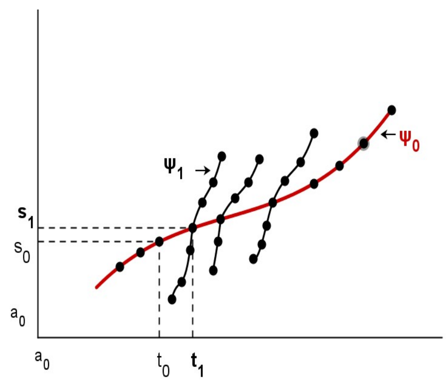


Figure 4.14: Points (t, s) on curves, with the map $f_{t,s}$ being Misiurewicz.

So, noting that $s_i \in (a_0, 1)$ with $1 - s_i \in 0_{n_0}$, and $\Lambda^+_{t_i,s_i}$ is arbitrarily close to $f_{t_i,s_i}(c_+)$, we have according to Remark 4.4.4 and estimate (4.21), that by varying (t, s) in ψ_i we will have the singular values $f_{t,s(t)}(c_+)$ crossing the continuation of $\Lambda^+_{t_i,s_i}$.

Then for each i, along the smooth curve ψ_i, arbitrarily close to (t_i, s_i), there is a set Γ_i of points $(t, s(t))$, such that $f_{t,s(t)}(c_+) = 1 - s(t) \in \Lambda_{t,s(t)}$ and also $f_{t,s(t)}(c_-) = t \in \Lambda_{t,s(t)}$. Therefore, $f_{t,s(t)}(c_+)$ and $f_{t,s(t)}(c_-)$ do not accumulate in c, by map $f_{t,s(t)}$.

□

Remark 4.4.9. *Note that for* (t_0, s_0) *obtained as in Proposition 4.4.3, considering the cone* Γ' *with the origin in* (t_0, s_0) *and obtained satisfying estimate* (4.23)*, from any point* $(t, s) \in \Gamma'$ *we can consider the set* $\Lambda_{t,s}$ *continuation of* Λ_{t_0,s_0} *given by Lemma 4.4.6, and then we can use the same reasoning as in Theorem 2 for a curve* ψ_0 *passing through* (t, s) *and satisfying estimate* (4.23).

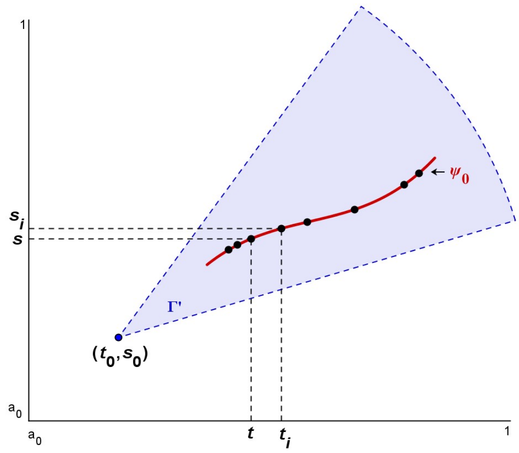


Figure 4.15: Curve ψ_0 in the cone.

4.5 Parameter Space Hausdorff Dimension

We will see in this section that the set Γ_0 and the sets Γ_i of parameters, obtained in Theorem 2, of the previous section, have a positive Hausdorff dimension in the curves ψ_0 and ψ_i, respectively. We saw that the maps associated to these sets Γ_i, has its singular values not accumulating in c, since each set of points $\Gamma_i = \{(t, s(t))\}$ found in each curve ψ_i, was determined in this way. While in the case of set Γ_0, we only guarantee the singular value of the right not accumulating in c.

In order to obtain estimates of the Hausdorff dimension in the parameter space, we will construct a bi-Hölder continuous map that projects the points of the phase space into the parameter space.

We know, from the hyperbolic continuation theorem, that there is a homeomorphism that conjugates the hyperbolic sets that are close, but in addition, we have that this map is bi-Hölder continuous, as we can see in the following theorem.

Theorem 3. *(see, for example, [KH] Theorem 19.1.2) Let Λ and $\tilde{\Lambda}$ be compact hyperbolic sets for diffeomorphisms uniformly expanding maps f and $\tilde{f}$, respectively, and $h = \tilde{f}hf^{-1} : \Lambda \to \tilde{\Lambda}$ a topological conjugacy. Then both h and h^{-1} are Hölder continuous.*

Thus, fix $f = f_{t_0,s_0}$, for some t_0 and s_0 as in Theorem 2 of the previous section, and let $\Lambda_0 = \Lambda_{t_0,s_0}$ be a compact hyperbolic invariant set for f, with $HD(\Lambda_0) > 1-\delta$, for some δ small enough. There is $\epsilon > 0$ such that, for each $t \in (t_0-\epsilon, t_0+\epsilon)$, there is a $\Lambda_{t,s(t)} \subset [0,1]$ compact hyperbolic invariant set for $f_{t,s(t)}$ and there is a bi-Hölder continuous map

$$\begin{aligned} \gamma_t : \Lambda_0 \subset [0,1] &\to \Lambda_{t,s(t)} \subset [0,1] \\ p &\mapsto \gamma_t(p) = \gamma(t,p) \end{aligned}$$

where $\gamma_{t_0}(p) = p$ for all $p \in \Lambda_0$. In addition, we have that $\dfrac{\partial \gamma}{\partial t}$ exists and is continuous.

Note that for each fixed t, the map γ_t is an increasing map. In fact, given $p < q \in \Lambda_0$, we will have to $\gamma(t,p) < \gamma(t,q)$, as shown on the left side of the Figure 4.16, because otherwise there will be a $t_0 < t_1 < t$, so that $\gamma(t_1,p) = \gamma(t_1,q)$ (see right side of Figure 4.16) and then γ_{t_1} would not be a conjugation between sets Λ_0 and $\Lambda_{t_1,s(t_1)}$.

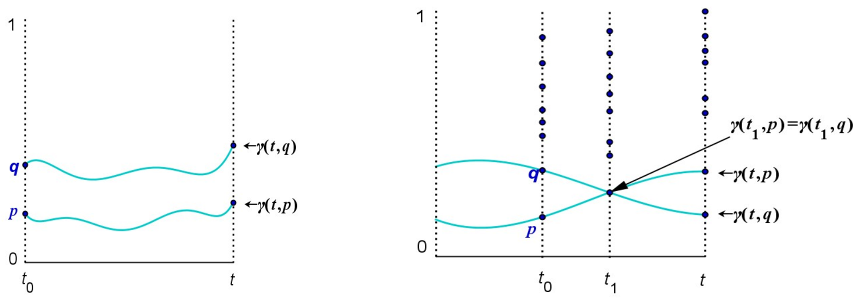


Figure 4.16: The map γ_t is an increasing map.

We define the map

$$\begin{aligned} \Psi : (t_0-\epsilon, t_0+\epsilon) \times \Lambda_0 &\to (t_0-\epsilon, t_0+\epsilon) \times [0,1] \\ (t,p) &\mapsto \Psi(t,p) := (t, \gamma(t,p)) \end{aligned}$$

(see Figure 4.17) and we will get an extension $\widetilde{\Psi}$, from Ψ , in the domain $(t_0-\epsilon, t_0+\epsilon) \times [0,1]$.

For this, we define an extension $\widetilde{\gamma}_t$, of γ_t, at the points of the gaps of the set Λ_0, as follows. If (a,b) is a connected component of $[0,1] \setminus \Lambda_0$ and $x \in (a,b)$, we have

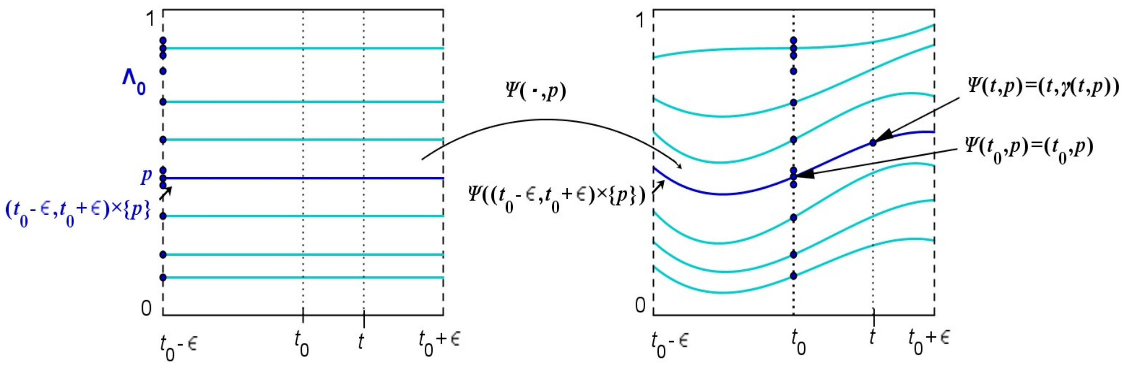


Figure 4.17: Map $\Psi(\cdot, p)$ between local leaves for points p of Λ_0.

$x = (1-r)a + rb$ for some $r \in (0, 1)$, then we will define

$$\widetilde{\gamma}_t(x) = \widetilde{\gamma}(t, x) = (1-r)\gamma(t, a) + r\gamma(t, b).$$

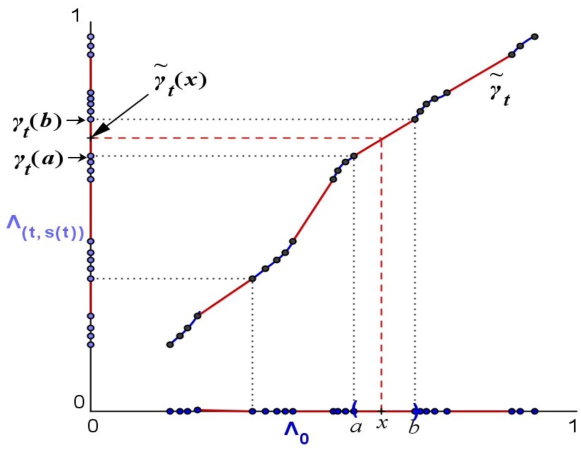


Figure 4.18: Map $\tilde{\gamma}_t$ extension of map γ_t in the points of the gaps.

So, as defined, we have that for each t, $\widetilde{\gamma}_t$ is also a continuous increasing map, therefore injective (see Figure 4.18). And then we will have for any two distinct points $x \neq y \in (a, b)$, the curves $\widetilde{\gamma}(t, x)$ and $\widetilde{\gamma}(t, y)$ with t varying in $(t_0 - \epsilon, t_0 + \epsilon)$, are differentiable curves that do not intersect.

We can also observe that $\widetilde{\gamma}_t$ remains bi-Hölder. Indeed, for fixed t, we have $\widetilde{\gamma}_t\mid_{\Lambda_0} = \gamma_t$ and let us denoted $g(x) = \widetilde{\gamma}(t, x)$. As $a, b \in \Lambda_0$ and $g\mid_{\Lambda_0}$ is Hölder continuous map, we have that there are $C > 0$ and $\alpha \in (0, 1)$ so that

$$|g(a) - g(b)| \leq C|a - b|^{\alpha}.$$

For $x < y \in (a, b)$, we have

$$x = (1 - r_1)a + r_1 b \text{ and } y = (1 - r_2)a + r_2 b,$$

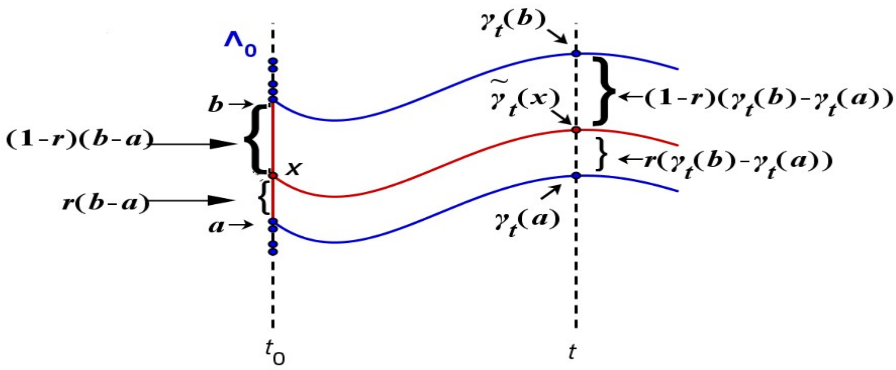


Figure 4.19: The curve $\widetilde{\gamma}(t,x)$ for t varying in $(t_0-\epsilon, t_0+\epsilon)$.

for some $r_1 < r_2 \in (0,1)$, thus,

$$g(x) = (1-r_1)g(a) + r_1 g(b) \text{ and } g(y) = (1-r_2)g(a) + r_2 g(b),$$

what gives,

$$\begin{aligned} |g(x)-g(y)| &= (r_2-r_1)|g(a)-g(b)| \\ &\leq (r_2-r_1)C|a-b|^{\alpha} \\ &\leq (r_2-r_1)^{1-\alpha}C|x-y|^{\alpha} \\ &< C|x-y|^{\alpha}. \end{aligned}$$

and then it follows.

Therefore, the map $\widetilde{\Psi}$, given by

$$\begin{aligned} \widetilde{\Psi} : (t_0-\epsilon, t_0+\epsilon) \times [0,1] &\rightarrow (t_0-\epsilon, t_0+\epsilon) \times [0,1] \\ (t,x) &\mapsto \widetilde{\Psi}(t,x) := (t, \widetilde{\gamma}(t,x)) \end{aligned}$$

is a bi-Hölder continuous mapping in each of the variables. And, for each $x \in [0,1]$, the curves $\widetilde{\Psi}((t_0-\epsilon, t_0+\epsilon) \times \{x\})$ define a foliation $\mathfrak{F}$ of $(t_0-\epsilon, t_0+\epsilon) \times [0,1]$, where these ones are the local leaves and $\widetilde{\Psi}(\{t\} \times [0,1])$ are the transverse leaves. So, we have a change of coordinates that rectifies the local leaves (Figure 4.20).

Initially we will discuss the case for the set Γ_0 contained in the curve $\psi_0(t) = (t, s(t))$, in the parameter space. We saw that for f_{t_0,s_0}, there is a subset of points of Λ_0, arbitrarily close to $f_{t_0,s_0}(c_-) = t_0$, having Hausdorff dimension greater than $1-\delta$, and that without loss of generality we will continue to denote by Λ_0. So that when we perturb f_{t_0,s_0} with (t,s) on the curve ψ_0, the singular values $f_{t,s(t)}(c_-) = t$ will cross $\Lambda_{t,s(t)}$, which is the continuation of Λ_0 for the maps $f_{t,s(t)}$, and that the points of Γ_0 corresponded to the moment when $f_{t,s(t)}(c_-)$ reached some point of $\Lambda_{t,s(t)}$.

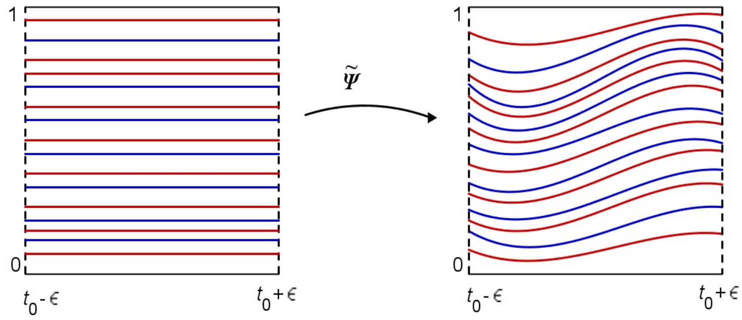


Figure 4.20: Map $\widetilde{\Psi}$ extension of map Ψ.

We will now restrict the map $\widetilde{\Psi}$ to the set $(t_0-\epsilon, t_0+\epsilon)\times(t_0-\epsilon, t_0+\epsilon)$ (see Figure 4.21), in order to obtain an estimate of the Hausdorff dimension in the parameter space, of the Misiurewicz maps obtained.

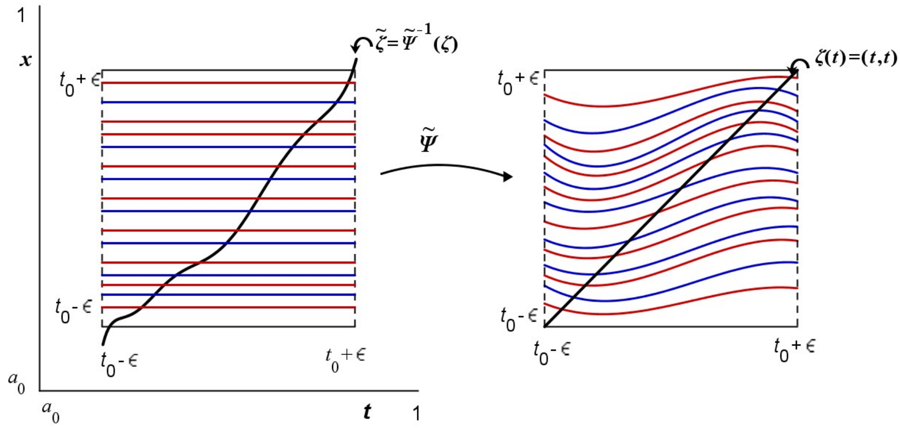


Figure 4.21: The map $\widetilde{\Psi}$ is restricted to $(t_0-\epsilon, t_0+\epsilon)\times(t_0-\epsilon, t_0+\epsilon)$.

Now, denoting by ζ the straight line $f_{t,s(t)}(c_-) = t$, i.e., $\zeta(t) = (t,t)$, we have that ζ is transversal to the foliation, i.e., each local leaf intersects ζ in only one point, and then the change of coordinates $\widetilde{\Psi}^{-1}$, gives us $\tilde{\zeta} = \widetilde{\Psi}^{-1}(\zeta) = \{(t, \widetilde{\gamma}^{-1}(t,t))\}$ a transverse C^r curve intersecting each rectilinear leaf $(t_0-\epsilon, t_0+\epsilon)\times\{x\}$, for $x \in (t_0-\epsilon, t_0+\epsilon)$, at a single point. Note that we are only interested in the points of intersection to which $x \in \Lambda_0$, i.e., $\widetilde{\gamma}^{-1}(t,t) = \widetilde{\gamma}_t^{-1}(t) \in \Lambda_0$ (Figure 4.22).

Let $T_0 = \{t; \widetilde{\gamma}_t^{-1}(t) \in \Lambda_0\}$ and consider $F(t) = \widetilde{\gamma}^{-1}(t,t)$. We have that F is a Hölder map and $F(T_0) = \Lambda_0$, thus, as $HD(\Lambda_0) > 1-\delta > 0$ it follows from Proposition 4.2.1 that $HD(T_0) > 0$. In addition, we can see in the parameter space that, $\psi_0(T_0) = \Gamma_0$ and then by Proposition 4.2.2, it follows that $HD(\Gamma_0) > 0$.

Now fixed i, we will see the case for the set of points Γ_i on the curve ψ_i. Remember that we have ψ_i defined in B_i and that for every $t \in B_i$, any point on the curve $\psi_i(t) =$

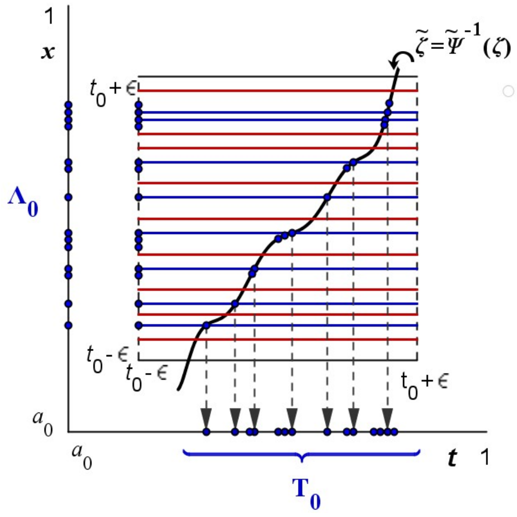


Figure 4.22: Hausdorff dimension of the projection on the axis t.

$(t, s(t))$, obtained in Theorem 2, has $f_{t,s(t)}(c_-) \in \Lambda_{t,s(t)}$.

However, only the points $(t, s(t)) \in \Gamma_i$ also have $f_{t,s(t)}(c_+) = 1 - s(t) \in \Lambda_{t,s(t)}$. In addition, we know that $s(t)$ was obtained by a diffeomorphism g from B_i to $g(B_i) = I_i$ ($g'(t) > 0$, for any $t \in B_i$ see inequality (4.27)). Thus, there is a set $S_i \subset I_i$ so that, for each $s \in I_i$, taking $\overline{\psi}_i(s) = (g^{-1}(s), s) = (t(s), s)$, we have $\overline{\psi}_i(S_i) = \Gamma_i$ (see Figure 4.23).

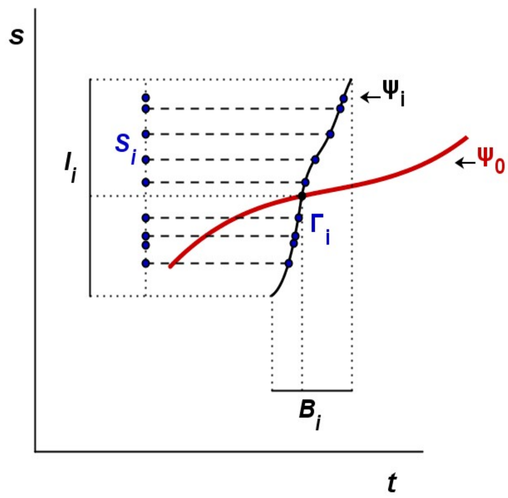


Figure 4.23: S_i projection of Γ_i on the axis s.

Constructing a foliation as before, given by $\widetilde{\Psi}$ in $I_i \times [0, 1]$, and observing the continuation of the set Λ_{t_i,s_i} arbitrarily close to $f_{t_i,s_i}(c_+) = 1 - s_i$, having Hausdorff dimension greater than $1 - \delta$, we can restrict the map $\widetilde{\Psi}$ to the set $(a_i, b_i) \times (1 - b_i, 1 - a_i)$, with

$I_i = (a_i, b_i)$. Thus, we will have the curve $\zeta(s) = (s, f_{t(s),s}(c_+)) = (s, 1-s)$ transversal the restricted foliation and then we get $\tilde{\zeta} = \widetilde{\Psi}^{-1}(\zeta)$ (see Figure 4.24). Proceeding in the same way as in T_0, we find for $S_i \subset I_i$, $HD(S_i) > 0$ and therefore $HD(\Gamma_i) = HD(\overline{\psi}_i(S_i)) > 0$ (Figure 4.25).

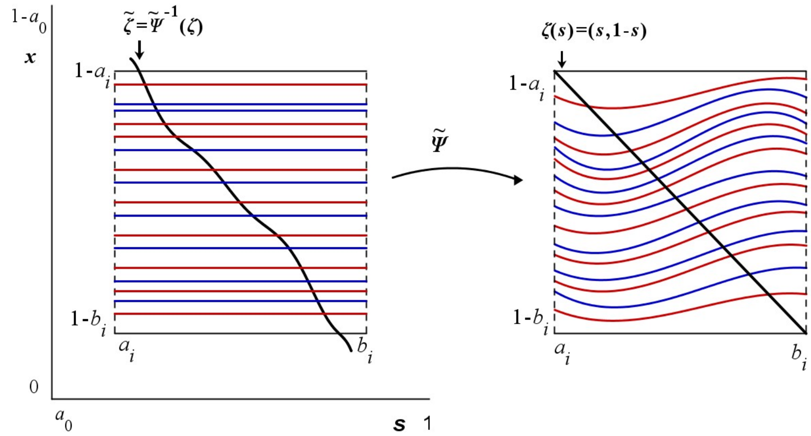


Figure 4.24: The map $\widetilde{\Psi}$ is restricted to $(a_i, b_i) \times (1 - b_i, 1 - a_i)$.

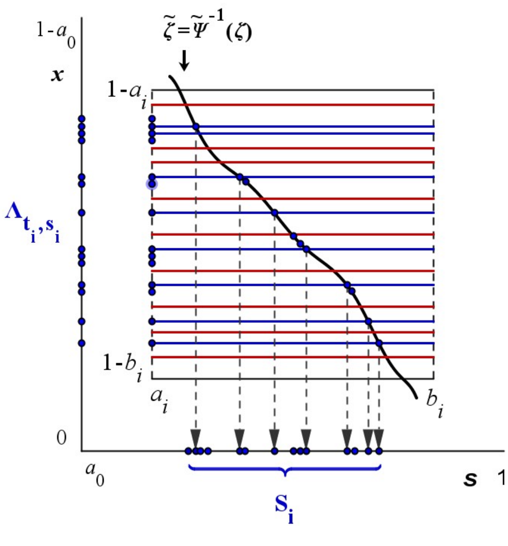


Figure 4.25: Hausdorff dimension of the projection on the axis s.

To finish the proof of Theorem D, note that for each $(t, s) \in \Gamma_i$, $c \notin \omega_{f_{t,s}}(f_{t,s}(c_\pm))$, so for each $(t, s) \in \Gamma_i$, $f_{t,s}(c_\pm)$ satisfies condition (3) of Theorem B and therefore it follows from Theorem B that for each $(t, s) \in \Gamma_i$, there is a constant $C_{t,s} > 0$ such that the Lyapunov exponent of every ergodic $f_{t,s}$-invariant probability $\mu_{t,s}$ is bounded by $C_{t,s}$.

Chapter 5

Thermodynamic formalism of non-uniformly expanding maps

In this chapter we will see the proof of Theorem E. We will see that for every (t, s) belonging to the set Γ_i, for each i, obtained in Theorem D, of the previous chapter, the Lorenz map $f_{t,s}$ has a unique equilibrium state for any given Hölder potential.

The existence of measures of equilibrium states is an important ingredient to be considered in a dynamics, because due to the variational principle, it will give us information about the entropy of the system and the topological entropy is a way to measure the rate of complexity of a dynamic system.

We will recall some results succinctly, for more details see for example [OV].

Theorem 4. *(Variational Principle) If $f : \mathbb{X} \to \mathbb{X}$ is a continuous map in a compact metric space then its topological entropy $h(f)$ coincides with the supremum metric entropy, $h_\mu(f)$, of the map f for all invariant probabilities.*

In addition, for a continuous potential $\phi : \mathbb{X} \to \mathbb{R}$, the variational principle is generalized to the context of pressure $P(f, \phi)$, i.e.,

$$P(f, \phi) = \sup\{h_\nu(f) + \int \phi d\nu : \nu \in \mathcal{M}^1(f)\}. \tag{5.1}$$

And then, an invariant probability μ is called *equilibrium states* for potential ϕ if it achieve the supremum in (5.1), that is,

$$h_\mu(f) + \int \phi d\mu = \sup\{h_\nu(f) + \int \phi d\nu : \nu \in \mathcal{M}^1(f)\}.$$

In systems containing critical or singular points for the case n.u.e. and potential Hölder, Pinheiro and Varandas obtained results for the existence of equilibrium states, Theorem 5. This result applies to our context, guaranteeing the existence of a unique equilibrium state for each map obtained in Theorem D.

Let $(\mathbb{X}, d)$ be a metric space, $A \subset \mathbb{X}$ an open set and $f : A \to \mathbb{X}$ be a continuous map. According to [Pi11], a ***zooming contraction*** is a sequence $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ of functions $\alpha_n \colon [0, +\infty) \to [0, +\infty)$ satisfying for every $m, n \in \mathbb{N}$ and $r > 0$ the conditions: (i) $\alpha_n(r) < r$, (ii) $\alpha_n(r) \le \alpha_n(r')$ whenever $r \le r'$, (iii) $\alpha_n \circ \alpha_m(r) \le \alpha_{m+n}(r)$ and (iv) $\sup_{r\in[0,1]} \sum_{n\ge 1} \alpha_n(r) < \infty$.

A zooming contraction $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ is called ***exponential*** if $\alpha_n(r) = e^{-\lambda n} r$ for some $\lambda > 0$. A zooming contraction $\alpha = \{\alpha_n(r)\}_n$ is called ***Lipschitz*** if $\alpha_n(r) = a_n r$ for some sequence a_n. In particular, all exponential zooming contraction is Lipschitz.

Following [PV], given a zooming contraction $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$, $\ell \in \mathbb{N}$ and $\delta > 0$, we say that n is a (α, δ, ℓ)***-zooming time*** for a point p when exists an open neighborhood $V_n(\alpha, \delta, \ell)(p)$ of p such that $f^{\ell n} : V_n(\alpha, \delta, \ell)(p) \to B_\delta(f^{\ell n}(p))$ is a homeomorphism that extends continuously to the boundary, and

$$d(f^{\ell j}(x), f^{\ell j}(y)) \le \alpha_{n-j}(d(f^{\ell n}(x), f^{\ell n}(y))), \quad \forall\, 0 \le j \le n-1 \text{ and } x, y \in V_n(\alpha, \delta, \ell)(p).$$

The neighborhood $V_n(\alpha, \delta, \ell)(p)$ is called a (α, δ, ℓ)***-zooming pre-ball*** of order n and $B_\delta(f^{\ell n}(p))$ is called a ***zooming ball***. The set of points having n as a (α, δ, ℓ)-zooming time is denoted by $\mathcal{Z}_n(\alpha, \delta, \ell)$.

A forward invariant measurable set Λ is called (α, δ, ℓ)***-weak zooming*** if

$$\Lambda \subset \limsup_{n\to+\infty} \mathcal{Z}_n(\alpha, \delta, \ell) := \bigcap_{k\ge 1} \bigcup_{n\ge k} \mathcal{Z}_n(\alpha, \delta.\ell).$$

The map f is called ***weak topologically mixing*** on a forward invariant set U whenever $U \times U \ni (x, y) \mapsto (f(x), f(y)) \in U \times U$ is transitive.

Definition 5.0.1 (Non-uniformly expanding set [PV])**.** *A set $U \subset \mathbb{X}$ is called a (α, δ)-**expanding set** for some exponential zooming contraction α and $\delta > 0$ if*

1. *U is a nonempty open set;*
2. *$f(U \cap A) \subset U \subset \overline{\limsup_n \mathcal{Z}_n(\alpha, \delta, 1)}$;*
3. *$U \subset \alpha_f(x)$ for every $x \in U$.*

*A set $U \subset \mathbb{X}$ is called **non-uniformly expanding** if U is open, forward invariant, weak topologically mixing and (α, δ)-expanding for some exponential zooming contraction α and $\delta > 0$.*

Let μ be a f-invariant Borel probability, α a zooming contraction and $\delta > 0$. Given $\ell \in \mathbb{N}$, we say that μ is a (α, δ, ℓ)***-weak zooming*** if $\mu\big(\mathbb{X} \setminus \limsup_{n\to+\infty} \mathcal{Z}_n(\alpha, \delta, \ell)\big) = 0$. If $\limsup_{n\to+\infty} \frac{1}{n} \#\{1 \le j \le n\,;, x \in \mathcal{Z}_n(\alpha, \delta, \ell)\} > 0$ for μ almost every $x \in \mathbb{X}$, we say that

μ is a (α, δ, ℓ)***-zooming measure***. A ***zooming*** measure is (α, δ, ℓ)-zooming measure for some zooming contraction α, $\delta > 0$ and $\ell \in \mathbb{N}$.

Given $\ell \in \mathbb{N}$, let $\mathcal{E}(f, \ell)$ be the set of all invariant (α, δ, ℓ)-zooming probability for some exponential zooming contraction α and $\delta > 0$. The set ***exponential zooming probabilities*** is defined as

$$\mathcal{E}(f) = \bigcup_{\ell \in \mathbb{N}} \mathcal{E}(f, \ell) \subset \mathcal{M}_f^1(\mathbb{X}).$$

Theorem 5 ([PV])**.** *Let $\mathbb{X}$ be a compact metric space and $\mathcal{C} \subset M$ a compact set with empty interior. Let $f : \mathbb{X} \setminus \mathcal{C} \to \mathbb{X}$ be a local bi-Lipchitz homeomorphism with $\# f^{-1}(x) < +\infty$ for every $x \in \mathbb{X}$. If $U \subset \mathbb{X}$ is non-uniformly expanding set and ϕ is a Hölder potential then there exists at most one $\mu \in \mathcal{E}(f)$ such that*

$$h_\mu(f) + \int \phi d\mu = \sup \left\{ h_\nu(f) + \int \phi d\nu \,;\, \nu \in \mathcal{E}(f) \right\}.$$

Let $f : [0,1] \setminus \{c\} \to [0,1]$, $0 < c < 1$, be an expanding Lorenz map. It follows from Lemma B.2 of [Pi20] that every ergodic f-invariant probability ν with finite Lyapunov exponent (i.e., $\int |\log |f'||d\nu < +\infty$) belongs to $\mathcal{E}(f)$. Moreover, as f admits an absolutely continuous invariant probability μ with $\operatorname{supp} \mu = [f(c_+), f(c_-)]$ and $\int |\log |f'||d\mu < +\infty$, we get that f is an non-uniformly expanding map, according to Definition 5.0.1. Therefore, we can use Theorem B to obtain the following theorem.

Theorem 6. *Let $f : [0,1] \setminus \{c\} \to [0,1]$ be a non-flat C^2 expanding Lorenz map with singular point $c \in (0,1)$. If*

$$limsup_n \frac{1}{n} \sum_{j=1}^{n} |\log |f^j(c_\pm) - c|| < +\infty$$

then f has one and only one equilibrium state for any given Hölder potential φ. In particular, if $f(c_\pm) \in B(\mu)$, then f has a unique equilibrium state for any potential Hölder.

5.1 Proof of Theorem E.

Theorem D assures us that, for each $(t,s) \in \Gamma_i$, there is a constant $C_{t,s} > 0$, so that, every ergodic $f_{t,s}$-invariant probability has Lyapunov exponent is bounded by $C_{t,s}$, i.e., has finite Lyapunov exponent. And so, it follows from Theorem 5, that for each $(t,s) \in \Gamma_i$, $f_{t,s}$ has a unique equilibrium state for any given Hölder potential $\varphi : [0,1] \to \mathbb{R}$. And then the Theorem E is established.

Chapter 6

Measures with fast-recurrence to the singularity of a expanding Lorenz map

In this chapter we will prove Theorem A, that is, we will see that there are expanding Lorenz maps having many ergodic measures with infinite Lyapunov exponent, whose entropy is positive and full support.

Let $f : [0,1] \setminus \{c\} \to [0,1]$, $0 < c < 1$, be a C^1 non-flat expanding Lorenz map. Let $\mathcal{M}^1(f)$ the set of all f-invariant Borel probabilities. A f-induced map defined on $A \subset [0,1] \setminus \{c\}$ with an induced time $R : A \to \mathbb{N} := \{1,2,3,\cdots\}$ is the map $F : A \to [0,1]$ defined by $F(x) = f^{R(x)}(x)$. An induced time $R : A \to \mathbb{N}$ is called ***exact*** if $R(x) = R(f^j(x)) + j$ for every $0 \le j < R(x)$. An induced map F is called ***orbit-coherent*** if $\mathcal{O}_f^+(x) \cap \mathcal{O}_f^+(y) \neq \emptyset \iff \mathcal{O}_F^+(x) \cap \mathcal{O}_F^+(y) \neq \emptyset$ for every $x, y \in \bigcap_{j \ge 0} F^{-j}([0,1])$.

A ***full induced Markov map*** is a triple $(F, B, \mathcal{P})$ where $B \subset [0,1] \setminus \{c\}$ is a connected open set, $\mathcal{P}$ is countable collection of disjoints open subsets of B and $F : A := \bigcup_{P \in \mathcal{P}} P \to B$ is an f induced map satisfying:

1. for each $P \in \mathcal{P}$, $F|_P$ is a diffeomorphism between P and B and it can be extended to a homeomorphism sending $\overline{P}$ onto $\overline{B}$;

2. $\lim_n \operatorname{diameter}(\mathcal{P}_n(x)) = 0$ for every $x \in \bigcap_{n \ge 1} F^{-n}(B)$, where $\mathcal{P}_n = \bigvee_{j=0}^{n-1} F^{-j}(\mathcal{P})$ and $\mathcal{P}_n(x)$ is the element of $\mathcal{P}_n$ containing x.

A ***mass distribution*** on $\mathcal{P}$ is a map $m : \mathcal{P} \to [0,1]$ such that $\sum_{P \in \mathcal{P}} m(P) = 1$. The ***$F$-invariant probability μ generated by the mass distribution*** m is the ergodic F invariant probability μ defined by

$$\mu(P_1 \cap F^{-1}(P_2) \cap \cdots \cap F^{n-1}(P_n)) = m(P_1)m(P_2)\cdots m(P_n),$$

where $P_1 \cap F^{-1}(P_2) \cap \cdots \cap F^{n-1}(P_n) \in \bigvee_{j=0}^{n-1} F^{-j}(\mathcal{P})$.

Lemma 6.0.1 (Lemma C.1 of [Pi20])**.** *Let f be a measure preserving automorphism on a probability space $(\mathbb{X}, \mathfrak{A}, \mu)$ and $F : A \subset \mathbb{X} \to \mathbb{X}$ a measurable induced map with induced time $R : A \to \mathbb{N}$. Suppose that μ is f ergodic and that ν is a F-lift of μ. If R is exact and $\mu(A) = 1$ then*

$$\frac{1}{2}\int R d\nu \int R d\mu \leq \int (R)^2 d\nu \leq 2\int R d\nu \int R d\mu.$$

Lemma 6.0.2 (Folklore result)**.** *If $F : A \to \mathbb{X}$ is a measurable f induced map with induced time R and ν is a F invariant probability then*

$$\mu := \sum_{n\geq 1}\sum_{j=0}^{n-1} f_*^j(\nu|_{\{R=n\}}) = \sum_{j\geq 0} f_*^j(\nu|_{\{R>j\}})$$

is a f invariant measure with $\mu(\mathbb{X}) = \int R d\nu$.

6.1 Proof of Theorem A

Proof. Let $p \in (c, c+r)$ be a periodic point such that $p = \min(\mathcal{O}_f^+(p) \cap (c,1)) < \min(\mathcal{O}_f^+(c_+) \cap (c,1))$. Let $J = (c,p)$ and note that $\mathcal{O}_f^+(\partial J) \cap J = \emptyset$. Let $A = \{x \in J\,;\, \mathcal{O}_f^+(f(x)) \cap J \neq \emptyset\}$, $R : A \to \mathbb{N}$ be the first return time to J and $F : A \to J$ the first return map, i.e., $F(x) = f^{R(x)}$.

Let $\mathcal{P}$ be the set of connected components of A.

Claim 1. *$F(P) = (c,p)$ for every $P \in \mathcal{P}$.*

Proof of the claim. Indeed, given $x \in A$, let $I = (a,b) \ni x$ be the maximal open interval such that $f^{R(x)}|_I$ is a homeomorphism and $f^{R(x)}(I) \subset J$. If $f^{R(x)}(I) \neq J$, then $f^{R(x)}(a_+) \in J$ or $f^{R(x)}(b_-) \in J$. Suppose for instance that $f^{R(x)}(a_+) \in J$. If $f^j(a_+) = c$ for some $1 \leq j < R(x)$ then $f^{R(x)-j}(c_+) = f^{R(x)}(a_+) \in J$ which is a contradiction with the definition of p. So, $f^j(a_+) \neq c$ for every $0 \leq j < R(x)$ and in this case, there is $\delta > 0$ such that $f^{R(x)}|_{B_\delta(a)}$ is a homeomorphism and $f^{R(x)}(B_\delta(a)) \subset J$, but this implies that $f^{R(x)}|_{(a-\delta,b)}$ is a homeomorphism with $f^{R(x)}((a-\delta,b)) \subset J$, contradicting the definition of $I(x)$. Now suppose that $f^{R(x)}(b_-) \in J$. If $f^j(b_-) = c$ for some $1 \leq j < R(x)$ then $f^{R(x)-j}(c_-) = f^{R(x)}(b_-) \in J$, contradicting that $\mathcal{O}_f^+(c_-) \cap (c, c+r) = \emptyset$. So, there is $\delta > 0$ such that $f^{R(x)}|_{B_\delta(b)}$ is a homeomorphism and $f^{R(x)}(B_\delta(b)) \subset J$, but this implies that $f^{R(x)}|_{(a,b+\delta)}$ is a homeomorphism with $f^{R(x)}((a,b+\delta)) \subset J$, contradicting the definition of $I(x)$. Hence, we must have $f^{R(x)}(I(x)) = (c,p)$, proving the claim. □

Claim 2. *$\exists q > c$ so that $(c,q) \in \mathcal{P}$ and $R((c,q)) = t_0 := \min\{j \geq 1\,;\, f^j(c_+) = c\}$.*

Proof of the claim. it follows from the definition of t_0 that exists $\delta > 0$ such that $f^{t_0}|_{(c,c+\delta)}$ is a homeomorphism and $f^{t_0}((c, c+\delta)) = (c, f^{t_0}(c+\delta))$. Moreover, $f^j((c, c+\delta)) \cap J = \emptyset$ for every $1 \le j < t_0$. Hence, $(c, c+\delta) \subset A$. Hence, the connected component of A containing $(c, c+\delta)$ must be (c, q) for some $c + \delta \le q \le p$ and $R((c, q)) = t_0$. $\square$

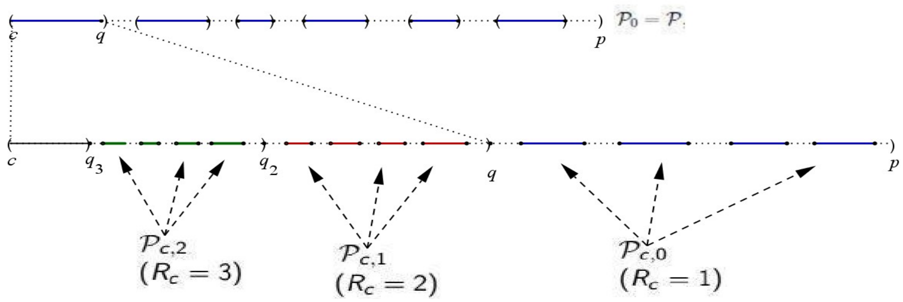
Let $P_0 := (c, q) \in \mathcal{P}$ be given by Claim 2. Let $\mathcal{P}_0 = \mathcal{P}$, $\mathcal{P}_n = \bigvee_{j=0}^{n} F^{-j}(\mathcal{P})$ for $n \ge 1$ and $\mathcal{P}_0(c) = J$, $\mathcal{P}_n(c) := (F_{P_0})^{-n}(J)$, that is, $\mathcal{P}_n(c)$ is the element of the partition $\mathcal{P}_{n-1}$ of J containing $(c, c+\delta_n)$ for some $\delta_n > 0$.

Let $R_c : A \to \mathbb{N}$ be an F-induced time given by

$$R_c(x) = \begin{cases} 1 & \text{if } x \in A \setminus P_0 \\ (n+1) & \text{if } x \in \mathcal{P}_n(c) \setminus \mathcal{P}_{n+1}(c) \text{ for } n \in \mathbb{N} \end{cases}$$

Set $\widetilde{A} = \bigcup_{n \ge 0} \left(\mathcal{P}_n(c) \setminus \mathcal{P}_{n+1}(c) \cap (\cap_{j=0}^{n} F^{-j}(A)) \right)$ and set $F_c : \widetilde{A} \to B$ by $F_c(x) = F^{R_c(x)}(x)$.

Let $\mathcal{P}_{c,0} = \mathcal{P}_0 \cap (J \setminus \mathcal{P}_1(c)) := \{P \in \mathcal{P}\,;\, P \subset J \setminus \mathcal{P}_1(c)\}$ and $\mathcal{P}_{c,n} = \mathcal{P}_n \cap (\mathcal{P}_n(c) \setminus \mathcal{P}_{n+1}(c))$ for $n \in \mathbb{N}$. Let $A^* = \{x \in J\,;\, \mathcal{O}_F^+(x) \cap (q, p) \cap A \neq \emptyset\}$. Note that $\mathcal{P}^* = \bigcup_{n \ge 0} \mathcal{P}_{c,n}$ is a partition of A^* and that $F_c(P) = J$ for every $P \in \mathcal{P}^*$.



Let μ be a f invariant ergodic probability with $\operatorname{supp} \mu = [f(c_+), f(c_-)]$. Note that $\mu_0 = \frac{1}{\mu((c,p))} \mu|_{(c,p)}$ is a F invariant ergodic probability and

$$\lim_n \frac{1}{n} \#\{0 \le j < n\,;\, F^j(x) \in \mathcal{O}_{F_c}^+(x)\} = \mu_0((q, p)) > 0 \tag{6.1}$$

for μ_0 almost every x. As F_c is orbit coherent F-induced map, it follows from Theorem A and B of [Pi20] that μ_0 has a unique F_c-lift μ_c and this μ_c is F_c-ergodic.

Given $\alpha \in (0, 1)$ and $\ell \in \mathbb{N}$, consider the mass distribution

$$m_\ell(P) = \begin{cases} \mu_c(P) & \text{if } P \in \bigcup_{j=0}^{\ell} \mathcal{P}_{c,j} \\ \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \frac{1}{(n-\ell)^{2+\alpha}} \frac{\mu_c(P)}{\mu_c(\mathcal{P}_n(c) \setminus \mathcal{P}_{n+1}(c))} & \text{if } P \in \mathcal{P}_{c,n} \text{ for } n > \ell \end{cases},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Note that $\sum_{P\in\mathcal{P}^*} m_\ell(P) = 1$. Moreover, taking

$$H(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\log(1/x) & \text{if } x > 0 \end{cases},$$

we have that

$$\sum_{n>\ell} R_c(P_{c,n}) m_\ell(P_{c,n}) = \sum_{n>\ell} (n+1)\, m_\ell(P_{c,n}) =$$

$$= \sum_{n>\ell} (n+1) \sum_{P\in\mathcal{P}_{c,n}} \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \frac{1}{(n-\ell)^{2+\alpha}} \frac{\mu_c(P)}{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))} =$$

$$= \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \sum_{n>\ell} \left(\frac{n+1}{(n-\ell)^{2+\alpha}} \frac{1}{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))} \underbrace{\sum_{P\in\mathcal{P}_{c,n}} \mu_c(P)}_{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))} \right) =$$

$$= \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \sum_{n>\ell} \frac{n+1}{(n-\ell)^{2+\alpha}} = \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \sum_{n=1} \frac{n+1+\ell}{n^{2+\alpha}} \leq$$

$$\leq \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} (\ell+2) \sum_{n=1} \frac{1}{n^{1+\alpha}} = (\ell+1)\mu_c(\mathcal{P}_{\ell+1}(c)) \left(\frac{\ell+2}{\ell+1} \frac{\zeta(1+\alpha)}{\zeta(2+\alpha)} \right) \leq$$

$$\leq 2(\ell+1)\mu_c(\mathcal{P}_{\ell+1}(c)) \frac{\zeta(1+\alpha)}{\zeta(2+\alpha)} \leq 2(\ell+1)\zeta(1+\alpha) < \infty$$

and

$$\sum_{n>\ell} H(m_\ell(P_{c,n})) = \sum_{n>\ell} m_\ell(P_{c,n}) \log(1/m_\ell(P_{c,n})) =$$

$$= \sum_{n>\ell} \sum_{P\in\mathcal{P}_{c,n}} H\left(\frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \frac{1}{(n-\ell)^{2+\alpha}} \frac{\mu_c(P)}{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))} \right) \leq$$

$$= \sum_{n>\ell} \sum_{P\in\mathcal{P}_{c,n}} H\left(\frac{1/\zeta(2+\alpha)}{(n-\ell)^{2+\alpha}} \right) \leq$$

$$\leq \sum_{j>\ell} \frac{\log(\zeta(2+\alpha)(n-\ell)^{2+\alpha})}{\zeta(2+\alpha)(n-\ell)^{2+\alpha}} =$$

$$= \frac{1}{\zeta(2+\alpha)} \sum_{n=1}^{+\infty} \frac{\log(\zeta(2+\alpha)n^{2+\alpha})}{n^{2+\alpha}} \leq$$

$$= \frac{\log(\zeta(2+\alpha))}{\zeta(2+\alpha)} \sum_{n=1}^{+\infty} \frac{1}{n^{2+\alpha}} + \frac{2+\alpha}{\zeta(2+\alpha)} \sum_{n=1}^{+\infty} \frac{\log(n)}{n^{2+\alpha}} =$$

$$= \log(\zeta(2+\alpha)) + \frac{2+\alpha}{\zeta(2+\alpha)} \sum_{n=1}^{+\infty} \frac{n}{n^{2+\alpha}} \leq$$

$$\leq \log(\zeta(2+\alpha)) + \frac{(2+\alpha)\zeta(1+\alpha)}{\zeta(2+\alpha)} \leq 3\zeta(1+\alpha) \leq 2(\ell+1)\zeta(1+\alpha) < \infty.$$

Taking $C = 2(\ell+1)\zeta(1+\alpha)$, we have that

$$\sum_{n>\ell}\sum_{P\in\mathcal{P}_{c,n}} R_c(P)\, m_\ell(P) \quad \text{and} \quad \sum_{n>\ell}\sum_{P\in\mathcal{P}_{c,n}} H(m_\ell(P)) \le C.$$

Hence,

$$\sum_{P\in\mathcal{P}^*} R_c(P)m_\ell(P) = \int_{\mathcal{P}_\ell(c)} R_c d\mu_c + \Big(\sum_{n>\ell}\sum_{P\in\mathcal{P}_{c,n}} (n+1)m_\ell(P)\Big) \le \int R_c d\mu_c + C < +\infty$$

and

$$\sum_{P\in\mathcal{P}^*} H(m_\ell(P)) = \Big(\sum_{n=0}^{\ell}\sum_{P\in\mathcal{P}_{c,n}} H(\mu_c(P))\Big) + \Big(\sum_{n>\ell}\sum_{P\in\mathcal{P}_{c,n}} H(m_\ell(P_{c,n}))\Big) \le$$

$$\le \Big(\sum_{P\in\mathcal{P}^*} H(\mu_c(P))\Big) + \Big(\sum_{n>\ell}\sum_{P\in\mathcal{P}_{c,n}} H(m_\ell(P_{c,n}))\Big) \le$$

$$\le h_{\mu_c}(F_c) + C < +\infty.$$

Thus, taking $\nu_{\alpha,\ell}$ as the ergodic F_c-invariant probability generated by the mass distribution m_ℓ, we get that,

$$h_{\nu_{\alpha,\ell}}(F_c) = \sum_{P\in\mathcal{P}^*} H(m_\ell(P)) < +\infty$$

and

$$\int R_c d\nu_{\alpha,\ell} = \sum_{P\in\mathcal{P}^*} R_c(P)\, m_\ell(P) \le \int R_c d\mu_c + C < +\infty.$$

It follows from Lemma 6.0.2 that

$$\eta_{\alpha,\ell} = \frac{1}{\int R_c d\nu_{\alpha,\ell}} \sum_{n\ge 1}\sum_{j=0}^{n-1} F_*^j(\nu_{\alpha,\ell}|_{\{R_c=n\}})$$

is an ergodic F invariant probability. Note that $\operatorname{supp}\eta_{\alpha,\ell} = [c,p]$. Moreover, as

$$F^j(\{R_c = n\}) \subset (c,q)$$

for every $0 \le j < n$ and $n \ge 2$, we get that

$$\eta_{\alpha,\ell}|_{(q,p)} = \frac{1}{\int R_c d\nu_{\alpha,\ell}} \nu_{\alpha,\ell}|_{(q,p)} \tag{6.2}$$

For similar reason, as μ_c is the F_c-lift of F-invariant probability μ_0, we have

$$\mu_0|_{(q,p)} = \frac{1}{\int R_c d\mu_c} \mu_c|_{(q,p)} \tag{6.3}$$

As $\int R_c d\nu_{\alpha,\ell}$ and $h_{\nu_{\alpha,\ell}}(F_c) < +\infty$, it follows from the generalized Abramov formula that

$$h_{\eta_{\alpha,\ell}}(F) = \frac{h_{\nu_{\alpha,\ell}}(F_c)}{\int R_c d\nu_{\alpha,\ell}}.$$

It is easy to see that R_c is a exact induced time. Hence, as A is the domain of R_c and $\eta_{\alpha,\ell}(A) = 1$, it follows from Lemma 6.0.1 that

$$\int R_c d\eta_{\alpha,\ell} \geq \frac{1}{2}\Big(\int (R_c)^2 d\nu_{\alpha,\ell}\Big)\Big/\Big(\int R_c d\nu_{\alpha,\ell}\Big) >$$

$$> \frac{1}{2(\int R_c d\mu_c + C)} \sum_{n>\ell} \sum_{P\in\mathcal{P}_{c,n}} (n+1)^2\, m_\ell(P) =$$

$$= \frac{1}{2(\int R_c d\mu_c + C)} \sum_{n>\ell} \sum_{P\in\mathcal{P}_{c,n}} \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \frac{(n+1)^2}{(n-\ell)^{2+\alpha}} \frac{\mu_c(P)}{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))} =$$

$$= \frac{1}{2(\int R_c d\mu_c + C)} \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \sum_{n>\ell} \frac{(n+1)^2}{(n-\ell)^{2+\alpha}} \underbrace{\sum_{P\in\mathcal{P}_{c,n}} \frac{\mu_c(P)}{\mu_c(\mathcal{P}_n(c)\setminus\mathcal{P}_{n+1}(c))}}_{1} =$$

$$= \frac{1}{2(\int R_c d\mu_c + C)} \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{\zeta(2+\alpha)} \sum_{n>\ell} \frac{(n+1)^2}{(n-\ell)^{2+\alpha}} =$$

$$= \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{2(\int R_c d\mu_c + C)\zeta(2+\alpha)} \sum_{n=1} \frac{(n+1+\ell)^2}{n^{2+\alpha}} \geq \frac{\mu_c(\mathcal{P}_{\ell+1}(c))}{2(\int R_c d\mu_c + C)\zeta(2+\alpha)} \sum_{n=1} \frac{1}{n^\alpha} = +\infty.$$

On the other hand, using (6.2) and (6.3), we have that

$$\int R d\eta_{\alpha,\ell} = \int_{(c,q)} R d\eta_{\alpha,\ell} + \int_{(q,p)} R d\eta_{\alpha,\ell} = t_0 \eta_{\alpha,\ell}((c,q)) + \int_{(q,p)} R d\Big(\frac{1}{\int R_c d\nu_{\alpha,\ell}} \nu_{\alpha,\ell}|_{(q,p)}\Big) =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \int_{J\setminus\mathcal{P}_0(c)} R\, d\nu_{\alpha,\ell} =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \sum_{P\in\mathcal{P}_{c,0}} R(P) m_\ell(P) =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \sum_{P\in\mathcal{P}_{c,0}} R(P) \mu_c(P) =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \int_{(q,p)} R d\mu_c =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \int_{(q,p)} R\, d\Big(\frac{1}{\int R_c d\mu_c} \mu_0|_{(q,p)}\Big) =$$

$$= t_0 \eta_{\alpha,\ell}((c,q)) + \frac{1}{\int R_c d\nu_{\alpha,\ell}} \frac{1}{\int R_c d\mu_c} \int_{(q,p)} R\, d\mu_0 < +\infty$$

Therefore,

$$\mu_{\alpha,\ell} = \frac{1}{\int R d\eta_{\alpha,\ell}} \sum_{n\geq 1} \sum_{j=0}^{n-1} f_*^j (\eta_{\alpha,\ell}|_{\{R=n\}})$$

is an ergodic f invariant probability and, as f is transitive on $[f(c_+), f(c_-)]$ and $\operatorname{supp} \mu_{\alpha,\ell} \supset (c,p)$,

$$\operatorname{supp} \mu_{\alpha,\ell} = [f(c_+), f(c_-)].$$

Claim 3. $\lim_\ell h_{\eta_{\alpha,\ell}}(F) \geq h_{\mu_0}(F)$

Proof of the claim. Indeed,

$$\lim_\ell h_{\eta_{\alpha,\ell}}(F) = \lim_\ell \frac{h_{\nu_{\alpha,\ell}}(F_c)}{\int R_c d\nu_{\alpha,\ell}} = \lim_{\ell\to+\infty} \frac{\sum_{P\in\mathcal{P}^*} H(m_\ell(P))}{\sum_{P\in\mathcal{P}^*} R_c(P)\, m_\ell(P)} =$$

$$= \frac{\lim_\ell \sum_{P\in\mathcal{P}^*} H(m_\ell(P))}{\lim_\ell \sum_{P\in\mathcal{P}^*} R_c(P)\, m_\ell(P)} = \frac{\sum_{P\in\mathcal{P}^*} H(\mu_c(P))}{\int R_c d\mu_c} \geq \frac{h_{\mu_c}(F_c)}{\int R_c d\mu_c} = h_{\mu_0}(F).$$

□

Claim 4. $\lim_\ell \int R\, d\eta_{\alpha,\ell} = \int R\, d\mu_0$

Proof of the claim. Let $\widetilde{R}(x)$ be the F_c-lift of R, that is, $\widetilde{R}(x) = \sum_{j=0}^{R_c(x)-1} R\circ F^j(x)$. We know that $\int R d\eta_{\alpha,\ell} = \frac{\int \widetilde{R} d\nu_{\alpha,\ell}}{\int R_c d\nu_{\alpha,\ell}}$ as well as $\int R d\mu_0 = \frac{\int \widetilde{R} d\mu_c}{\int R_c d\mu_c}$. As $\lim_\ell \int R_c d\nu_{\alpha,\ell} = \int R_c d\mu_c < +\infty$, we need to show that $\lim_\ell \int \widetilde{R} d\nu_{\alpha,\ell} = \int \widetilde{R} d\mu_c$. To prove so, observe that $\widetilde{R}(x)$ is constant on each $P \in \mathcal{P}_{c,n}$ and every $n \geq 0$. Indeed, $\widetilde{R}(x) = t_0 n + R(f^{t_0 n}(x))$ and $f^{t_0 n}(P) \in \mathcal{P}_{c,0} = \mathcal{P} \cap (q,p)$. As R is constant on the elements of $\mathcal{P}$, we get that $\widetilde{R}$ is constant on $P \in \mathcal{P}_{c,n}$. Therefore,

$$\lim_\ell \int \widetilde{R} d\nu_{\alpha,\ell} = \lim_\ell \sum_{n=0}^{+\infty} \sum_{P\in\mathcal{P}_{c,n}} \widetilde{R}(P)\nu_{\alpha,\ell}(P) = \sum_{n=0}^{+\infty} \sum_{P\in\mathcal{P}_{c,n}} \widetilde{R}(P)\mu_c(P) = \int \widetilde{R} d\mu_c,$$

concluding the proof of the claim. □

Using Claim 3 and Claim 4 we can conclude that $\sup\{h_{\mu_{\alpha,\ell}}(f)\,;\, \ell \in \mathbb{N}\} \geq h_\mu(f)$. Indeed,

$$\lim_\ell h_{\mu_{\alpha,\ell}}(f) = \lim_\ell \frac{h_{\eta_{\alpha,\ell}}(F)}{\int R d\eta_{\alpha,\ell}} = \frac{\lim_\ell h_{\eta_{\alpha,\ell}}(F)}{\lim_\ell \int R d\eta_{\alpha,\ell}} = \frac{\lim_\ell h_{\eta_{\alpha,\ell}}(F)}{\int R d\mu_0} \geq \frac{h_{\mu_0}(F)}{\int R d\mu_0} = h_\mu(f).$$

As we can choose any f invariant probability μ, with $\operatorname{supp}\mu = [f(c_+), f(c_-)]$, to construct $\mu_{\alpha,\ell}$ and as $\sup\{h_\mu(f)\,;\, \mu \in \mathcal{M}^1(f) \text{ and } \operatorname{supp}\mu = [f(c_+), f(c_-)]\} = h_{top}(f)$, we get that

$$\sup\{h_\mu(f)\,;\, \mu \in \mathcal{M}_c\} = h_{top}(f),$$

where

$$\mathcal{M}_c := \{\mu_{\alpha,\ell}\,;\, \mu \in \mathcal{M}^1(f) \text{ with } \operatorname{supp}\mu = [f(c_+), f(c_-)], \alpha \in (0,1) \text{ and } \ell \in \mathbb{N}\}$$

Claim 5. *There are $K > 0$ and $x_0 \in (c,q)$ such that $|\log|x-c|| \geq K R_c(x)$ for every $x \in (c, x_0)$.*

Proof of the claim. As f is non-flat, there is $1 < \theta_1 \le \theta_2$ and $a \ge 1$ such that

$$(1/a)(x-c)^{1/\theta_1} \le f(x) - f(c_+) \le a(x-c)^{1/\theta_2}$$

for every $x > c$. As $f^{t_0}(c_+) = c$ and $0 < f'(f^j(c_+)) < +\infty$ for every $1 \le j < t_0$, there exist $b \ge 1$ and $\delta_0 > 0$ such that

$$(1/b)(x-c)^{1/\theta_1} \le f^{t_0}(x) - c \le b(x-c)^{1/\theta_2}.$$

Taking $1 < \theta_1' < \theta_1 \le \theta_2 < \theta_2'$ and $0 < \delta_0' < \delta_0$ small, we get that

$$(x-c)^{1/\theta_1'} \le f^{t_0}(x) - c \le (x-c)^{1/\theta_2'}.$$

for every $c < x < c + \delta_0'$. Hence, taking $g = (f^{t_0}|_{(c,c+\delta_0')})^{-1}$, we have

$$(x-c)^{\theta_2'} \le g(x) - c \le (x-c)^{\theta_1'}$$

and so

$$(x-c)^{(\theta_2')^n} \le g^n(x) - c \le (x-c)^{(\theta_1')^n}.$$

Let n_0 be the smaller $n \ge 1$ such that $P_{c,n} \subset (c, f(c+\delta_0'))$. Taking $\gamma = \sup P_{c,n_0}$, we get that $P_{c,n} = [g^{n-n_0+1}(\gamma), g^{n-n_0}(\gamma))$ for every $n \ge n_0$. Thus, $(\gamma - c)^{\theta_2'} \le |x-c| \le (\gamma - c)$ for $x \in P_{c,n_0}$ and

$$(\gamma - c)^{(\theta_2')^{(n-n_0+1)}} \le |x-c| \le (\gamma-c)^{(\theta_1')^{(n-n_0)}},$$

For every $x \in P_{c,n_0}$ and $n > n_0$. This means that

$$\log(1/(\gamma-c))(\theta_1')^{(n-n_0)} \le |\log|x-c|| \le \log(1/(\gamma-c))(\theta_2')^{(n-n_0+1)},$$

$x \in P_{c,n}$ and $n \ge n_0$. As $R_c(P_{c,n}) = n$, if we take $n_1 \ge n_0$ such that $(\theta_1')^n \ge n$ for every $n \ge n_1$, $K = \frac{1}{2(\theta_1')^{n_1}} \log(\frac{1}{\gamma - c})$ and $x_0 = \sup P_{c,n_0+1} = \inf P_{c,n_0}$ then

$$|\log|x-c|| \ge K\, {\theta_1'}^{R_c(x)} \ge K R_c(x)$$

for every $c < x < x_0$. $\square$

As

$$\int_J R_c d\mu_{\alpha,\ell} \ge \frac{1}{\int R d\eta_{\alpha,\ell}} \int R_c d\eta_{\alpha,\ell} = +\infty$$

and c is the unique pole of R_c, we conclude that $\int_{x\in(c,x_0)} R_c(x) d\mu_{\alpha,\ell} = \infty$ and so,

$$\int_{x\in[0,1]} |\log|x-c|| d\mu_{\alpha,\ell} \ge \int_{x\in(c,x_0)} |\log|x-c|| d\mu_{\alpha,\ell} \ge$$

$$\ge K \int_{x\in(c,x_0)} R_c(x) d\mu_{\alpha,\ell} = +\infty.$$

Note that, if $0 < \alpha_0 < \alpha_1 < 1$, then $\nu_{\alpha_0,\ell}(P_{c,n}) \neq \nu_{\alpha_1,\ell}(P_{c,n})$ for any $n > \ell$. In particular, $\nu_{\alpha_0,\ell} \neq \nu_{\alpha_1,\ell}$. As R_c is exact, F_c is orbit-coherent (see Lemma 2.6 of [Pi20]). Thus, it follows from Theorem B of [Pi20] that an ergodic F-invariant probability μ has at most one F_c-lift. That is, setting $\mathcal{U} = \{\nu \in \mathcal{M}^1(F_c)\,;\, \nu \text{ is } F_c \text{ ergodic and } \int R_c d\nu < +\infty\}$, we have that

$$\mathcal{U} \ni \nu \mapsto \frac{1}{\int R d\nu} \sum_{n \geq 1} \sum_{j=0}^{n-1} F_*^j(\nu|_{\{R_c = n\}}) \in \mathcal{M}^1(F)$$

is injective. Therefore, $\eta_{\alpha_0,\ell} \neq \eta_{\alpha_1,\ell}$ and, by the same argument, $\mu_{\alpha_0,\ell} \neq \mu_{\alpha_1,\ell}$. This implies that $M = \{\mu_{\alpha,\ell}\,;\, \alpha \in (0,1) \text{ and } \ell \geq 1\}$ is uncountable. Finally, it follows from f be non-flat that there are constants $c_0, c_1, c_2 > 0$ such that $-c_0 + c_1|\log|x - c|| \leq \log|f'(x)| \leq c_0 + c_2|\log|x - c||$ for every $x \in [0,1] \setminus \{c\}$ and so $\int \log|f'|d\mu_{\alpha,\ell} = +\infty$. Hence, it follows from Birkhoff and the ergodicity of $\mu_{\alpha,\ell}$ that $\lim_n \frac{1}{n}\log(f^n)'(x) = \lim_n \frac{1}{n}\sum_{j=0}^{n-1}\log(f' \circ f^j(x)) = \int \log|f'|d\mu_{\alpha,\ell} = +\infty$ for $\mu_{\alpha,\ell}$ almost every x.

$\square$

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Universidade Federal da Bahia - UFBA
Instituto de Matemática / Colegiado da Pós-Graduação em Matemática

Av. Adhemar de Barros, s/n, Campus Universitário de Ondina, Salvador - BA
CEP: 40170 - 110
<http://www.pgmat.ufba.br>