SIMPLICITY OF LYAPUNOV SPECTRUM FOR LINEAR COCYCLES OVER NON-UNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. We prove that generic fiber-bunched and Hölder continuous linear cocycles over a non-uniformly hyperbolic system endowed with a u-Gibbs measure have simple Lyapunov spectrum. This gives an affirmative answer to a conjecture proposed by Viana in the context of fiber-bunched cocycles.

1. Introduction

The notion of (uniform) hyperbolicity was introduced in the context of dynamical systems by Smale in [28] and since then has played a major rule in this area of research. This notion is expressed in terms of (uniform) rates of contraction and expansion by the dynamics along complementary directions. At first, it was conjectured that uniform hyperbolicity is quite frequent among all dynamical systems. Newhouse [21] then proved that this was not the case: he exhibited a C^2-open set of diffeomorphisms on the 2-sphere where none of its elements were hyperbolic.

In order to describe the majority of the dynamical systems, weaker notions of hyperbolicity have been introduced and intensively studied. These include partially hyperbolic and non-uniform hyperbolic dynamical systems. The notion of non-uniform hyperbolicity is defined in terms of *Lyapunov exponents*: a system is said to be non-uniformly hyperbolic if it has no zero Lyapunov exponents. These last objects measure the *asymptotic* rates of contractions and expansions along different directions and are one of the most fundamental notions in dynamical systems. As such, they have received a great deal of attention in the last decades and are also the focus of our present work. Among many interesting questions that one could ask about this objects we could cite

- What are the regularity properties of Lyapunov exponents?
- How frequently are systems with *at least one* non-zero Lyapunov exponent?
- How frequently are systems with *all* Lyapunov exponents different from zero?
- How frequently are systems whose Lyapunov exponents are *all different*?

The context of *linear cocycles* has provided a fruitful playground for addressing the previous questions, by the possibility of detaching the underlying dynamics from an action, induced by it, on a vector space. In this context the abundance of non-uniform hyperbolicity, foreseen since the pioneering works of Furstenberg [15], Guivarch and Raugi [17], and Goldsheid and Margulis [16] on random i.i.d. products of matrices, was later extended by Bonatti and Viana [12], Viana [29] and Avila and Viana [2] to include a much broader class of (Hölder continuous)

2010 *Mathematics Subject Classification.* 37H15,37D30,37D25.

Key words and phrases. Lyapunov exponents, non-uniform hyperbolicity, linear cocyles.

cocycles over hyperbolic maps. More recently, Matheus, Möller and Yoccozz [20] considered the Kontsevich–Zorich cocycle, Poletti and Viana [25] established a criterion for simplicity of Lyapunov spectrum for cocycles over partially hyperbolic maps, and Bessa et al [9] considered the top Lyapunov exponent for linear cocycles on semisimple Lie groups over non-uniformly hyperbolic diffeomorphisms. One should mention that coincidence of all Lyapunov exponents (in opposition to the abundance of non-uniform hyperbolicity) occurs for generic continuous cocycles over ergodic automorphisms as proved by Bochi in [10]. In the present note, we address the question of how frequently are systems whose Lyapunov exponents are *all different*. In fact, we prove that

Theorem 1.1. *Generic fiber-bunched linear cocycles over a non-uniformly hyperbolic system have simple Lyapunov spectrum.*

The precise statement of our results, which provide an affirmative answer to a conjecture of Viana in [29], p. 648, in the fiber-bunched context, will appear by the end of Section 2 after some preliminary definitions. It means that the set linear cocycles whose Lyapunov exponents are all different contains an open and dense set of fiber-bunched cocycles. It is worth noticing that those other questions have also attracted the attention of many mathematicians and have already been answered in some specific contexts (see for instance [3, 4, 7, 8, 11, 12, 14, 24, 30] and references therein). One should mention that in the case of the dynamical cocycle, corresponding to the cocycle Df over the diffeomorphism f, fewer results are known.

2. Definitions and Statements

At this section we introduce some preliminary notions and state our main result precisely. Let M be a compact smooth manifold, $f : M \to M$ a $C^{1+\beta}$ diffeomorphism and μ an ergodic f-invariant measure.

2.1. **Linear cocycles and Lyapunov exponents.** Given a positive integer $d \geq 1$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the *linear cocycle generated by the matrix-valued map* $A : M \to GL(d, \mathbb{K})$ *over* f is the (invertible) map $F_A : M \times \mathbb{K}^d \to M \times \mathbb{K}^d$ given by

$$F_A\,(x, v) = (f(x), A(x)v)\,.$$

Its iterates are given by $F_A^n\,(x, v) = (f^n(x), A^n(x)v)$ where

$$A^n(x) = \begin{cases} A(f^{n-1}(x)) \ldots A(f(x))A(x) & \text{if } n > 0 \\ Id & \text{if } n = 0 \\ A(f^n(x))^{-1} \ldots A(f^{-1}(x))^{-1} & \text{if } n < 0. \end{cases}$$

Sometimes we denote this cocycle by (f, A) or when there is no risk of ambiguity simply by A. The *projectivized cocycle* $f_A : M \times \mathbb{P}^{d-1}(\mathbb{K}) \to M \times \mathbb{P}^{d-1}(\mathbb{K})$ is defined simply by

$$f_A\,(x, v) = \left(f(x), \frac{A(x)v}{\|A(x)v\|} \right).$$

A natural example of linear cocycle is given by the derivative of a diffeomorphism: the cocycle generated by $A(x) = Df(x)$ over f is called the *derivative cocycle*.

Under some integrability conditions, namely the integrability of $\log \|A\|$ and $\log \left\|A^{-1}\right\|$, it follows from a famous theorem due to Oseledets [22] (see also [30]) that there exists a full μ-measure set $\mathcal{R}(\mu) \subset M$, whose points are called *μ-regular points*, such that for every $x \in \mathcal{R}(\mu)$ there exist real numbers $\lambda_1(A,x) > \cdots > \lambda_k(A,x)$ and a direct sum decomposition $\mathbb{K}^d = E_x^{1,A} \oplus \ldots \oplus E_x^{k,A}$ into vector subspaces such that

$$A(x)E_x^{i,A} = E_{f(x)}^{i,A} \text{ and } \lambda_i(A,x) = \lim_{n\to\infty} \frac{1}{n} \log \| A^n(x)v \|$$

for every non-zero vector $v \in E_x^{i,A}$ and $1 \leq i \leq k$. Moreover, since our measure μ is ergodic, the *Lyapunov exponents* $\lambda_i(A,x)$ are constant on a full μ-measure subset of M (and thus we denote it just by $\lambda_i(A,\mu)$) as well as the dimensions of the *Oseledets subspaces* $E_x^{i,A}$. This last notion is called the *multiplicity* of $\lambda_i(A,\mu)$. We say that the cocycle (f,A) has *simple Lyapunov spectrum* with respect to the measure μ if every Lyapunov exponent has multiplicity one. This means that A has d distinct Lyapunov exponents.

2.2. **Non-uniformly hyperbolic systems and Gibbs states.** An ergodic measure μ is said to be *hyperbolic* if all Lyapunov exponents $\{\lambda_i(Df,\mu)\}_{i=1}^k$ are non-zero. In such case the system (f,μ) is called *non-uniformly hyperbolic.* If $0 < \chi < \min\{| \lambda_i(Df,\mu) | : 1 \leq i \leq k\}$ then μ is called a *χ-hyperbolic measure.* The non-uniform hyperbolicity condition implies the existence of a very rich geometric structure of the dynamics f given by Pesin's stable manifold theorem (see [5]): there exists a full μ-measure set $H(\mu) \subset M$ so that through every point $x \in H(\mu)$ there exist C^1 embedded disks $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ such that

i) $W^s_{\mathrm{loc}}(x)$ is tangent to E^s_x and $W^u_{\mathrm{loc}}(x)$ is tangent to E^{cu}_x;

ii) given $0 < \tau_x < \min_{1\leq i\leq k} | \lambda_i(Df,\mu) |$ there exists $C_x > 0$ such that

$$d(f^n(y), f^n(z)) \leq C_x e^{-\tau_x n} d(y,z) \text{ for all } y,z \in W^s_{\mathrm{loc}}(x) \text{ and } n \geq 0$$

and

$$d(f^{-n}(y), f^{-n}(z)) \leq C_x e^{-\tau_x n} d(y,z) \text{ for all } y,z \in W^u_{\mathrm{loc}}(x) \text{ and } n \geq 0;$$

iii) $f(W^s_{\mathrm{loc}}(x)) \subset W^s_{\mathrm{loc}}(f(x))$ and $f(W^u_{\mathrm{loc}}(x)) \supset W^u_{\mathrm{loc}}(f(x))$;

and

iv) $W^s(x) = \cup_{n=0}^{\infty} f^{-n}(W^s_{\mathrm{loc}}(f^n(x)))$ and $W^u(x) = \cup_{n=0}^{\infty} f^{n}(W^s_{\mathrm{loc}}(f^{-n}(x)))$.

Moreover, the local stable and unstable sets $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ depend measurably on x, as C^1 embedded disks, as well as the constants τ_x and C_x. Therefore, we may find compact *hyperbolic blocks* $\mathcal{H}(C,\tau)$ whose measure can be made arbitrarily close to 1 by increasing C and decreasing τ such that

$$\tau_x > \tau \text{ and } C_x < C \text{ for every } x \in \mathcal{H}(C,\tau)$$

and the disks $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ vary continuously with $x \in \mathcal{H}(C,\tau)$. In particular, the sizes of $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ are uniformly bounded from zero on $\mathcal{H}(C,\tau)$ as well as the angles between two disks. The drawback is that hyperbolic blocks are in general not invariant by the diffeomorphism. We define the set

$$NUH := \bigcup_{\chi>0} \bigcup_{\{\mu \text{ is } \chi-\text{hyperbolic}\}} H(\mu)$$

as the set of points that have some non-uniformly hyperbolicity. By construction this set has full measure with respect to all non-uniformly hyperbolic invariant measures.

Definition 2.1. We say that an f-invariant measure μ is an *u-Gibbs state* (respectively, *s-Gibbs state*) if its disintegrations along unstable (respectively, stable) manifolds are absolutely continuous with respect to the Lebesgue measure.

One should mention that the class of u-Gibbs measures is physically relevant. More specifically, all *SRB measures* (meaning ergodic measures whose basin of attraction have positive Lebesgue measure) for uniformly hyperbolic and many partially hyperbolic attractors are u-Gibbs states (see e.g. [1, 13, 23] and references therein).

Remark 2.2. By [18, 19] an f-invariant measure μ is a u-Gibbs state if and only if satisfies the following relation

$$h_\mu(f) = \int \sum_{\lambda_i(x)>0} \lambda_i(x) d\mu.$$

Analogously, μ is a s-Gibbs state if and only if $h_\mu(f) = \int \sum_{\lambda_i(x)<0} -\lambda_i(x) d\mu$. As an example, it is clear that if f is volume preserving then the Lebesgue measure is both a s-Gibbs and a u-Gibbs state.

2.3. **Fiber-bunched cocycles.** Given $r \in \mathbb{N}$ and $\alpha \in [0,1]$ let $C^{r,\alpha}(M, GL(d,\mathbb{K}))$ denote the set of C^r-maps $A : M \to GL(d,\mathbb{K})$ so that $D^r A$ is α-Hölder continuous. Endow $C^{r,\alpha}(M, GL(d,\mathbb{K}))$ with the norm

$$\|A\|_{r,\alpha} := \max_{0\le j\le r} \sup_{x\in M} \|D^j A(x)\| + \sup_{x\neq y\in M} \frac{\|D^r A(x) - D^r A(y)\|}{d(x,y)^\alpha}.$$

Assume that μ is an f-invariant and ergodic χ-hyperbolic measure. We say that the cocycle $A \in C^{r,\alpha}(M, GL(d,\mathbb{K}))$ over (f,μ) is $\frac{\chi}{2}$*-fiber-bunched* if there are constants $C > 0$ and $\theta \in (0,1)$ such that

$$\| A^n(x) \| \| A^n(x)^{-1} \| (e^{-\frac{\chi}{2}})^{|n|\alpha} \le C\theta^{|n|} \tag{1}$$

for every $x \in M$ and $n \in \mathbb{Z}$. A simple remark is that the previous notion does not depend on the invariant measure μ but on the hyperbolicity constant $\chi > 0$. The set $\mathcal{B}^{r,\alpha}_\chi(M)$ of $C^{r,\alpha}$-cocycles that are $\frac{\chi}{2}$-fiber-bunched over (f,μ) forms a C^0-open subset of $C^{r,\alpha}(M, GL(d,\mathbb{K}))$.

2.4. **Main theorem.** The main result of this work is that typical fiber-bunched cocycles over u-Gibbs measures have simple Lyapunov spectrum. More precisely,

Theorem A. Let $f : M \to M$ be a $C^{1+\beta}$ diffeomorphism of a compact smooth manifold and μ be a χ-hyperbolic ergodic f-invariant u-Gibbs measure. Given $r \ge 0$ and $\alpha \ge 0$ so that $r + \alpha > 0$, there exists an open and dense subset $\mathcal{O} \subset \mathcal{B}^{r,\alpha}_\chi(M)$ such that for any $A \in \mathcal{O}$, the cocycle (f, A) has simple Lyapunov spectrum.

It is clear that the previous theorem also applies for s-Gibbs measures. Indeed, while on the one hand any s-Gibbs measure μ for f is a u-Gibbs measure for f^{-1}, on the other hand, a cocycle is fiber-bunched for (f,μ) if and only if it is fiber-bunched for (f^{-1},μ). Moreover, similar calculations to those of [12] gives us that the set $\mathcal{B}^{r,\alpha}_\chi(M) \setminus \mathcal{O}$ has infinite codimension, that is, it is locally contained in finite unions of closed submanifolds with arbitrarily large codimension. Theorem A should be compared to [29], where the author proved that an open and dense set of $C^{r,\alpha}$ cocycles over a χ-hyperbolic and ergodic measure with local product structure have at least one positive Lyapunov exponent.

At this point it would be interesting to exhibit non-trivial examples of open sets of cocycles which are not fiber-bunched and having simple Lyapunov spectrum. By non-trivial examples we mean, for instance, not having dominated decomposition.

The proof of our theorem makes use of the simplicity criterion established by Avila and Viana [2] and the symbolic description of non-uniformly hyperbolic systems given by Sarig [27] and Ben Ovadia [6].

3. Preliminaries

At this section we recall some notions and results that are going to be used in the proof of our main theorem.

3.1. **Topological Markov Shifts.** Let $\mathcal{G}$ be a directed graph with a countable collection of vertices $\mathcal{V}$ such that every vertex has at least one edge coming in, and at least one edge coming out. The *topological Markov shift* associated to $\mathcal{G}$ is the set

$$\hat{\Sigma} = \hat{\Sigma}(\mathcal{G}) := \{(x_i)_{i\in\mathbb{Z}} \in \mathcal{V}^{\mathbb{Z}} : x_i \to x_{i+1} \ \forall i \in \mathbb{Z}\},$$

equipped with the left-shift $\hat{\sigma} : \hat{\Sigma} \to \hat{\Sigma}$, $\hat{\sigma}((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ and the metric

$$d(\hat{x}, \hat{y}) := e^{-\frac{\chi}{2}\min\{|n|:\, x_n \neq y_n\}} \tag{2}$$

where $\hat{x} = (x_i)_{i\in\mathbb{Z}}$ and $\hat{y} = (y_i)_{i\in\mathbb{Z}}$. Moreover, the choice of the metric in (2) guarantees that $\hat{\sigma}$ is a hyperbolic homeomorphism (with hyperbolicity constant $e^{\frac{\chi}{2}}$). Thus, $\hat{\Sigma}$ is a complete separable metric space, $\hat{\Sigma}$ is compact if and only if $\mathcal{G}$ is finite and $\hat{\Sigma}$ is locally compact if and only if every vertex of $\mathcal{G}$ has finite degree. Given a topological Markov shift $\hat{\Sigma}$, we define

$$\hat{\Sigma}^{\#} := \{(x_i)_{i\in\mathbb{Z}} \in \hat{\Sigma} : \exists\, v', w' \in \mathcal{V} \ , \exists\, n_k, m_k \uparrow \infty \text{ s.t. } x_{n_k} = v' \text{ and } x_{-m_k} = w'\}.$$

Observe that by the Poincaré recurrence theorem every $\hat{\sigma}$-invariant probability measure gives to $\hat{\Sigma}^{\#}$ full measure. Furthermore, notice that every periodic point of $\hat{\sigma}$ is in $\hat{\Sigma}^{\#}$.

The next result plays an important part in our proof. It was first established by Sarig in [27] for surface diffeomorphisms and latter generalized to any dimension by Ovadia [6].

Theorem 3.1 (Markov Partitions)**.** *Let $f : M \to M$ be a $C^{1+\beta}$ diffeomorphism and μ an f-invariant ergodic measure. Assume also that (f, μ) is χ-hyperbolic. Then, there exists a locally compact topological Markov shift $\hat{\Sigma}_\chi$ and a Hölder continuous map $\hat{\pi} = \hat{\pi}_\chi : \hat{\Sigma}_\chi \to M$ such that*

(1) $\hat{\pi}_\chi \circ \hat{\sigma} = f \circ \hat{\pi}_\chi$.
(2) $\hat{\pi}_\chi[\hat{\Sigma}^{\#}_\chi]$ *is of full measure with respect to any χ-hyperbolic measure; moreover, every point in $\hat{\pi}_\chi[\hat{\Sigma}^{\#}_\chi]$ has finitely many pre-images in $\hat{\Sigma}^{\#}_\chi$.*
(3) *There exists a $\hat{\sigma}$-invariant measure $\hat{\mu}$ such that $(\hat{\pi}_\chi)_*\hat{\mu} = \mu$.*

Remark 3.2. It follows from [6, Theorem 1.3.21] that the Hölder exponent of $\hat{\pi}_\chi$ is $\frac{\chi}{2}$ with respect to the canonical metric $d'(\hat{x}, \hat{y}) = e^{-\min\{|i|:x_i\neq y_i\}}$ (cf. proof of [6, Proposition 1.3.20]). Thus, using our adapted χ-metric defined by (2) we get that $\hat{\pi}_\chi$ is actually Lipschitz with respect to this metric.

3.2. **Product structure and continuous product structure.** Let us consider

$$\hat{\Sigma}^+ = \{(x_n)_{n\geq 0} : \text{ there exists } \hat{y} = (y_n)_{n\mathbb{Z}} \in \hat{\Sigma} \text{ such that } (x_n)_{n\geq 0} = (y_n)_{n\geq 0}\}$$

and

$$\hat{\Sigma}^- = \{(x_n)_{n\leq 0} : \text{ there exists } \hat{y} = (y_n)_{n\mathbb{Z}} \in \hat{\Sigma} \text{ such that } (x_n)_{n\leq 0} = (y_n)_{n\leq 0}\}.$$

Points in $\hat{\Sigma}^+$ we denote by x^+ while points in $\hat{\Sigma}^-$ we denote by x^-. We have natural projections $P^+ : \hat{\Sigma} \to \hat{\Sigma}^+$ and $P^- : \hat{\Sigma} \to \hat{\Sigma}^-$ obtained by dropping all of the negative coordinates and all of the positive coordinates, respectively, of a sequence in $\hat{\Sigma}$.

We define the *local stable set* of $\hat{x} \in \hat{\Sigma}$ to be

$$W^s_{loc}(\hat{x}) = W^s_{loc}(P^+(\hat{x})) = \{(y_n)_{n\in\mathbb{Z}} \in \hat{\Sigma} : x_n = y_n \text{ for all } n \geq 0\}$$

and the *local unstable set* to be

$$W^u_{loc}(\hat{x}) = W^u_{loc}(P^-(\hat{x})) = \{(y_n)_{n\in\mathbb{Z}} \in \hat{\Sigma} : x_n = y_n \text{ for all } n \leq 0\}.$$

We think of $\hat{\Sigma}^-$ and $\hat{\Sigma}^+$, respectively, as parametrizations of the local stable and unstable sets. Observe that for $\hat{y} \in W^s_{loc}(\hat{x})$ we have

$$d(\hat{\sigma}(\hat{x}), \hat{\sigma}(\hat{y})) \leq e^{-\frac{\chi}{2}} d(\hat{x}, \hat{y})$$

and similarly for $\hat{y} \in W^u_{loc}(\hat{x})$ when changing $\hat{\sigma}$ by $\hat{\sigma}^{-1}$.

Given a symbol $R \in \mathcal{V}$, let us consider the *cylinder*

$$[R] = \{(y_n)_{n\in\mathbb{Z}} \in \hat{\Sigma} : y_0 = R\}.$$

Definition 3.3. We say that an $\hat{\sigma}$-invariant probability measure $\hat{\mu}$ has *product structure* if the normalized restriction of $\hat{\mu}$ to every cylinder $[R]$ is of the form $\hat{\mu}\mid_{[R]} = \hat{\rho}\,(\hat{\nu}^s \times \hat{\nu}^u)$ where $\hat{\nu}^s = P^-_*\hat{\mu}$ and $\hat{\nu}^u = P^-_*\hat{\mu}$ and the density $\hat{\rho}$ is measurable. Moreover, we say that $\hat{\mu}$ has *continuous product structure* if the density $\hat{\rho} : \hat{\Sigma} \to \mathbb{R}_+$ is uniformly continuous and bounded away from zero and infinity.

3.3. **Invariant holonomies.** A (α-Hölder) *stable holonomy* for the linear cocycle generated by $\hat{A} : \hat{\Sigma} \to GL(d, \mathbb{K})$ over $\hat{\sigma}$ is a collection of linear maps $H^{s,\hat{A}}_{\hat{x}\hat{y}} \in GL(d, \mathbb{K})$ defined for $\hat{y} \in W^s_{loc}(\hat{x})$ which satisfy the following properties for some $L > 0$,

- $H^{s,\hat{A}}_{\hat{y}\hat{z}} = H^{s,\hat{A}}_{\hat{x}\hat{z}} H^{s,\hat{A}}_{\hat{y}\hat{x}}$ and $H^{s,\hat{A}}_{\hat{x}\hat{x}} = Id$;
- $H^{s,\hat{A}}_{\hat{\sigma}(\hat{y})\hat{\sigma}(\hat{z})} = \hat{A}(\hat{z}) H^{s,\hat{A}}_{\hat{y}\hat{z}} \hat{A}(\hat{y})^{-1}$;
- $\parallel H^{s,\hat{A}}_{\hat{y}\hat{z}} - id \parallel \leq L d(\hat{y}, \hat{z})^{\alpha}$.

By replacing the cocycle generated by $\hat{A}$ over $\hat{\sigma}$ by the cocycle generated by $\hat{A}$ over $\hat{\sigma}^{-1}$ we get an analogous definition of *unstable holonomies* $H^{u,\hat{A}}_{\hat{x}\hat{y}}$ for $\hat{y} \in W^u_{loc}(\hat{x})$.

Examples of linear cocycles admitting stable and unstable holonomies are given by *locally constant cocycles* and $\frac{\chi}{2}$*-fiber-bunched cocycles*. In such cases, a family of stable and unstable holonomies is given by

$$H^{s,\hat{A}}_{\hat{x}\hat{y}} = \lim_{n\to+\infty} \hat{A}^n(\hat{y})^{-1} \hat{A}^n(\hat{x}), \;\; y \in W^s_{loc}(\hat{x})$$

and

$$H^{u,\hat{A}}_{\hat{x}\hat{y}} = \lim_{n\to+\infty} \hat{A}^{-n}(\hat{y})^{-1} \hat{A}^{-n}(\hat{x}), \;\; y \in W^u_{loc}(\hat{x}),$$

respectively. See for instance [11, 29].

4. Translation to the symbolic setting

The objective of this section is to prove the following proposition which gives us a translation of our problem to the symbolic setting.

Proposition 4.1. *Let $f : M \to M$ be a $C^{1+\beta}$ diffeomorphism, μ an ergodic f-invariant χ-hyperbolic u-Gibbs measure and $A \in \mathcal{B}^{r,\alpha}_{\chi}(M)$. Then, there exist a topological Markov shift $\hat{\Sigma}$ and a Hölder continuous map $\hat{\pi} := \hat{\pi}_{\chi} : \hat{\Sigma} \to M$ such that $\hat{\pi} \circ \hat{\sigma} = f \circ \hat{\pi}$. Moreover,*

i) *there exists an ergodic $\hat{\sigma}$-invariant measure $\hat{\mu}$ with continuous product structure such that $\mu = \hat{\pi}_* \hat{\mu}$*

and

ii) *the map $\hat{A} : \hat{\Sigma} \to GL(d, \mathbb{K})$ given by $\hat{A} = A \circ \hat{\pi}$ is Hölder continuous and admits invariant holonomies.*

The proof of Proposition 4.1 will occupy this section. The existence of the semi-conjugacy $\hat{\pi} : \hat{\Sigma} \to M$ as stated above follows readily from our assumptions and Theorem 3.1. So all we have to do is to prove items i) and ii). We start with ii).

Let $\hat{A} : \hat{\Sigma} \to GL(d, \mathbb{K})$ be the map given by $\hat{A}(\hat{x}) = A(\hat{\pi}(\hat{x}))$. We start observing that, since $\hat{\pi}$ is Lipschitz (see Remark 3.2), $\hat{A}$ is a α-Hölder continuous map. Moreover, by definition of $\hat{A}$ and our assumptions on A, it follows that

$$\| \hat{A}^n(\hat{x}) \| \| \hat{A}^n(\hat{x})^{-1} \| (e^{-\frac{\chi}{2}})^{|n|\alpha} \leq \theta^{|n|}$$

for every $\hat{x} \in \hat{\Sigma}$ and $n \in \mathbb{Z}$. Thus, since $d(\hat{\sigma}(\hat{x}), \hat{\sigma}(\hat{y})) \leq e^{-\frac{\chi}{2}} d(\hat{x}, \hat{y})$ for any $\hat{x}$ and $\hat{y}$ in $\hat{\Sigma}$ in the same stable manifold and similarly for points in the same unstable manifold under iteration of $\hat{\sigma}^{-1}$, it follows that $\hat{A}$ is fiber-bunched in the sense of [2]. Thus, by Proposition A.6 of [2], $\hat{A}$ admits stable and unstable holonomies proving ii).

In order to prove i) let us consider the measure $\hat{\mu}$ given by

$$\hat{\mu}(E) := \int_M \left(\frac{1}{|\hat{\pi}^{-1}(x)|} \sum_{\hat{\pi}(\hat{x}) = x} 1_E(\hat{x}) \right) d\mu(x) \tag{3}$$

where $|X|$ is the number of elements of the set $X \subset \hat{\Sigma}$. The fact that $\hat{\mu}$ is actually a probability measure is straightforward from the argument in [27, Proposition 13.2]. Moreover, the same proposition shows that $\hat{\mu}$ is $\hat{\sigma}$-invariant and satisfies $\hat{\pi}_* \hat{\mu} = \mu$. Furthermore, every ergodic component $\tilde{\mu}$ of $\hat{\mu}$ also projects on μ. Thus, all that is left to do is to prove that $\hat{\mu}$ has continuous product structure.

4.1. **Extensions of Gibbs measures have continuous product structure.**

We say that a measurable set $Q \subset NUH$ has *local produt structure* if for every x and y in Q, $W^s_{loc}(x)$ and $W^u_{loc}(y)$ intersect in a unique point which we denote by $[x, y]$ and, moreover, $[x, y] \in Q$. Observe that fixed any $x_0 \in Q$, there are measurable sets $\mathcal{N}^u_Q \subset W^u_{loc}(x_0)$ and $\mathcal{N}^s_Q \subset W^s_{loc}(x_0)$ such that Q is homeomorphic to $\mathcal{N}^u_Q \times \mathcal{N}^s_Q$ via $(x, y) \to [x, y]$.

Definition 4.2. We say that a hyperbolic measure μ has *local product structure* if for every measurable set $Q \subset M$ with local product structure and satisfying $\mu(Q) > 0$, the restriction $\nu = \mu_{|Q}$ of μ to Q is equivalent to $\nu^u \times \nu^s$, where ν^u and ν^s are the projections of μ to $\mathcal{N}^u_Q$ and $\mathcal{N}^s_Q$, respectively.

We would like to stress that the notions of product structure and continuous product structure given by Definition 3.3 and the notion of local product structure given above do not coincide, in general.

Remark 4.3. Observe that by the absolute continuity of stable and unstable holonomies any Gibbs state (u-Gibbs and s-Gibbs) has local product structure.

The rest of this section is devoted to prove that the measure $\hat{\mu}$ given by (3) has continuous product structure. We start with some auxiliary results.

Lemma 4.4. *For every $R \in \mathcal{V}$, $\hat{\pi}([R]) \subset M$ is a compact subset with local product structure. Moreover, $\hat{\pi}\mid_{[R]} \colon [R] \to \hat{\pi}([R])$ preserves $[\cdot,\cdot]$. That is, for any $\hat{x}, \hat{y} \in \hat{\Sigma}$,*

$$\hat{\pi}([\hat{x}, \hat{y}]) = [\hat{\pi}(\hat{x}), \hat{\pi}(\hat{y})].$$

Proof. The proof of this result is based on [27, Proposition 12.6] whose higher dimensional version is analogous. For completeness we explain it here. We use all the notations introduced in [27].

Since $\hat{\Sigma}$ is locally compact we get that $[R]$ is compact for every $R \in \mathcal{V}$. Thus, since $\hat{\pi}$ is Lipschitz continuous, it follows that $\hat{\pi}([R])$ is compact. Moreover, from the proof of [27, Proposition 12.6] we see that $\hat{\pi}([R])$ is contained in some set $Z = Z(v) \subset M$ which has product structure.

Given $\hat{x}, \hat{y} \in [R]$, let us consider $x = \hat{\pi}(\hat{x})$, $y = \hat{\pi}(\hat{y})$, $z \in W^s(x, Z) \cap W^u(y, Z)$ and $\hat{z} \in W^s_{loc}(\hat{x}) \cap W^u_{loc}(\hat{y})$. We claim that $z = \hat{\pi}(\hat{z})$. Defining $R_i := z_i = x_i$ for $i \geq 0$ it follows by [27, Lemma 12.2] that there exists a chain $(v_i)_{i\in\mathbb{Z}} \in \hat{\Sigma}$ such that $R_i \subset Z(v_i)$ and ${}_{-n}[R_{-n}, \dots, R_0] \subset Z_{-n}(v_{-n}, \dots, v_0)$ for every n. Thus, $\{f^n(x), f^n(\hat{\pi}(\hat{z}))\} \subset \overline{Z(v_i)}$ for every n. So, if $v_n = \psi_{x_n}^{p^u_n, p^s_n}$, then $\overline{Z(v_n)} \subset \psi_{x_n}^{p^u_n, p^s_n}[R_{p^u_n \wedge p^s_n}]$ and $x, \hat{\pi}(\hat{z}) \in V^u[(v_i)_{i\leq 0}]$. Analogously, $y, \hat{\pi}(\hat{z}) \in V^s[(v_i)_{i\geq 0}]$. Consequently, $\hat{\pi}(\hat{z}) \in W^s(x, Z) \cap W^u(y, Z) = \{z\}$. □

Given a cylinder $[R] \subset \hat{\Sigma}$ and considering $\hat{\mu}_R := \hat{\mu}\mid_{[R]}$, by Rokhlin's disintegration theorem (see [31]), we can disintegrate $\hat{\mu}_R$ along local stable sets $W^s_{loc}(x^+)$

$$\hat{\mu}_R = \int_{\hat{\Sigma}^+} \hat{\mu}^s_{x^+} d\hat{\mu}^u(x^+).$$

Similarly, as the set $\tilde{R} = \hat{\pi}([R]) \subset M$ has local product structure (recall Lemma 4.4), considering $\mathcal{N}^u \times \mathcal{N}^s := \mathcal{N}^u_{\tilde{R}} \times \mathcal{N}^s_{\tilde{R}}$ a system of coordinates given by the product structure as described above, we can disintegrate $\mu_{\tilde{R}} := \mu\mid_{\tilde{R}}$ along stable manifolds as

$$\mu_{\tilde{R}} = \int_{\mathcal{N}^u} \mu^s_{x^u} d\mu^u(x^u).$$

Our next result gives us a relation between the disintegrations of $\hat{\mu}_R$ and $\mu_{\tilde{R}}$. In order to state it precisely we use the following notation: given a point $\hat{x} \in \hat{\Sigma}$ we write $\hat{x} = (x^+, x^-)$ where $x^- = P^-(\hat{x})$ and $x^+ = P^+(\hat{x})$. Moreover, given $x = \hat{\pi}(\hat{x}) \in M$, using the local coordinates $\mathcal{N}^u \times \mathcal{N}^s$ we may write $x = (x^u, x^s)$ where $x^u \in \mathcal{N}^u$ and $x^s \in \mathcal{N}^s$. Thus, we define $\hat{\pi}^-(x^-) = x^s$ and $\hat{\pi}^+(x^+) = x^u$.

Lemma 4.5. *For $P^+_*\hat{\mu}$-almost every x^+ and every measurable set $B_{x^+} \subset \{x^+\} \times \hat{\Sigma}^-$,*

$$\hat{\mu}^s_{x^+}(B_{x^+}) = \int_{\mathcal{N}^s} \frac{1}{|\hat{\pi}^{-1}(x^u, x^s)|} \sum_{\hat{\pi}(x^+, x^-) = (x^u, x^s)} 1_{B_{x^+}}(x^-) d\mu^s_{\hat{\pi}^+(x^+)}(x^s).$$

Proof. Given a symbol $R \in \mathcal{V}$, let us consider $\tilde{R} := \hat{\pi}([R])$, $\hat{p} \in [R]$ and $p = \hat{\pi}(\hat{p})$. Let $\mathcal{N}^u \times \mathcal{N}^s$ be coordinates on $\tilde{R}$ centered at p which are given by the local product structure. By Lemma 4.4 the map $\hat{\pi}_{|[R]} : [R] \to \tilde{R}$ preserves $[\cdot,\cdot]$. Thus, using the product structure coordinates $\hat{\Sigma}^+ \times \hat{\Sigma}^-$ on $[R]$ we have that

$$\hat{\pi}_{|[R]} : [R] \to \mathcal{N}^u \times \mathcal{N}^s \text{ is given by } \hat{\pi}(x^+, x^-) = (\hat{\pi}^+(x^+), \hat{\pi}^-(x^-)) = (x^u, x^s).$$

Let $B \subset [R]$ be a measurable set of positive $\hat{\mu}$-measure. Thus,

$$\hat{\mu}_R(B) = \int_M \left(\frac{1}{|\hat{\pi}^{-1}(x^u, x^s)|} \sum_{\hat{\pi}^+(x^+)=(x^u)} \sum_{\hat{\pi}^-(x^-)=x^s} 1_{B_{x^+}}(x^-) \right) d\mu_{\tilde{R}},$$

where $B_{x^+} = \{y^- \in \hat{\Sigma}^-, \text{ such that } (x^+, y^-) \in B\}$. Consequently,

$$\hat{\mu}_R(B) = \int_{\mathcal{N}^u} \sum_{\hat{\pi}^+(x^+)=(x^u)} \hat{\mu}^s_{x^+}(B_{x^+}) d\mu^u(x^u),$$

where

$$\hat{\mu}^s_{x^+}(B_{x^+}) = \int_{\mathcal{N}^s} \frac{1}{|\hat{\pi}^{-1}(x^u, x^s)|} \sum_{\hat{\pi}^-(x^-)=(x^s)} 1_{B_{x^+}}(x^-) d\mu^s_{x^u}(x^s).$$

The result follows because disintegration are uniquely defined almost everywhere. $\square$

As a simple consequence we have

Corollary 4.6. *If μ has local product structure then $\hat{\mu}$ has product structure. Moreover, if μ has local product structure with density ρ then $\hat{\mu}$ has product structure with density $\hat{\rho}$ satisfying $\hat{\rho}(\hat{x}) = \rho(\hat{\pi}(\hat{x}))$.*

Proof. The first part of our claim follows easily from the previous lemma. In order to prove the second one assume $\mu_{|\hat{\pi}([R])} = \rho\nu$ where $\nu := \nu^s \times \nu^u$ and ν^s, ν^u are as in Definition 4.2. Given $\hat{x} \in \hat{\Sigma}$, let $\hat{F}_n$ be a sequence of sets with diameter going to zero such that $\lim_n \hat{F}_n = \hat{x}$. On the one hand, since $\hat{\pi}$ is at most countable to one then $F_n := \hat{\pi}(\hat{F}_n)$ are measurable sets (see e.g. [27, Proposition 13.2] for a reference on this fact). On the other hand, since $\hat{\pi}$ is Lipschitz continuous, we have that its diameters also go to zero and, consequently, $\lim_n F_n = \hat{\pi}(\hat{x})$. Thus, if $\hat{\nu}$ defines the lift of ν as in (3) then

$$\begin{aligned}
\hat{\rho}(\hat{x}) = \lim_n \frac{\hat{\mu}(\hat{F}_n)}{\hat{\nu}(\hat{F}_n)} &= \lim_n \frac{\int_M \frac{1}{|\hat{\pi}^{-1}(x)|} \left(\sum_{\hat{\pi}(\hat{x})=x} 1_{\hat{F}_n}(\hat{x}) \right) \rho(x)\, d\nu(x)}{\int_M \frac{1}{|\hat{\pi}^{-1}(x)|} \left(\sum_{\hat{\pi}(\hat{x})=x} 1_{\hat{F}_n}(\hat{x}) \right) d\nu(x)} \\
&= \lim_n \frac{\int_{F_n} \frac{1}{|\hat{\pi}^{-1}(x)|} \left(\sum_{\hat{\pi}(\hat{x})=x} 1_{\hat{F}_n}(\hat{x}) \rho(\hat{\pi}(\hat{x})) \right) d\nu(x)}{\int_{F_n} \frac{1}{|\hat{\pi}^{-1}(x)|} \left(\sum_{\hat{\pi}(\hat{x})=x} 1_{\hat{F}_n}(\hat{x}) \right) d\nu(x)} \\
&= \rho(\hat{\pi}(\hat{x}))
\end{aligned}$$

for $\hat{\mu}$-almost every $\hat{x}$. Finally, as $\hat{\nu} = \hat{\nu}^s \times \hat{\nu}^u$ we conclude the proof of the lemma. $\square$

Now we are ready to prove that $\hat{\mu}$ has continuous product structure. For sake of definiteness we assume that μ is an s-Gibbs state. The case of u-Gibbs states is analogous.

Proposition 4.7. *If μ is an s-Gibbs state then $\hat{\mu}$ has product structure with density function $\hat{\rho}:\hat{\Sigma}\to\mathbb{R}_+$ uniformly continuous and bounded away from zero and infinity.*

The fact that $\hat{\mu}$ has product structure and its density $\hat{\rho}$ satisfies $\hat{\rho}(\hat{x})=\rho(\hat{\pi}(\hat{x}))$ follows from the previous corollary. So all that is left to do is to observe that $\hat{\rho}$ is uniformly continuous and bounded away from zero and infinity.

We start by recalling that, since μ is an s-Gibbs state, it admits a disintegration $(\mu^s_p)_p$ that is absolutely continuous with respect to the Lebesgue measure on stable leaves. Thus, by [18] we have that if $\mu^s_p=\psi\, m^s_p$, where m^s_p is the Lebesgue measure in the stable manifold of $p\in M$, and the density ψ is such that

$$\Delta(x,y):=\frac{\psi(x)}{\psi(y)}=\lim_{n\to\infty}\frac{JDf^n\mid_{E^s(x)}}{JDf^n\mid_{E^s(y)}} \tag{4}$$

for every x and y in the same stable manifold, where $JDf^n\mid_{E^s(x)}$ denotes the Jacobian of Df along E^s_x.

Let $\Gamma\subset\hat{\Sigma}\times\hat{\Sigma}$ be defined by $\Gamma=\{(\hat{x},\hat{y}),x_i=y_i \text{ for } i\geq 0\}$ and consider the function $\hat{\Delta}:\Gamma\to\mathbb{R}$ given by $\hat{\Delta}(\hat{x},\hat{y})=\Delta(\hat{\pi}(\hat{x}),\hat{\pi}(\hat{y}))$. We now study the regularity properties of $\hat{\Delta}$. In order to do that let us recall the notion of weak Hölder continuity.

Definition 4.8. Let (X,dist_X) be a metric space. We say that a function $g:X\to\mathbb{R}$ is *weak Hölder* if there exists $\gamma>0$ such that for every $\epsilon>0$ there exists $C(\epsilon)$ such that

$$|g(x)-g(y)|\leq C(\epsilon)\,\mathrm{dist}_X(x,y)^\gamma+\epsilon.$$

Lemma 4.9. *The function $\hat{\Delta}$ is weak Hölder (and consequently uniformly continuous) and bounded away from zero and infinity.*

Proof. Define $Jf^s(\hat{x}):=JDf\mid_{E^s(\hat{\pi}(\hat{x}))}$ and consider

$$\hat{\Delta}_n(\hat{x},\hat{y})=\frac{JDf^n\mid_{E^s(\hat{\pi}(\hat{x}))}}{JDf^n\mid_{E^s(\hat{\pi}(\hat{y}))}}.$$

By [27, Proposition 12.6] (recall the comment in the proof of Lemma 4.4) the map $\hat{x}\to E^s(\hat{\pi}(\hat{x}))$ is Hölder continuous. Since f is $C^{1+\beta}$, the Jacobian is non-negative and M is compact, then $\log Jf^s$ is also Hölder continuous. Let γ be its Hölder exponent and C be the corresponding constant. Thus, for every $(\hat{x},\hat{y})\in\Gamma$,

$$\begin{aligned}
|\log\hat{\Delta}_n(\hat{x},\hat{y})| &= \Big|\sum_{i=0}^{n-1}\log Jf^s(\hat{\sigma}^i\hat{x})-\log Jf^s(\hat{\sigma}^i\hat{y})\Big|\\
&\leq \sum_{i=0}^{n-1}|\log Jf^s(\hat{\sigma}^i\hat{x})-\log Jf^s(\hat{\sigma}^i\hat{y})|\\
&\leq C\sum_{i=0}^{n-1} d(\hat{\sigma}^i(\hat{x}),\hat{\sigma}^i(\hat{y}))^\gamma\leq C\frac{1}{1-e^{-1/2\chi\gamma}}d(\hat{x},\hat{y})^\gamma.
\end{aligned}$$

Therefore, $\log\hat{\Delta}_n$ converges uniformly to $\log\hat{\Delta}$ and

$$e^{-C/(1-e^{-1/2\chi\gamma})}\leq\hat{\Delta}(\hat{x},\hat{y})\leq e^{C/(1-e^{-1/2\chi\gamma})}. \tag{5}$$

We now claim that $\log \hat{\Delta}$ is weak Hölder. Indeed, if $(\hat{x}, \hat{y})$ and $(\hat{z}, \hat{p})$ are points in Γ then

$$\log \hat{\Delta}_n(\hat{x}, \hat{y}) - \log \hat{\Delta}_n(\hat{z}, \hat{p}) = \sum_{i=0}^{n-1} \big(\log Jf^s(\hat{\sigma}^i \hat{x}) - \log Jf^s(\hat{\sigma}^i \hat{y}) - \log Jf^s(\hat{\sigma}^i \hat{z}) + \log Jf^s(\hat{\sigma}^i \hat{p}) \big).$$

Consequently, recalling $\log Jf^s$ is (C, γ)-Hölder,

$$| \log \hat{\Delta}_n(\hat{x}, \hat{y}) - \log \hat{\Delta}_n(\hat{z}, \hat{p}) | \leq C \min_{1 \leq j \leq n} \Bigg(\sum_{i=0}^{j-1} d(\hat{\sigma}^i(\hat{x}), \hat{\sigma}^i(\hat{z}))^\gamma + d(\hat{\sigma}^i(\hat{y}), \hat{\sigma}^i(\hat{p}))^\gamma + \sum_{i=j}^{n-1} d(\hat{\sigma}^i(\hat{x}), \hat{\sigma}^i(\hat{y}))^\gamma + d(\hat{\sigma}^i(\hat{z}), \hat{\sigma}^i(\hat{p}))^\gamma \Bigg),$$

where the first summands in the right hand side are related to distances for points that are not necessarily in the same stable leaf and the second summands are related with points in the same stable manifolds. Now, since the map $\hat{\sigma}$ is Lipschitz with constant $L = e^{\frac{1}{2}\chi}$, it follows that

$$| \log \hat{\Delta}(\hat{x}, \hat{y}) - \log \hat{\Delta}(\hat{z}, \hat{p}) | \leq C \min_{n \in \mathbb{N}} \Bigg(\frac{L^{n\gamma} - 1}{L^\gamma - 1} (d(\hat{x}, \hat{z})^\gamma + d(\hat{y}, \hat{p})^\gamma) + \frac{2e^{-\frac{1}{2}\chi\gamma n}}{1 - e^{-\frac{1}{2}\chi\gamma}} \Bigg).$$

Thus, given $\epsilon > 0$ if we take n such that $2C \frac{e^{-\frac{1}{2}\chi\gamma n}}{1-e^{-\frac{1}{2}\chi\gamma}} \leq \epsilon$ and $C(\epsilon) = 2C \frac{L^{\gamma n}-1}{L^\gamma - 1}$ the claim follows. Consequently, using (5) and observing that $\hat{\Delta} = e^{\log \hat{\Delta}}$ we complete the proof of the proposition. □

Fix a cylinder $[R] \subset \hat{\Sigma}$ and let $\hat{\mu}^s_{x^+}$ be the measure in $\{x^+\} \times \hat{\Sigma}^-$ given by Lemma 4.5. Recalling $\mu^s_p = \psi\, m^s_p$, take $\hat{\psi}(\hat{x}) = \psi(\hat{\pi}(\hat{x}))$. We normalize the map $\hat{\psi}$ so that $\int_{\hat{\Sigma}^-} \hat{\psi}(\hat{x}) d\hat{m}^s_{x^+} = 1$. Then, using (4) it follows that, for every $\hat{y} \in \hat{\Sigma}$,

$$\hat{\psi}(\hat{y}) = \frac{1}{\int_{\hat{\Sigma}^-} \hat{\Delta}(\hat{y}, (x^-, y^+)) d\hat{m}^s_{x^+}(x^-)}.$$

In particular, it follows from the previous lemma that $\hat{\psi} : [R] \to \mathbb{R}_+$ is uniformly continuous and bounded away from zero and infinity.

Let $\tilde{R} := \hat{\pi}([R])$. By Lemma 4.4 we know that $\tilde{R}$ has local product structure. Thus, working again in the coordinates $\mathcal{N}^u \times \mathcal{N}^s$, by Rokhlin's Disintegration Theorem we have that

$$\mu \,|_{\tilde{R}} = \int_{\mathcal{N}^u} \psi(x) m^s_{x^u} \times \nu^u$$

where $m^s_{x^u}$ is the volume measure in the manifold $W^s(x^u) := \{x^u\} \times \mathcal{N}^s$. Moreover, in the product structure coordinates the unstable holonomy $h^u_{x^u, y^u} : W^s(x^u) \to W^s(y^u)$ is given by

$$h^u_{x^u,y^u} \colon \{x^u\} \times \mathcal{N}^s \to \{y^u\} \times \mathcal{N}^s \quad \text{and} \quad h^u_{x^u,y^u}(x^u, x^s) = (y^u, x^s).$$

By the absolute continuity of the unstable foliation we have that $(h^u_{x^u,y^u})_* m^s_{x^u} = Jh^u_{x^u,y^u} m^s_{y^u}$ and, by [26],

$$Jh^u_{x^u,y^u}(x^s) = \lim_{n\to\infty} \frac{JDf^{-n}\mid_{E^s(x^u,x^s)}}{JDf^{-n}\mid_{E^s(y^u,x^s)}}.$$

We now fix some point $(y^u, y^s) \in \mathcal{N}^u \times \mathcal{N}^s$ and define

$$\phi : \tilde{R} \to \mathbb{R} \quad \text{by} \quad \phi(x^u, x^s) = Jh^u_{x^u,y^u}(x^s).$$

Then, $\mu_{\tilde{R}} = \psi \cdot \phi m^s_{y^u} \times \nu^u$. Moreover, considering

$$\hat{\phi} : [R] \to \mathbb{R} \quad \text{given by} \quad \hat{\phi}(\hat{x}) = \phi(\hat{\pi}(\hat{x}))$$

where $\hat{\pi}(\hat{x}) = \hat{\pi}(x^+, x^-) = (x^u, x^s)$, an analogous calculation to the one we did in Lemma 4.9 shows that $\hat{\phi}$ is uniformly continuous and bounded away from zero and infinity. Thus, using Corollary 4.6 and the fact that $\rho = \psi \cdot \phi$ we get that $\hat{\rho} = \hat{\psi} \cdot \hat{\phi}$ and conclude the proof of Proposition 4.7.

Now, to conclude the proof of Proposition 4.1 we only have to observe that the measure $\hat{\mu}$ may be taken ergodic. This follows from our next lemma.

Lemma 4.10. *Almost every ergodic component of $\hat{\mu}$ projects to μ and has continuous product structure.*

Proof. By [27, Proposition 13.2] almost every ergodic component of $\hat{\mu}$ projects to μ. We claim that the ergodic decomposition of $\hat{\mu}$ is actually a sum of restrictions of $\hat{\mu}$ to union of cylinders. Indeed, let $\hat{\Sigma}_0 \subset \hat{\Sigma}$ be a full $\hat{\mu}$-measure subset such that for each point in $\hat{\Sigma}_0$ the Birkhoff theorem is satisfied for every continuous function. Divide $\hat{\Sigma}_0$ in equivalence classes: $\hat{x} \sim \hat{y}$ if the Birkhoff averages are the same for every continuous function. For any cylinder $[R]$, using a Hopf argument we conclude that $\hat{\mu}$-almost every point in $[R]$ are in the same equivalence class. So, every equivalence class is a union (module zero measure sets) of cylinders. In particular, there are at most countably many classes, which we denote by Γ_j, $j \in \mathbb{N}$. Thus, $\hat{\mu} = \sum_j \frac{1}{\hat{\mu}(\Gamma_j)} \hat{\mu}\mid_{\Gamma_j}$ and every $\hat{\mu}_j := \frac{1}{\hat{\mu}(\Gamma_j)} \hat{\mu}\mid_{\Gamma_j}$ is ergodic. Moreover, the restriction of $\hat{\mu}_j$ to a cylinder of positive measure is a multiple of the restriction of $\hat{\mu}$ to this cylinder. In particular, every $\hat{\mu}_j$ has continuous product structure. □

The proof of Proposition 4.1 is now complete.

5. Simplicity is typical

At this section we conclude the proof of Theorem A. We start by recalling a criterion established by Avila and Viana in [2] to get simplicity of the Lyapunov spectrum for fiber-bunched cocycles over topological Markov shifts.

5.1. **Simplicity Criterion.** Let $\hat{p} \in \hat{\Sigma}$ be a $\hat{\sigma}$-periodic point and denote by $q \geq 1$ its period. A point $\hat{z} \in W^u_{loc}(\hat{p})$ is called *homoclinic* if there exists some multiple $l \geq 1$ of q such that $\hat{\sigma}^l(\hat{z}) \in W^s_{loc}(\hat{p})$. The *transition map* $\psi^{\hat{A}}_{\hat{p},\hat{z}} : \mathbb{K}^d \to \mathbb{K}^d$ is then defined by

$$\psi^{\hat{A}}_{\hat{p},\hat{z}} = H^{s,\hat{A}}_{\hat{\sigma}^l(\hat{z})\hat{p}} \hat{A}^l(\hat{z}) H^{u,\hat{A}}_{\hat{p}\hat{z}}.$$

Definition 5.1. A cocycle $\hat{A} : \hat{\Sigma} \to GL(d, \mathbb{K})$ is *simple* if there exists a $\hat{\sigma}$-periodic point $\hat{p} \in \hat{\Sigma}$ of period $q \geq 1$ and some homoclinic point $\hat{z} \in W^u_{loc}(\hat{p})$ such that

(P) all eigenvalues of $\hat{A}^q(\hat{p})$ have distinct absolute values;

(T) for any invariant subspaces (sums of eigenspaces) E and F of $\hat{A}^q(\hat{p})$ with $\dim E + \dim F = d$, we have that $\psi^{\hat{A}}_{\hat{p},\hat{z}}(E) \cap F = \{0\}$.

Property (P) is called *pinching* while property (T) is called *twisting*. It was proven in [2, Theorem A] that pinching and twisting are indeed sufficient conditions to guarantee simplicity of the Lyapunov spectrum. More precisely,

Theorem 5.2. *If $\hat{A} : \hat{\Sigma} \to GL(d, \mathbb{K})$ is simple then the cocycle generated by $\hat{A}$ over $\hat{\sigma}$ has simple Lyapunov spectrum.*

We observe that, in order to apply Theorem 5.2, we only need $(\hat{\Sigma}, \hat{\sigma})$ to be a topological Markov shift and the measure $\hat{\mu}$ to have continuous product structure in the sense of Definition 3.3. Indeed, this follows from the comments on the Appendix of [2]. In particular, given (f, μ) satisfying the hypothesis of Theorem A it follows by Proposition 4.1 that the associated system $(\hat{\sigma}, \hat{\mu})$ satisfies the hypothesis of Avila-Viana's simplicity criterion.

5.2. **Conclusion of the proof.** To conclude the proof the idea is to combine Proposition 4.1 with Avila-Viana's simplicity criterion. We retain all the notation introduced at the previous sections. Before we go into the proof, let us recall a theorem from [6] which is going to be useful in what follows.

Theorem 5.3 (Theorem 0.1.3 of [6])**.** *There exists a function $\varphi : \mathcal{V} \times \mathcal{V} \to \mathbb{N}$ such that for every $x \in M$, which can be written as $x = \hat{\pi}(\hat{x})$ for some $\hat{x} = (x_n)_{n\in\mathbb{N}} \in \hat{\Sigma}$ with $x_{-i} = u$ and $x_j = v$ for infinitely many values of i and j in $\mathbb{N}$, we have that $\left|\hat{\pi}^{-1}(x)\right| \leq \varphi(u, v)$.*

Proposition 5.4. *Let $f : M \to M$ and $A : M \to GL(d, \mathbb{K})$ be as in Theorem A. Then, there exists $B : M \to GL(d, \mathbb{K})$ $C^{r,\alpha}-$arbitrarily close to A such that $\hat{B} = B \circ \hat{\pi}$ is simple.*

Proof. We start dealing with the case when $\mathbb{K} = \mathbb{C}$. Let $\hat{p} \in \hat{\Sigma}$ be a $\hat{\sigma}$-periodic point and denote by q its period, $\hat{z} \in W^u_{loc}(\hat{p})$ a homoclinic point such that $\hat{\sigma}^l(\hat{z}) \in W^s_{loc}(\hat{p})$ and consider $p = \hat{\pi}(\hat{p})$ and $z = \hat{\pi}(\hat{z})$. Since our map A takes values in $GL(d, \mathbb{C})$ we can perform a $C^{r,\alpha}$-small perturbation of A on a neighborhood of p obtaining a map $B' : M \to GL(d, \mathbb{C})$ arbitrarily close to A such that all eigenvalues of $B'^q(p)$ have distinct absolute values. In particular, $\hat{B}' = B' \circ \hat{\pi}$ satisfies the pinching condition.

Now, to get the twisting property we start observing that $\hat{\pi}(\hat{\sigma}^{nq}(\hat{z})) \neq \hat{\pi}(\hat{\sigma}^{jq}(\hat{z}))$ for every pair of different integers n and j. Indeed, if there were $n \neq j \in \mathbb{Z}$ so that $\hat{\pi}(\hat{\sigma}^{nq}(\hat{z})) = \hat{\pi}(\hat{\sigma}^{jq}(\hat{z}))$ then, using that $\hat{\pi} \circ \hat{\sigma}^\ell = f^\ell \circ \hat{\pi}$ for all $\ell \in \mathbb{Z}$, one could conclude that $f^{q(j-n)}(z)) = z$, which would imply that z is periodic and, recalling the properties of $\hat{\pi}$, contradicts the choice of $\hat{z}$. Thus, there exists a small neighborhood $V \subset M$ of z such that $\hat{z} \notin \hat{\sigma}^{nq}(\hat{\pi}^{-1}(V))$ for every $n \in \mathbb{Z} \setminus \{0\}$. In particular, modifying B' in this neighborhood V does not change the holonomies $H^{s,\hat{B}'}_{\hat{\sigma}^l(\hat{z})\hat{p}}$ and $H^{u,\hat{B}'}_{\hat{p}\hat{z}}$. So, $\psi^{\hat{B}}_{\hat{p}\hat{z}} = H^{s,\hat{B}'}_{\hat{\sigma}^l(\hat{z})\hat{p}} B^l(z) H^{u,\hat{B}'}_{\hat{p}\hat{z}}$ for any $C^{r,\alpha}$-perturbation $B : M \to GL(d, \mathbb{C})$ of B' that is supported on V. Consequently, we can find a cocycle B that is $C^{r,\alpha}$-arbitrarily close to B' , coinciding with B' outside V, such that $\psi^{\hat{B}}_{\hat{p}\hat{z}}$ does not preserve the invariant subspaces of $\hat{B}^q(\hat{p})$. This proves that $\hat{B}$ has the twisting property. Finally, observing that, since B coincides with B' outside V, the new map $\hat{B}$ still satisfies the pinching condition we conclude the proof of the proposition in the case when $\mathbb{K} = \mathbb{C}$.

To deal with the case when $\mathbb{K}=\mathbb{R}$ we observe that, up to a small initial perturbation, we can assume that $\hat{A}$ satisfies the twisting property at a periodic point $\hat{p}$ of period $q\geq 1$. Indeed, the same perturbation as the one done in the case when $\mathbb{K}=\mathbb{C}$ may be carried out in this new context to get twisting. On the other hand, to get pinching when $\mathbb{K}=\mathbb{R}$ is much more subtle because we can have pairs of complex eigenvalues. To bypass this issue, we explain how we can adapt ideas from [12, Section 9] to our context. We retain the notations already introduced.

After a small perturbation, if necessary, we can assume that there exists a splitting $\mathbb{R}^d=E^1(\hat{p})\oplus\cdots\oplus E^k(\hat{p})$ in invariant subspaces of $A^q(\hat{\pi}(\hat{p}))$ where each $E^j(\hat{p})$ is a one or two dimensional eigenspace and the eigenvalues corresponding to different subspaces have different absolute values. This perturbation can be done in a way that $\hat{A}$ still satisfies the twisting condition at $\hat{p}$. If all subspaces $E^j(\hat{p})$ are one dimensional then we are done. So, let us assume there exists j for which $\dim E^j(\hat{p})=2$. That is, $E^j(\hat{p})$ is associated to a complex eigenvalue of $A^q(\hat{\pi}(\hat{p}))$. As the cocycle $\hat{A}$ admits stable and unstable holonomies and $\hat{A}$ is twisting at $\hat{p}$, there exists a horseshoe H containing $\hat{p}$ and $\hat{z}$ and a dominated decomposition $E^1\oplus\cdots\oplus E^k$ over H that extends $E^1(\hat{p})\oplus\cdots\oplus E^k(\hat{p})$ (see [12, Section 9]). For $t\in[0,1]$, let $R_{t\delta}:M\to GL(d,\mathbb{R})$ be a C^∞ cocycle on M such that the matrix $R_{t\delta}(p)$ restricted to the plane $E^j(\hat{p})\subset\mathbb{R}^d$ is a rotation of angle $t\delta$ while restricted to the other subspaces $E^i(\hat{p})$ for $i\neq j$ is just the identity map. Consider the continuous family of cocycles $A_{\delta,t}:=R_{t\delta}A\in\mathcal{B}^{r,\alpha}_\chi(M)$.

By the symbolic dynamics, there exists a sequence $(\hat{x}_n)_n$ of periodic points in H such that each $\hat{x}_n$ has period $nq+l$ and such that $\hat{\sigma}^i(\hat{x}_n)$ is close to $\hat{\sigma}^i(\hat{z})$ for every $0\leq i\leq l$, and $\hat{\sigma}^{l+i}(\hat{x}_n)$ is close to $\hat{\sigma}^i(\hat{p})$ for every $0\leq i\leq qn$.

As we will perturb the cocycle generated by $A:M\to GL(d,\mathbb{R})$ over f we need real eigenvalues for the perturbation of $A^{\mathrm{per}(x_n)}(x_n)$, where $x_n:=\hat{\pi}(\hat{x}_n)$. The point x_n is clearly periodic for f. If the projection $\hat{\pi}$ restricted to the orbit $\{\hat{\sigma}^j(\hat{x}_n),\, j\geq 1\}$ is injective, then the argument given in [12, Section 9] works directly in our context. If this is not the case, we need to estimate the period of x_n.

If $u\in\mathcal{V}$ denotes the symbol such that $\hat{p}\in[u]$ then the symbol u appears infinitely many times in the coding of the periodic point $\hat{x}_n$ and, by Theorem 5.3, $\hat{\pi}^{-1}(x_n)$ has cardinality less than or equal to $m=\varphi(u,u)<\infty$. Then, the period of x_n satisfies $\mathrm{per}(x_n)\geq\frac{nq+l}{m}$. Now, the argument of [12, Section 9] shows that the variation of the rotation number of $A^{\mathrm{per}(x_n)}_{\delta,t}(x_n)$ is at least $\frac{nt\delta}{2m}$. Thus, for n sufficiently large we can find t close to 0 so that $A^{\mathrm{per}(x_n)}_{\delta,t}(x_n)$ has a real eigenvalue of multiplicity 2 in the plane $E^j(\hat{x}_n)$. Then, making an extra $C^{r,\alpha}$-small perturbation near the point x_n we obtain 2 different real eigenvalues on $E^j(\hat{x}_n)$.

Repeating this process a finite number of times (in fact, no more than d times) we find a cocycle B close to A and a periodic point $\hat{p}\in\hat{\Sigma}$ that has both the twisting and pinching properties. $\square$

In other terms, Proposition 5.4 informs that the subset of cocycles $B\in\mathcal{B}^{r,\alpha}_\chi(M)$ for which $\hat{B}$ satisfies the pinching and twisting conditions is dense in the set of fiber-bunched cocycles. We now observe that such set is also open. Indeed, let us consider the map

$$\hat{\pi}^*:\mathcal{B}^{r,\alpha}_\chi(M)\to H^\alpha(\hat{\Sigma})\text{ given by }\hat{\pi}^*(A)=\hat{A}=A\circ\hat{\pi}$$

which is a Lipschitz continuous map. In fact, the $C^{0,\alpha}$-norm of $\hat{\pi}^*A$ on $\hat{\Sigma}$ can be bounded by

$$\begin{aligned}
\|\hat{\pi}^*A\|_{0,\alpha} &= \sup_{\hat{x}\in\hat{\Sigma}} \left\|\hat{A}(\hat{x})\right\| + \sup_{\hat{x}\neq\hat{y}} \frac{\left\|\hat{A}(\hat{x}) - \hat{A}(\hat{y})\right\|}{d(\hat{x},\hat{y})^\alpha} \\
&= \sup_{x\in M} \|A(x)\| + \sup_{\hat{\pi}(\hat{x})\neq\hat{\pi}(\hat{y})} \left(\frac{\|A(\hat{\pi}(\hat{x})) - A(\hat{\pi}(\hat{y})\|}{\operatorname{dist}(\hat{\pi}(\hat{x}),\hat{\pi}(\hat{y}))^\alpha}\right)\left(\frac{\operatorname{dist}(\hat{\pi}(\hat{x}),\hat{\pi}(\hat{y}))}{d(\hat{x},\hat{y})}\right)^\alpha \\
&\leq \max\{1, \operatorname{Lip}(\hat{\pi})^\alpha\}\|A\|_{0,\alpha}.
\end{aligned}$$

Thus, since the pinching and twisting conditions are open conditions in $H^\alpha(\hat{\Sigma})$, given a cocycle $B \in \mathcal{B}^{r,\alpha}_\chi(M)$ such that $\hat{\pi}^*B$ satisfies these two conditions it follows that for every $B' \in \mathcal{B}^{r,\alpha}_\chi(M)$ that is $C^{r,\alpha}$ sufficiently close to B, $\hat{\pi}^*B'$ also satisfies pinching and twisting once $\hat{\pi}^*B'$ is close to $\hat{\pi}^*B$. This proves our claim.

Consequently, since (A, f, μ) and $(\hat{A}, \hat{\sigma}, \hat{\mu})$ have the same Lyapunov spectrum, combining the previous observations with Avila-Viana's simplicity criterion (Theorem 5.2) it follows that the set of cocycles with simple Lyapunov spectrum contains an open and dense subset of the space of fiber-bunched cocycles in $\mathcal{B}^{r,\alpha}_\chi(M)$ concluding the proof of the Theorem A.

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