

TOPOLOGICAL ENTROPY OF LEVEL SETS OF EMPIRICAL MEASURES FOR NON-UNIFORMLY EXPANDING MAPS

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ABSTRACT. In this article we obtain a variational principle for saturated sets for maps with some non-uniform specification properties. More precisely, we prove that the topological entropy of saturated sets coincides with the smallest measure theoretical entropy among the invariant measures in the accumulation set. Using this fact we provide lower bounds for the topological pressure of the irregular set and the level sets in the multifractal analysis of Birkhoff averages for continuous observables. The topological entropy estimates use as tool a non-uniform specification property on topologically large sets, which we prove to hold for open classes of non-uniformly expanding maps. In particular we prove some multifractal analysis results for C^1 -open classes of non-uniformly expanding local diffeomorphisms and Viana maps [2, 30].

1. INTRODUCTION

The study of the thermodynamic formalism and multifractal analysis for maps with some hyperbolicity has drawn the attention of many researchers from the theoretical physics and mathematics communities in the last decades. The general concept of multifractal analysis, that can be traced back to Besicovitch, is to decompose the phase space in subsets of points which have a similar dynamical behavior and to describe the size of each of such subsets from the geometrical or topological viewpoint. We refer the reader to [19, 21] and references therein. A first natural problem is the multifractal analysis of Birkhoff averages. Given a continuous map T of a compact metric space X and an observable $\phi : X \rightarrow \mathbb{R}$, it is natural to decompose

$$X = \bigcup_{\alpha \in \mathbb{R}} X_\alpha \cup I_\phi(T)$$

where $X_\alpha = \{x \in M : \lim_{n \rightarrow \infty} \frac{1}{n} S_n \psi(x) = \alpha\}$ are level sets of convergence for Birkhoff averages and the irregular set $I_\phi(T)$ is the set of points for which the Birkhoff averages for ϕ does not converge. The description of these level sets arise in the analysis of level sets in several important quantities in dynamics. For instance, in the special case of a C^1 interval map f , the level sets on the multifractal decomposition associated to the Birkhoff averages of the potential $\phi = \log |f'|$ coincides with the set of points with the same Lyapunov exponent.

From the measure theoretical viewpoint, Birkhoff's ergodic theorem guarantees that the irregular set has zero measure for every invariant measure. Nevertheless, irregular sets may have full topological entropy and full Hausdorff dimension [21]. A multifractal analysis program has been carried out successfully to deal e.g. with the Lyapunov spectrum, self-similar measures and local entropies [25, 19, 20, 4, 26, 9, 13, 14, 5, 32] in contexts of uniform and non-uniform hyperbolicity. Some methods to describe the topological entropy of level sets include the use of the differentiability and convexity of the pressure function, the existence of large deviations rate functions or the use of a specification property. Despite the fact that it provides stronger results, the description of the level sets of Birkhoff averages using the strict convexity of the pressure function is often possible only for observables that are at least Hölder continuous.

A major question in multifractal analysis is to describe the topological entropy or Hausdorff dimension of the so called saturated sets. A saturated set in X is the subset of points $x \in X$ whose accumulation points $V_T(x)$, in the weak* topology, of the empirical measures

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)}$$

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in the space of T -invariant probability measures coincides with a prescribed subset of invariant probability measures. Saturated sets can be used to describe convergence properties of Birkhoff averages with respect to every continuous observable. On the one hand, by weak* convergence, $V_T(x)$ is a singleton if and only if the Birkhoff averages of every continuous observable are convergent at x . On the other hand, if the accumulation set $V_T(x)$ contains invariant measures μ_1, μ_2 that distinguish a continuous potential ϕ (i.e. so that $\int \phi d\mu_1 \neq \int \phi d\mu_2$) then x is a Birkhoff irregular point associated to ϕ . Therefore, the global understanding of saturated sets in the space of probability measures allows to derive as a consequence, important results on the multifractal analysis of Birkhoff averages for continuous observables. One of the key difficulties to estimate the topological entropy of saturated sets consists of the fact that, since these enclose information on the Birkhoff averages of all *continuous* observables, no information on the strict convexity and differentiability of the pressure function can be used, even in a uniformly hyperbolic context. These difficulties were overcome by Takens and Verbistki [25] and by Pfister and Sullivan [23] that used some notions similar to specification, to characterize the topological pressure of saturated sets in the case of maps with some hyperbolicity, including the Maneville-Pomeau map and β -shifts. Some of the difficulties that arise in the use of the previous methods to obtain multifractal analysis results for multidimensional non-uniformly expanding maps are that the pressure function is much harder to describe and that specification is most likely to fail in the absence of uniform hyperbolicity (see e.g. [17, 24] and references therein).

In the present paper we contribute to the description of saturated sets for non-uniformly expanding maps. Although expanding and hyperbolic measures satisfy a non-uniform specification property [18, 28] (in rough terms, any finite pieces of orbits of generic points can be shadowed by a true orbit of the dynamics and that the time lag between pieces of orbits grow sublinearly on the size of the pieces of orbits) these notions have been established for generic points of ergodic and invariant measures. For that reason, such notions are not a suitable tool to describe multifractal analysis since ‘shadowable points’ consist of Birkhoff regular with respect to every continuous observable. For that reason, we provide a criterium for non-uniformly expanding maps to admit a topologically large set of points with (topological) non-uniform specification properties (cf. Theorem B). In particular, this provides a criterium for creating Birkhoff irregular points using orbits of points which are generic for different ergodic measures, something that was not possible using [18, 28]. This consists of a method different from the improved shadowing lemma developed by C. Liang, G. Liao, W. Sun and X. Tian [16] in the context of non-uniformly hyperbolic diffeomorphisms. Our main result (Theorem A) is that, under (topological) non-uniform specification properties, the topological entropy of the set of points whose empirical measures accumulates on two ergodic measures is bounded below by the minimum entropy among both measures. Using this, we provide a description of multifractal analysis of Birkhoff averages for continuous potentials for both irregular set and level sets (Corollaries A, B and C). We use our main results to study the multifractal analysis of Birkhoff averages for multidimensional non-uniformly expanding maps including Viana maps (we refer the reader to Section 2.3 for precise statements).

2. STATEMENT OF THE MAIN RESULTS

2.1. Topological entropy of level sets of empirical measures. Let $T : X \rightarrow X$ be a continuous map on a compact metric space X and $\mathcal{M}_T(X)$, $\mathcal{M}_T^e(X)$ denote the space of T -invariant, T -ergodic probability measures respectively. Given $x \in X$, let $V_T(x) \subset \mathcal{M}_T(X)$ be the set of accumulation points of the empirical measures

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)}.$$

For any $K \subset \mathcal{M}_T(X)$ it is natural to consider both sets

$$G^K = \{x \in X : V_T(x) \cap K \neq \emptyset\} \quad \text{and} \quad G_K = \{x \in X : V_T(x) = K\}.$$

It follows from the work of Pfister and Sullivan [23] that for any non-empty, compact set $K \subset \mathcal{M}_T(X)$,

$$h_{top}(T, G^K) \leq \sup\{h_\mu(f) : \mu \in K\} \tag{2.1}$$

and, if in addition $K \subset \mathcal{M}_T(X)$ is connected,

$$h_{top}(T, G_K) \leq \inf\{h_\mu(f) : \mu \in K\}. \tag{2.2}$$

Here we provide an extension of this result that makes use of some notions of non-uniform specification (we refer the reader to Subsection 4.3 for the definitions). Our main result here is as follows. 2

Theorem A. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X and let $\Delta \subseteq X$. Given two invariant measures $\mu_1, \mu_2 \in \mathcal{M}_T(\Delta)$,

(1) if T has non-uniform specification on Δ ,

$$\lim_{\epsilon \rightarrow 0} h_{top}(T, QG_{\mu_1, \mu_2}(\epsilon)) \geq \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}, \quad (2.3)$$

where $QG_{\mu_1, \mu_2}(\epsilon) = \{x \in X \mid d_{Haus}(V_T(x), K) < \epsilon\}$ and $K = \{t\mu_1 + (1-t)\mu_2 \mid t \in [0, 1]\}$.

(2) if T has strong non-uniform specification on Δ , then

$$h_{top}(T, G_{\mu_1, \mu_2}) = \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}, \quad (2.4)$$

where $G_{\mu_1, \mu_2} = \{x \in X \mid V_T(x) = \{t\mu_1 + (1-t)\mu_2 \mid t \in [0, 1]\}\}$.

Some comments are in order. If T has strong non-uniform specification on Δ then it is expected that if $K \subset \mathcal{M}_T(X)$ is a non-empty, compact and convex set then

$$h_{top}(T, G_K) = \inf\{h_{\mu}(T) : \mu \in K\} = \inf\{h_{\mu}(T) : \mu \in \partial K\}.$$

This is coherent with the second item of Theorem A since, in the case that $G_{\mu_1, \mu_2} = \{x \in X \mid V_T(x) = \{t\mu_1 + (1-t)\mu_2 \mid t \in [0, 1]\}\}$ the affine property of the measure theoretical entropy implies that $\min\{h_{t\mu_1 + (1-t)\mu_2}(T) : t \in [0, 1]\} = \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}$. Although we do not pursue that here, it is also expected our method to produce estimates on the topological pressure of saturated sets.

The classical approach to prove that the irregular set of continuous observables that are not cohomologous to a constant has full topological entropy uses specification to create points whose Birkhoff averages oscillate between space averages with respect to two ergodic measures μ_1, μ_2 . In this article we describe the set of points whose empirical measures have a prescribed accumulation set. In rough terms, given ergodic measures μ_1 and μ_2 , the proof of Theorem A follows the strategy of Pfister and Sullivan [23] to bound the entropy of the set of points whose empirical measures accumulate on a convex set K by the entropy of a Cantor sets whose topological entropy is bounded below by the smallest entropy of the measures in K . The construction of such a Cantor set uses the non-uniform specification property on some set that contains the basin of different ergodic and expanding measures. In particular, we can use these results to provide some results on the multifractal analysis of open classes of multidimensional non-uniformly expanding maps derived from expanding and Viana maps [30, 29].

2.2. Multifractal analysis of Birkhoff averages. Given a continuous function $\phi : X \rightarrow \mathbb{R}$ let

$$R_{\phi}(T) := \left\{ x \in X \mid \text{Birkhoff averages } \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) \text{ converge as } n \rightarrow +\infty \right\}.$$

For convenience, we refer to $R_{\phi}(T)$ as the *regular set with respect to ϕ* (or simply ϕ -regular set). The ϕ -irregular set is defined as $I_{\phi}(T) = X \setminus R_{\phi}(T)$. These two sets describe different asymptotic behavior under the observation of the given continuous function ϕ . Clearly, if ϕ is cohomologous to a constant, meaning that $\phi = u \circ T - u + c$ for some $c \in \mathbb{R}$ and $u : X \rightarrow \mathbb{R}$ continuous, then the Birkhoff averages converge at every point and the irregular set is empty. The next results show that if the dynamics satisfies some specification property then the topological entropy of the irregular set is bounded below by the entropy of the measures supported in a set with non-uniform specification. The notions of non-uniform specification are defined in Subsection 4.3 below.

2.2.1. Maps with strong non-uniform specification. For any continuous function $\phi : X \rightarrow \mathbb{R}$ and any $a \in \mathbb{R}$ the ϕ -regular set can be further decomposed in level sets. Indeed, given $a \in \mathbb{R}$ let

$$R_{\phi, a}(T) := \left\{ x \in X \mid \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) = a \right\},$$

and observe that $R_{\phi}(T)$ can be written as the following disjoint union

$$R_{\phi}(T) = \bigsqcup_{a \in \mathbb{R}} R_{\phi, a}(T).$$

Corollary A. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X , let $\Delta \subseteq X$ and let $\phi : X \rightarrow \mathbb{R}$ be a continuous function. Given any real number $a \in \mathbb{R}$, if T has strong non-uniform specification on Δ , then

$$h_{top}(T, R_{\phi, a}(T)) \geq \sup\{h_{\rho}(T) \mid \rho \in \mathcal{M}_T(\Delta) \text{ and } \int \phi d\rho = a\}.$$

2.2.2. *Maps with non-uniform specification.* In what follows we also consider dynamics where do not require the strong non-uniform specification. The first result concerns the irregular set of continuous observables.

Corollary B. *Let $T : X \rightarrow X$ be a continuous map of a compact metric space X , let $\Delta \subseteq X$ and $\phi : X \rightarrow \mathbb{R}$ be a continuous function. Suppose that T has non-uniform specification on Δ . If*

$$\inf_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu < \sup_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu, \quad (2.5)$$

then

$$h_{\text{top}}(T, I_\phi(T)) \geq \sup\{h_\rho(T) \mid \rho \in \mathcal{M}_T(\Delta)\}.$$

In particular, if T admits a maximal entropy measure supported on Δ then $I_\phi(T)$ carries full entropy.

Some comments are in order. It is well known that if T satisfies the specification property then the following are equivalent: (i) $\inf_{\mu \in \mathcal{M}_T} \int \phi d\mu < \sup_{\mu \in \mathcal{M}_T} \int \phi d\mu$, (ii) the Birkhoff averages of ϕ are not uniformly convergent to a constant, and (iii) the Birkhoff irregular set $I_\phi(T)$ is non-empty. Moreover, the Livsic theorem for mixing uniformly expanding maps assures that the later conditions hold if and only if ϕ is not cohomologous to a constant, or equivalently, there are expanding periodic points p_1 and p_2 so that $\int \phi d\delta_{\mathcal{O}(p_1)} < \int \phi d\delta_{\mathcal{O}(p_2)}$. In many applications of our results to non-uniformly expanding maps, the set $\Delta \subseteq X$ of points where nonuniform specification holds will be taken as the set of points whose sequence of hyperbolic times is non-lacunar, a notion that is intimately related to the integrability of the first hyperbolic time map (cf. Subsection 7.1).

Returning to the multifractal analysis, if, instead of its strong version, the dynamical system satisfies the non-uniform specification property then one could not estimate the topological entropy of level sets. Nevertheless, given a continuous potential ϕ , $a \in \mathbb{R}$ and $\sigma > 0$ one can estimate the size of the following sets obtained as approximations of level sets

$$R_{\phi,a,\sigma}(T) := \{x \in X \mid a - \sigma < \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(x)) < a + \sigma\}.$$

More precisely

Corollary C. *Let $T : X \rightarrow X$ be a continuous map of a compact metric space X , let $\Delta \subseteq X$ and $\phi : X \rightarrow \mathbb{R}$ be a continuous function. Given any real number $a \in \mathbb{R}$, if T has non-uniform specification on Δ , then*

(1) for any $\sigma > 0$,

$$h_{\text{top}}(T, R_{\phi,a,\sigma}(T)) \geq \sup\{h_\rho(T) \mid \rho \in \mathcal{M}_T(\Delta) \text{ and } \int \phi d\rho \in (a - \sigma, a + \sigma)\}.$$

(2) for any $\sigma > 0$,

$$\lim_{\sigma \rightarrow 0} h_{\text{top}}(T, R_{\phi,a,\sigma}(T)) \geq \sup\{h_\rho(T) \mid \rho \in \mathcal{M}_T(\Delta) \text{ and } \int \phi d\rho = a\}.$$

2.3. Non-uniformly expanding maps. Throughout this subsection let T be a C^1 -map on X which behaves like a power of the distance to a critical/singular region \mathcal{C} : there exists $B \geq 1, \beta > 0$ so that for all $x \in X \setminus \mathcal{C}$ and all $v \in T_x X$

$$(C1) \quad \frac{1}{B} \text{dist}(x, \mathcal{C}) \leq \frac{\|DT(x)v\|}{\|v\|} \leq B \text{dist}(x, \mathcal{C})$$

and

$$(C2) \quad \left| \log \|DT(x)^{-1}\| - \log \|DT(y)^{-1}\| \right| \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,\mathcal{C})^\beta}$$

$$(C3) \quad \left| \log |\det DT(x)| - \log |\det DT(y)| \right| \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,\mathcal{C})^\beta}.$$

for all points $x, y \in X \setminus \mathcal{C}$ satisfying $\text{dist}(x, y) < \text{dist}(x, \mathcal{C})/2$. Let B, β be given by condition (C2) and take $0 < b < \{\frac{1}{2}, \frac{1}{2\beta}\}$. The choice of these constants is important for the construction of hyperbolic times (see e.g. [2]).

Definition 2.1. *Assume that $\eta \in \mathcal{M}_T(X)$. We say that (T, η) is non-uniformly expanding if there exists $\sigma > 1$ such that η -almost every x satisfies*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|DT(T^j(x))^{-1}\| \leq -2 \log \sigma < 0 \quad (2.6)$$

and the slow recurrence condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for η -almost every point $x \in X$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(T^j(x), \mathcal{C}) < \varepsilon, \quad (2.7)$$

where $\text{dist}_\delta(x, \mathcal{C}) = 1$ if $\text{dist}(x, \mathcal{C}) > \delta$ and $\text{dist}_\delta(x, \mathcal{C}) = \text{dist}(x, \mathcal{C})$ otherwise.

The non-uniform hyperbolicity property for an ergodic measure μ implies on the exponential non-uniform specification property on its basin of attraction $B(\mu)$ [18, 28]. We are interested in extending this fact to sets that contains the basin of several invariant measures. This motivates the following definition.

We say that T is *non-uniformly expanding* if

$$\mathbb{M} = \bigcup_{\sigma > 1, \delta > 0} \mathbb{M}_{\sigma, \delta} \neq \emptyset$$

where, for any $\sigma > 1$ and $\delta > 0$, $\mathbb{M}_{\sigma, \delta} \subset \mathcal{M}_T(X)$ denotes the space of T -invariant probability measures η that have a non-lacunar sequence of (σ, δ) -hyperbolic times. It is known that if the first hyperbolic time is integrable then the sequence of hyperbolic times is non-lacunar (see e.g. [29]). We refer the reader to Subsection 7.1 for the definitions. The following result provides a criterium for a non-uniformly expanding map to satisfy the (strong) exponential non-uniform specification property.

Theorem B. *Let X be a compact Riemannian manifold and let $T : X \rightarrow X$ be a topologically exact C^1 -endomorphism. If $\mathbb{M} \neq \emptyset$ then T satisfies the (strong) exponential non-uniform specification property on a set $\Delta \subset X$ such that $\eta(\Delta) = 1$ for every $\eta \in \mathbb{M}$. More precisely, for any points x_1, x_2, \dots, x_k in Δ there exists $\delta > 0$ and $\sigma > 1$ (depending on the points) so that the following property holds: for any $n_1, \dots, n_k \geq 1$ there exists $y \in X$ so that*

$$d(T^j(x_i), T^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(y)) < \varepsilon \sigma^{-\frac{1}{2}(n_i-j)}$$

for every $0 \leq j \leq n_i$, $1 \leq i \leq k$ and $0 < \varepsilon < \delta$.

We observe that the dependence of the constants δ, σ on the points can be dropped out, for instance in the case that the set Δ consists of generic points for a finite number of ergodic and expanding maps (cf. Remark 7.1).

3. SOME EXAMPLES

It is clear both the non-uniform and strong non-uniform specification properties hold for uniformly hyperbolic diffeomorphisms and expanding maps. In particular, our results extend some well known results on the multifractal analysis in the uniformly hyperbolic setting. Moreover, our results can be applied for the one-dimensional Maneville-Pomeau transformations and Benedicks-Carleson quadratic maps. In what follows we focus on providing some new results on the multifractal analysis of multidimensional non-uniformly expanding maps.

3.1. Local diffeomorphisms with non-uniform expansion. The first class of examples we shall consider are multidimensional local diffeomorphisms obtained by local bifurcation of expanding maps, considered originally by Alves, Bonatti and Viana [2].

Let T_0 be an expanding map in \mathbb{T}^n and take a periodic point $p \in \mathbb{T}^n$ for T_0 . Let \mathcal{U} be a C^1 -open set of local diffeomorphisms so that every $T \in \mathcal{U}$ is a C^1 -local diffeomorphism obtained from T_0 by a bifurcation in a small neighborhood V of p in such a way that:

- (1) every point $x \in \mathbb{T}^n$ has some preimage outside V ;
- (2) V can be covered by $q < \deg(T)$ domains of injectiveness for T ;
- (3) $\|DT(x)^{-1}\| \leq \sigma^{-1}$ for every $x \in \mathbb{T}^n \setminus V$, and $\|DT(x)^{-1}\| \leq L$ for every $x \in \mathbb{T}^n$ where $\sigma > 1$ is large enough or $L > 0$ is sufficiently close to 1 (cf. [2]);
- (4) T is topologically exact: for every open set $W \subset \mathbb{T}^n$ there is $N \geq 1$ for which $T^N(W) = \mathbb{T}^n$.

Although the original expanding maps satisfy the specification property it is expected that most local diffeomorphisms in \mathcal{U} should not satisfy the specification property (we refer the reader to [24] for such a discussion in the case of partially hyperbolic diffeomorphisms). The coexistence of expanding and contracting behavior constitutes an obstruction to a global description of the multifractal description of the dynamics. We describe some applications in two situations:

(a) *Equilibrium states for potentials with low variation.* If $\phi : \mathbb{T}^n \rightarrow \mathbb{R}$ is a Hölder continuous potential so that

$$\sup \phi - \inf \phi < \log \deg(f) - \log q \quad (3.1)$$

then there exists a unique equilibrium state μ_ϕ for T with respect to ϕ , it is fully supported and it is an expanding measure with an integrable first hyperbolic time map ([29]). In particular T has a unique maximal entropy measure. Theorem B implies that there exists a dense set $\Delta \subset \mathbb{T}^n$ where T satisfies the strong (exponential) non-uniform specification property. There exists $t_0 > 1$ so that the potential $t\phi$ also satisfies the variation condition (3.1) for every $t \in [-t_0, t_0]$. To the best of our knowledge the following was the only result known on the multifractal analysis of these class of dynamics: the pressure function $[-t_0, t_0] \ni t \mapsto P_{top}(T, t\phi)$ is C^1 and, if $\alpha_1 = dP_{top}(T, t\phi)/dt|_{t=-t_0}$ and $\alpha_2 = dP_{top}(T, t\phi)/dt|_{t=t_0}$ then

$$h_{top}(T, R_{\phi, \alpha}(T)) = \inf_{t \in \mathbb{R}} \{P_{top}(T, t\phi) - \alpha t\}$$

for every $\alpha \in [\alpha_1, \alpha_2]$ (see [9, Proposition 7.2]). Our main result, Theorem A (2) implies that

$$h_{top}(T, G_{\mu_\phi}) = h_{\mu_\phi}(T) = P_{top}(T, \phi) - \int \phi d\mu_\phi$$

Moreover, if $\psi : \mathbb{T}^n \rightarrow \mathbb{R}$ is any continuous observable so that $\int \psi d\mu_1 \neq \int \psi d\mu_2$ for some expanding measures $\mu_1 \neq \mu_2$ then it follows from Corollary B that the irregular set carries full topological entropy, that is, $h_{top}(T, I_\psi(T)) = h_{top}(T)$.

(b) *SRB measures.* It follows from [2] that T has a unique (ergodic) probability measure μ_1 absolutely continuous with respect to Lebesgue, with density bounded away from zero and infinity and integrable first hyperbolic time map. Moreover, μ_1 is an equilibrium state for the potential $\phi = -\log |\det Df|$ (that might not satisfy the low variation condition) and is an expanding measure. If Δ is as above and

$$\inf_{\mu \in \mathcal{M}_T(\Delta)} \int \log |\det DT| d\mu < \sup_{\mu \in \mathcal{M}_T(\Delta)} \int \log |\det DT| d\mu$$

then Corollary B implies that $h_{top}(T, I_\phi(T)) \geq h_{\mu_1}(T)$.

3.2. Non-uniformly expanding endomorphisms with critical region. We also prove that the C^3 -robust class of multidimensional non-uniformly expanding maps with singularities known as Viana maps also satisfies the non-uniform specification property on a dense set.

In [30], Viana introduced a robust class of multidimensional non-uniformly hyperbolic maps with singularities. More precisely, these are obtained as C^3 small perturbations of the skew product T of the cylinder $S^1 \times I$ given by

$$T(\theta, x) = (d\theta \pmod{1}, 1 - ax^2 + \alpha \cos(2\pi\theta)),$$

where $d \geq 16$ is an integer, a is a Misiurewicz parameter for the quadratic family, and α is small. These maps admit a unique SRB measure μ (it is absolutely continuous with respect to $m = Leb$, has only positive exponents and $d\mu/dm \in L^p(m)$ where $p = d/(d-1)$) and are strong topologically mixing on the attractor $\Lambda = \bigcap_{n \geq 0} T^n(S^1 \times I)$: for every open set W there exists $n \geq 1$ such that $T^n(W) = \Lambda$. See [30, 1] for more details. Since T carries full supported expanding measures then there exists a dense set $\Delta \subset \Lambda$ of points where the strong non-uniformly expanding property holds. We observe that there are infinitely many expanding measures, including periodic Dirac masses at repelling periodic points. Moreover, it follows from [22] that T has a unique expanding map that is a maximal entropy measure and, consequently,

$$h_{top}(T, I_\phi(T)) = h_{top}(T)$$

for any continuous observable ϕ so that the function $\mu \mapsto \int \phi d\mu$ over all expanding measures is not constant.

4. SOME PRELIMINARIES

4.1. Entropy. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X . Now let us to recall the definition of topological entropy in [6] by Bowen. Let $E \subseteq X$ be a T -invariant set and $\mathfrak{F}_n(E, \epsilon)$ be the collection of all finite or countable covers of E by sets of the form $B_m(x, \epsilon)$ with $m \geq n$. We set

$$C(E; t, n, \epsilon, T) := \inf \left\{ \sum_{B_m(x, \epsilon) \in \mathcal{C}} 2^{-tm} : \mathcal{C} \in \mathfrak{F}_n(E, \epsilon) \right\},$$

and $C(E; t, \epsilon, T) := \lim_{n \rightarrow \infty} C(E; t, n, \epsilon, T)$. Then

$$h_{top}(E, \epsilon, T) := \inf\{t : C(E; t, \epsilon, T) = 0\} = \sup\{t : C(E; t, \epsilon, T) = \infty\}$$

and the *topological entropy* of E is defined as $h_{top}(T, E) := \lim_{\epsilon \rightarrow 0} h_{top}(E, \epsilon, T)$. In particular, if $E = X$, we also denote $h_{top}(T, X)$ by $h_{top}(T)$. It is known from [6] that if E is an invariant compact subset, then the topological entropy $h_{top}(T, E)$ is same as the classical definition using the exponential growth rate of the maximal cardinality of separated points (see Chapter 7 in [33]). Let us recall two basic facts about topological entropy from [6]: (i) for any subsets $Y_1 \subseteq Y_2 \subseteq X$,

$$h_{top}(T, Y_1) \leq h_{top}(T, Y_2), \quad (4.1)$$

and (ii) if one considers a collection $\{Y_i\}_{i=1}^{+\infty}$ of subsets of X then

$$h_{top}(T, \bigcup_{i=1}^{+\infty} Y_i) = \sup_{i \geq 1} h_{top}(T, Y_i). \quad (4.2)$$

Let $\mathcal{M}(X)$ denote the space of all Borel probability measures supported on X . Let $\xi = \{V_i | i = 1, 2, \dots, k\}$ be a finite partition of measurable sets of X . The entropy of $\nu \in \mathcal{M}(X)$ with respect to ξ is

$$H(\nu, \xi) := - \sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i).$$

Given $\Lambda \subset \mathbb{Z}$ we write $T^{\vee n} \xi := \bigvee_{k \in \Lambda} T^{-k} \xi$. The entropy of $\nu \in \mathcal{M}_T(X)$ with respect to ξ is

$$h(T, \nu, \xi) := \lim_{n \rightarrow \infty} \frac{1}{n} H(\nu, T^{\vee n} \xi),$$

and the *metric entropy* of ν is $h_\nu(T) := \sup_\xi h(T, \nu, \xi)$. For more information of metric entropy we refer to Chapter 4 of [33].

Let $\mu \in \mathcal{M}(X)$. The *measure-theoretical lower and upper entropies* of μ are defined respectively by

$$\underline{h}_\mu(T) = \int \underline{h}_\mu(T, x) d\mu(x), \quad \bar{h}_\mu(T) = \int \bar{h}_\mu(T, x) d\mu(x),$$

where

$$\begin{aligned} \underline{h}_\mu(T, x) &= \lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow +\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)), \\ \bar{h}_\mu(T, x) &= \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)). \end{aligned}$$

Brin and Katok [8] proved that for any $\mu \in \mathcal{M}_T^\epsilon(X)$, $\underline{h}_\mu(T, x) = \bar{h}_\mu(T, x)$ for μ a.e. $x \in X$, and $\int \underline{h}_\mu(T, x) d\mu(x) = h_\mu(T)$. So for $\mu \in \mathcal{M}_T(X)$,

$$\underline{h}_\mu(T) = \bar{h}_\mu(T) = h_\mu(T).$$

From [12, Proposition 1.2] we know that if $E \subseteq X$ is non-empty and compact, then

$$h_{top}(T, E) = \sup\{\underline{h}_\mu(T) : \mu \in \mathcal{M}(X), \mu(E) = 1\}. \quad (4.3)$$

We also need to recall Katok's definition of metric entropy (see [15]). It is defined for ergodic measures and equivalent to the classical one. Let $Z \subset X$. A set S is (n, ϵ) -separated for Z if $S \subset Z$ and $d_n(x, y) > \epsilon$ for any $x, y \in S$ and $x \neq y$. A set $S \subset Z$ is (n, ϵ) -spanning for Z if for any $x \in Z$, there exists $y \in S$ such that $d_n(x, y) \leq \epsilon$. Let $\mu \in \mathcal{M}_T^\epsilon(X)$. For $\epsilon > 0$, $\rho \in (0, 1)$ and $l \geq 1$, let $N_l^\mu(\epsilon, \rho)$ be the minimal number of ϵ -balls $B_l(x, \epsilon)$ in the d_l -metric, which cover a set $Z \subseteq X$ of measure at least $1 - \rho$. Define

$$h_\mu(T, \epsilon) = \liminf_{l \rightarrow \infty} \frac{1}{l} \log N_l^\mu(\epsilon, \rho), \quad h'_\mu(T, \epsilon) = \limsup_{l \rightarrow \infty} \frac{1}{l} \log N_l^\mu(\epsilon, \rho).$$

Then from [15] we know

$$h_\mu(T) = \lim_{\epsilon \rightarrow 0} h_\mu(T, \epsilon) = \lim_{\epsilon \rightarrow 0} h'_\mu(T, \epsilon). \quad (4.4)$$

4.2. Invariant measures. Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a dense subset of $C(X, \mathbb{R})$ and set $\|\varphi_i\| = \max\{|\varphi_i(x)| : x \in X\}$. The expression

$$\rho(\xi, \tau) = \sum_{j=1}^{\infty} \frac{|\int \varphi_j d\xi - \int \varphi_j d\tau|}{2^j \|\varphi_j\|}$$

defines a metric on $\mathcal{M}(X)$ which is compatible with the *weak** topology (see [33]). Observe that

$$\rho(\xi, \tau) \leq 2 \text{ for any } \xi, \tau \in \mathcal{M}(X). \quad (4.5)$$

It is well known that the natural projection $x \mapsto \delta_x$ is continuous and, if we define the push-forward operator T_f on $\mathcal{M}(X)$ by $T_f \mu(A) = \mu(T^{-1}(A))$, we can identify (X, f) with T_f restricted to the set of Dirac measures (these are conjugate). Therefore, assume without loss of generality that the metric d is inherited from ρ , defined by $d(x, y) = \rho(\delta_x, \delta_y)$. The following facts are simple consequences of (4.5) and the fact that $\rho(\delta_x, \delta_y) = d(x, y)$:

Lemma 4.1. [11, Lemma 2.1] *Let (X, T) be a continuous dynamical system and let $x \in X$.*

- (1) *If $0 \leq k < n < m$ then $\rho(\mathcal{E}_m(x), \mathcal{E}_n(T^k(x))) \leq \frac{2}{m}(m - n + k)$;*
- (2) *Given $\varepsilon > 0$ and $p \in \mathbb{N}$, for every $y \in B_p(x, \varepsilon)$ we have $\rho(\mathcal{E}_p(y), \mathcal{E}_p(x)) < \varepsilon$;*
- (3) *Given $\varepsilon > 0$ and $p, q \in \mathbb{N}$ satisfying $p \leq q \leq (1 + \varepsilon/2)p$, for every $y \in B_p(x, \varepsilon)$ we have $\rho(\mathcal{E}_q(y), \mathcal{E}_p(x)) < 2\varepsilon$.*

4.3. Some forms of specification. In the last years some different versions of non-uniform specification have been introduced, many of them unrelated. Here we define the notions that will be used throughout this paper. For any $n \in \mathbb{N}$, the d_n -distance between $x, y \in X$ is defined as

$$d_n(x, y) := \max_{0 \leq i \leq n-1} \{d(T^i x, T^i y)\}.$$

Let $x \in X$, $n \geq 1$ and $\varepsilon > 0$. The dynamical ball $B_n(x, \varepsilon)$ is defined as the set

$$B_n(x, \varepsilon) := \{y \in X \mid d_n(x, y) \leq \varepsilon\}.$$

The following definition is the topological counterpart of [28, Definition 2.2].

Definition 4.2. We say that T satisfies the *non-uniform specification property on the set $\Delta \subseteq X$* , if there exists $\delta > 0$ such that for every $x \in \Delta$, every $n \geq 1$ and every $0 < \varepsilon < \delta$ there exists a positive integer $p(x, n, \varepsilon) \geq 1$ so that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} p(x, n, \varepsilon) = 0$$

and the following holds: given points x_1, \dots, x_k in Δ and positive integers n_1, \dots, n_k , if $p_i \geq p(x_i, n_i, \varepsilon)$ then there exists $z \in X$ that ε -shadows the orbits of each x_i during n_i iterates with a time lag of $p(x_i, n_i, \varepsilon)$ in between $T^{n_i}(x_i)$ and $x_i + 1$, that is, $z \in B(x_1, n_1, \varepsilon)$ and $T^{n_1+p_1+\dots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$ for every $2 \leq i \leq k$.

In the case that T is a topologically exact C^1 -local diffeomorphism and μ is a T -invariant and ergodic probability measure with infinitely many hyperbolic times then f satisfies the non-uniform specification property on the set $B(\mu)$, where

$$B(\mu) = \left\{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} \rightarrow \mu \text{ in the weak*-topology}\right\}$$

stands for the ergodic basin of attraction of μ (see [28]). Here we also consider two similar-flavored notions with a (summable) control of the distances during the ‘shadowing’.

Definition 4.3. We say that T satisfies the *strong non-uniform specification property on some set $\Delta \subseteq X$* , if the non-uniform specification property holds on Δ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} p(x, n, \varepsilon) = 0$$

for every $0 < \varepsilon < \delta$ and the shadowing sizes during the non-uniform specification process satisfy

$$\sum_{j=0}^{n_i-1} d(T^j(x_i), T^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(y)) < \varepsilon, \quad \forall 1 \leq i \leq k.$$

This condition is weaker than the following one, which requires the shadowing distances to be exponentially small and that holds for broad classes of non-uniformly expanding maps (recall Theorem B).

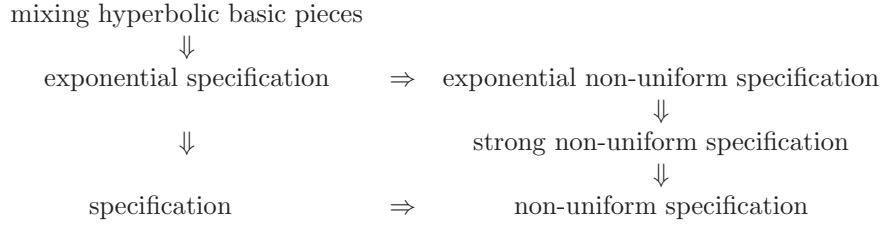
Definition 4.4. We say that T satisfies the *exponential non-uniform specification property with respect to the exponent $\lambda > 0$ on the set $\Delta \subseteq X$* , if non-uniform specification property holds on Δ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} p(x, n, \varepsilon) = 0$$

for every $0 < \varepsilon < \delta$ and the shadowing sizes in the non-uniform specification can be taken exponentially small, i.e.,

$$d(T^j(x_i), T^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(y)) < \varepsilon e^{-\lambda \min\{j, n_i-j\}}, \quad 0 \leq j \leq n_i, \quad 1 \leq i \leq k.$$

In the previous definitions the term ‘non-uniform’ refers to the time lag between the shadowing of the pieces while the terms ‘strong’ and ‘exponential’ refer to sharper approximations of the distances involved in the shadowing. Specification properties have been proved to be strongly related to hyperbolicity. The original specification property was first introduced by Bowen in [7] (see also [10]) and requires the function $p(x, n, \varepsilon)$ to depend just on ε . Thus, in comparison with non-uniform specification properties, the time required to be able to shadow the finite pieces of orbits does not depend on their size. Similarly, exponential specification is stronger than exponential non-uniform specification in the same sense (see e.g. [27]). Indeed the following relations hold:



5. PROOF OF THEOREM A

5.1. Proof of Theorem A (1). We assume without loss of generality that the measures μ_1, μ_2 are ergodic. In general, since one can use the ergodic decomposition of invariant measures exactly as in [23, Lemma 6.2] we shall omit the details here. Let $h^* = \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}$. We will show that for any $\gamma > 0$, $\varepsilon_0 > 0$, there is some $\varepsilon \in (0, \varepsilon_0)$ such that

$$h_{\text{top}}(T, QG_{\mu_1, \mu_2}(\varepsilon)) \geq h^* - 5\gamma. \quad (5.1)$$

This implies that for any $\gamma > 0$, $\varepsilon_0 > 0$,

$$h_{\text{top}}(T, QG_{\mu_1, \mu_2}(\varepsilon_0)) \geq h^* - 5\gamma,$$

because $QG_{\mu_1, \mu_2}(\varepsilon) \subseteq QG_{\mu_1, \mu_2}(\varepsilon_0)$. Then (2.3) holds. Now we fix $\gamma > 0$, $\varepsilon_0 > 0$ and start to prove (5.1).

Step 1. Choice of separated sets.

For $\varepsilon > 0, l \geq 1, \theta > 0$, define

$$\Delta_{\varepsilon, l}^{\theta} := \{x \in \Delta \mid \frac{p(x, n, \varepsilon)}{n} < \theta \text{ holds for any } n \geq l\}.$$

Note that if $l \geq l'$, then $\Delta_{\varepsilon, l}^{\theta} \supseteq \Delta_{\varepsilon, l'}^{\theta}$ and for any $\mu \in \mathcal{M}_T(\Delta)$ and $\theta > 0$, one has

$$\lim_{\varepsilon \rightarrow 0} \lim_{l \rightarrow \infty} \mu(\Delta_{\varepsilon, l}^{\theta}) = 1.$$

Fix a rational number $\theta \in (0, \frac{1}{3}\varepsilon_0)$ such that

$$\frac{1}{1+\theta}(h^* - 3\gamma) > h^* - 4\gamma. \quad (5.2)$$

Fix a number $\rho_0 \in (0, 1)$. For the measures μ_1 and μ_2 , by Katok’s definition of metric entropy, take $\varepsilon \in (0, \frac{1}{3}\varepsilon_0)$ such that for $i = 1, 2$,

$$h_{\mu_i}(T, 4\varepsilon) = \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_n^{\mu_i}(4\varepsilon, \rho_0) > h_{\mu_i}(T) - \gamma; \quad (5.3)$$

and simultaneously, $\lim_{l \rightarrow \infty} \mu_i(\Delta_{\varepsilon, l}^{\theta}) > 1 - \rho_0$. Let $\varepsilon = \varepsilon + 2\theta$. By the choice of ε and θ , one has $\varepsilon \in (0, \varepsilon_0)$.

Take l large enough such that

$$\mu_1(\Delta_{\varepsilon,l}^\theta) > 1 - \rho_0, \quad \mu_2(\Delta_{\varepsilon,l}^\theta) > 1 - \rho_0.$$

Let $g : \mathbb{N} \rightarrow \{1, 2\}$ be given by $g(k) = (k + 1) \pmod{2} + 1$. Choose a strictly decreasing sequence $\delta_k \rightarrow 0$ and a strictly increasing sequence $L_k \rightarrow \infty$ so that the set

$$J_k := \{x \in \Delta_{\varepsilon,l}^\theta : \rho(\mathcal{E}_n(x), \mu_{g(k)}) < \delta_k \text{ for all } n \geq L_k\}$$

satisfies $\mu_{g(k)}(J_k) > 1 - \rho_0$ for all $k \geq 1$. This is a simple consequence of the fact that the ergodic basin of attraction of μ_i is a full μ_i -measure set. Let E_n^k be a maximal $(n, 4\varepsilon)$ -separated set for J_k . Notice that a maximal $(n, 4\varepsilon)$ -separated set for J_k is also a $(n, 4\varepsilon)$ -spanning set for J_k and thus E_n^k is also a $(n, 4\varepsilon)$ -spanning set for J_k . Since $\mu_{g(k)}(J_k) > 1 - \rho_0$, then by (5.3)

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#E_n^k \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_n^{\mu_{g(k)}}(4\varepsilon, \rho_0) > h_{\mu_{g(k)}}(T) - \gamma. \quad (5.4)$$

Take n_k be a strictly increasing sequence such that $n_k \geq L_k$, $\frac{1}{n_k} \log \#E_{n_k}^k > h_{\mu_{g(k)}}(T) - 3\gamma$ and $\frac{n_k}{n_{k+1}} \rightarrow 0$. We may assume in addition that each $n_k\theta$ is always integer. Let

$$\mathcal{S}_k := E_{n_k}^k, \quad \mu_k = \mu_{g(k)}, \quad m_k = n_k\theta.$$

From above construction, we have

$$\#\mathcal{S}_k \geq \exp((h_{\mu_k}(T) - 3\gamma)n_k). \quad (5.5)$$

Step 2. Construction of the fractal F .

The purpose here is to construct a fractal F with topological entropy bounded below by h^* and in such a way that the accumulation points of empirical measures associated to points of F remain ε -close to the line segment $K = \{t\mu_1 + (1-t)\mu_2 : t \in [0, 1]\}$. Let us choose a sequence with $N_0 = 0$ and N_k increasing to ∞ sufficiently quickly so that

$$\lim_{k \rightarrow \infty} \frac{n_{k+1} + m_{k+1}}{N_k} = 0, \quad \lim_{k \rightarrow \infty} \frac{N_1(n_1 + m_1) + \dots + N_k(n_k + m_k)}{N_{k+1}} = 0. \quad (5.6)$$

Let $\underline{x}_i = (x_1^i, \dots, x_{N_i}^i) \in \mathcal{S}_i^{N_i}$. For any $(\underline{x}_1, \dots, \underline{x}_k) \in \mathcal{S}_1^{N_1} \times \dots \times \mathcal{S}_k^{N_k}$, by non-uniform specification property, we have

$$B(\underline{x}_1, \dots, \underline{x}_k) := \bigcap_{i=1}^k \bigcap_{j=1}^{N_i} T^{-\sum_{l=0}^{i-1} N_l(n_l + m_l) - (i-1)N - (j-1)(n_i + m_i)} B_{n_i}(x_j^i, \varepsilon) \neq \emptyset. \quad (5.7)$$

For every $k \geq 1$, consider the set F_k defined as

$$F_k = \overline{B(\underline{x}_1, \dots, \underline{x}_k)} : (\underline{x}_1, \dots, \underline{x}_k) \in \mathcal{S}_1^{N_1} \times \dots \times \mathcal{S}_k^{N_k}.$$

Note that each set of F_k is non-empty and compact. Moreover, since $F_{k+1} \subseteq F_k$ for every $k \geq 1$ then $F := \bigcap_{k=1}^{\infty} F_k$ is compact and non-empty.

Lemma 5.1. *For any $p \in F$, $d_{Haus}(V_T(p), K) < \varepsilon$. In other words, $F \subseteq QG_{\mu_1, \mu_2}(\varepsilon)$.*

Proof. The proof is standard. Define $t_k = \sum_{i=0}^k N_i(n_i + m_i)$. Choose $p \in F$ and let $p_k := T^{t_{k-1}}p$. Then there exists $(x_1^k, \dots, x_{N_k}^k) \in \mathcal{S}_k^{N_k}$ such that

$$p_k \in \bigcap_{j=1}^{N_k} T^{-(j-1)(n_k + m_k)} \overline{\bigcap_{t=0}^{n_k-1} T^{-t} B(T^t x_j^k, \varepsilon)}.$$

By Lemma 4.1 items (1) and (2), the distance between empirical measures of each of the x'_i 's can be bounded as:

$$\begin{aligned}
& \rho\left(\frac{1}{N_k} \sum_{j=1}^{N_k} \mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{t_k-t_{k-1}}(p_k)\right) \\
& \leq \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k+m_k}(T^{(j-1)(n_k+m_k)} p_k)) \\
& \leq \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k)) \\
& \quad + \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k), \mathcal{E}_{n_k+m_k}(T^{(j-1)(n_k+m_k)} p_k)) \\
& \leq \varepsilon + 2 \frac{m_k}{n_k + m_k} \leq \varepsilon + 2\theta.
\end{aligned} \tag{5.8}$$

Using Lemma 4.1 (1) one more we deduce that

$$\begin{aligned}
& \rho(\mathcal{E}_{t_k}(p), \mu_k) \\
& \leq \rho(\mathcal{E}_{t_k}(p), \mathcal{E}_{t_k-t_{k-1}}(p_k)) + \rho\left(\frac{1}{N_k} \sum_{j=1}^{N_k} \mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{t_k-t_{k-1}}(p_k)\right) + \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mu_k) \\
& \leq 2 \frac{t_k - (t_k - t_{k-1}) + t_{k-1}}{t_k} + (\varepsilon + 2\theta) + \delta_k.
\end{aligned}$$

Then $\limsup_{k \rightarrow \infty} \rho(\mathcal{E}_{t_k}(p), \mu_k) \leq \varepsilon$. For other subsequences the strategy of the proof follows well known ideas by constructing an appropriate convex sum of the measures μ_1 and μ_2 . Here we omit the details. \square

Step 3. *Construction of a special sequence of measures ω_k .*

We must first undertake an intermediate construction. For each $\underline{x} = (x_1, \dots, x_k) \in \mathcal{S}_1^{N_1} \times \dots \times \mathcal{S}_k^{N_k}$, we choose one point $z = z(\underline{x})$ such that $z \in B(x_1, \dots, x_k)$. If \mathcal{T}_k is the set of all points constructed in this way we compute its cardinality below.

Lemma 5.2. *Given $k \geq 1$, if $\underline{x}, \underline{y} \in \mathcal{S}_1^{N_1} \times \dots \times \mathcal{S}_k^{N_k}$ are distinct then any $z_1 := z(\underline{x})$ and $z_2 := z(\underline{y})$ are different points. In particular $\#\mathcal{T}_k = \#\mathcal{S}_1^{N_1} \dots \#\mathcal{S}_k^{N_k}$.*

Proof. Since $\underline{x} \neq \underline{y}$, there exists i, j such that $x_j^i \neq y_j^i$. We have

$$d_{n_i}(x_j^i, T^h z_1) < \varepsilon \text{ and } d_{n_i}(y_j^i, T^h z_2) < \varepsilon,$$

where $h = \sum_{l=0}^{i-1} N_l(n_l + m_l) + (j-1)(n_i + m_i)$. Since $d_{n_i}(x_j^i, y_j^i) > 4\varepsilon$ then $d_{n_i}(T^h z_1, T^h z_2) > 2\varepsilon$, which guarantees that the points are distinct. \square

Now we start to define the measures on F . For each $k \geq 1$ consider the measure

$$\nu_k := \sum_{z \in \mathcal{T}_k} \delta_z$$

obtained as the sum of Dirac measure at points of \mathcal{T}_k . Consider the probability measure $\omega_k := \frac{1}{\#\mathcal{T}_k} \nu_k$ obtained by normalization of ν_k .

Lemma 5.3. *Let ω be an accumulation point of $(\omega_k)_{k \geq 1}$ in the weak* topology. Then $\omega(F) = 1$.*

Proof. For any fixed l and every $p \geq 0$, notice that $\omega_{l+p}(F_{l+p}) = 1$ and $F_{l+p} \subseteq F_l$. Then $\omega_{l+p}(F_l) = 1$. By assumption there exists $l_k \rightarrow \infty$ such that $\omega = \lim_{k \rightarrow \infty} \omega_{l_k}$. Since each F_l is closed, by weak* convergence,

$$\omega(F_l) \geq \limsup_{k \rightarrow \infty} \omega_{l_k}(F_l) = 1.$$

This implies that $\omega(F) = \lim_{l \rightarrow \infty} \omega(F_l) = 1$. \square

Step 4. Estimate of $h_{top}(T, QG_{\mu_1, \mu_2}(\epsilon))$.

The strategy of the proof is to use (cf. equation (4.3)) that

$$h_{top}(T, F) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(X), \mu(F) = 1\}$$

and to provide lower bounds on the lower entropy of probability measures supported on the fractal F . Let $\mathcal{B} := B_n(q, \epsilon)$ be an arbitrary dynamical ball with $\mathcal{B} \cap F \neq \emptyset$. Let k be the integer number such that $t_k \leq n < t_{k+1}$. We firstly consider n with $t_k \leq n < t_{k+1} - m_k$. Let $j \in \{0, \dots, N_{k+1} - 1\}$ be the number such that

$$t_k + (n_{k+1} + m_{k+1})j \leq n < t_k + (n_{k+1} + m_{k+1})(j + 1).$$

We suppose that $j \geq 1$ (and leave the details of the easier case $j = 0$ to the reader). Now we start to estimate the number of points in the set $\mathcal{B} \cap \mathcal{T}_{k+p}$.

Lemma 5.4. For every integer $p \geq 1$, one has $\omega_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$.

Proof. Firstly let us assume that $p = 1$ and prove that $\omega_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$. We need an upper bound estimate for the number of points which can be in the set $\mathcal{T}_{k+1} \cap \mathcal{B}$. If $\omega_{k+1}(\mathcal{B}) = 0$ we are done. Thus we may assume throughout that $\omega_{k+1}(\mathcal{B}) > 0$ and, in particular, $\mathcal{T}_{k+1} \cap \mathcal{B} \neq \emptyset$.

Since $\mathcal{T}_{k+1} \cap \mathcal{B} \neq \emptyset$ there exist $\underline{x} \in \mathcal{S}_1^{N_1} \times \dots \times \mathcal{S}_k^{N_k}$ and $\underline{x}_{k+1} \in \mathcal{S}_{k+1}^{N_{k+1}}$ and some point $z = z(\underline{x}, \underline{x}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$. Fix \underline{x} and \underline{x}_{k+1} as above and consider the set

$$\mathcal{C}_{\underline{x}; x_1, \dots, x_j} = \{z(\underline{x}, y_1, \dots, y_{N_{k+1}}) \in \mathcal{T}_{k+1} : y_1 = x_1, \dots, y_j = x_j\}.$$

This corresponds roughly to the ‘cylinder’ formed by all possible choices of points that shadow the coordinate elements that determine the point \underline{x} and the first j components of the point \underline{x}_{k+1} .

Claim: If $z' = z(\underline{y}, \underline{y}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ for some \underline{y} and \underline{y}_{k+1} then $z' \in \mathcal{C}_{\underline{x}; x_1, \dots, x_j}$.

Proof of the claim: Take $z' = z(\underline{y}, \underline{y}_{k+1}) \in \mathcal{T}_{k+1} \cap \mathcal{B}$ for some \underline{y} and \underline{y}_{k+1} . We prove that $x_l = y_l$ for $l \in \{1, 2, \dots, j\}$ (since the proof that $\underline{x} = \underline{y}$ is completely similar we shall omit it). Since $z, z' \in \mathcal{B} = B_n(q, \epsilon)$ then

$$d_n(z, z') < 2\epsilon, \quad (5.9)$$

Assume by contradiction that there exists $1 \leq l \leq j$ so that $y_l \neq x_l$, and let $a_l = t_k + (l - 1)(n_{k+1} + m_{k+1})$. Recall that $d_{n_{k+1}}(x_l, y_l) > 4\epsilon$. Using

$$d_{n_{k+1}}(T^{a_l}z, x_l) < \epsilon \text{ and } d_{n_{k+1}}(T^{a_l}z', y_l) < \epsilon,$$

then we obtain that

$$\begin{aligned} d_n(z, z') &\geq d_{n_{k+1}}(T^{a_l}z, T^{a_l}z') \\ &\geq d_{n_{k+1}}(x_l, y_l) - d_{n_{k+1}}(T^{a_l}z, x_l) - d_{n_{k+1}}(T^{a_l}z', y_l) > 2\epsilon, \end{aligned}$$

which contradicts (5.9). This proves the claim. \square

We proceed with the proof of the lemma. By the claim we have that

$$\nu_{k+1}(\mathcal{B}) \leq \#\mathcal{C}_{\underline{x}; x_1, \dots, x_j} = (\#\mathcal{S}_{k+1})^{N_{k+1}-j}$$

and, consequently,

$$\omega_{k+1}(\mathcal{B}) \leq (\#\mathcal{T}_{k+1})^{-1}(\#\mathcal{S}_{k+1})^{N_{k+1}-j} = (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}.$$

This proves the lemma in the case that $p = 1$. In the case that $p > 1$, similar estimates as above yield that $\nu_{k+p}(\mathcal{B}) \leq \#\mathcal{C}_{\underline{x}; x_1, \dots, x_j}(\#\mathcal{S}_{k+2})^{N_{k+1}} \dots (\#\mathcal{S}_{k+p})^{N_{k+p}}$. Dividing by $\#\mathcal{T}_{k+p}$, it follows that $\omega_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_k)^{-1}(\#\mathcal{S}_{k+1})^{-j}$ proving the lemma. \square

To finalize the proof of Theorem A (1) we are left to give the estimate on the topological entropy of the set $QG_{\mu_1, \mu_2}(\epsilon)$. Using (5.5), (5.2) and $m_i = n_i\theta$, we have

$$\begin{aligned} \#\mathcal{T}_k(\#\mathcal{S}_{k+1})^j &\geq \exp\{(h^* - 3\gamma)(N_1n_1 + N_2n_2 + \dots + N_kn_k + jn_{k+1})\} \\ &= \exp\left\{\frac{(h^* - 3\gamma)}{1 + \theta}n\right\} \\ &\geq \exp\{(h^* - 4\gamma)n\}. \end{aligned} \quad (5.10)$$

Thus

$$\omega_{k+p}(\mathcal{B}) \leq (\#\mathcal{T}_{k-1})^{-1}(\#\mathcal{S}_{k+1})^{-j} \leq \exp\{-(h^* - 4\gamma)n\}. \quad (5.11)$$

This proves the theorem in this first case that $t_k \leq n < t_{k+1} - m_k$.

Now we consider the case that $t_{k+1} - m_k \leq n < t_{k+1}$. Using $\mathcal{B} \subseteq B_{t_{k+1}-m_k-1}(q, \varepsilon)$ together with Lemma 5.4 and (5.10), we have that for $p \geq 1$,

$$\omega_{k+p}(\mathcal{B}) \leq \omega_{k+p}(\mathcal{B}_{t_{k+1}-m_k-1}(q, \varepsilon)) \leq \exp\{-(h^* - 4\gamma)(t_{k+1} - m_k - 1)\}.$$

Thus, if k is large (by using $\frac{m_k}{t_{k+1}} \rightarrow 0$) then

$$\omega_{k+p}(\mathcal{B}) \leq \exp\{-(h^* - 4\gamma)(t_{k+1} - m_k - 1)\} \leq \exp\{-(h^* - 5\gamma)n\}. \quad (5.12)$$

Combining (5.11) and (5.12) we have for all n ,

$$\limsup_{l \rightarrow \infty} \omega_l(B_n(q, \varepsilon)) \leq \exp\{-n(h^* - 5\gamma)\}.$$

The later implies that $\omega(B_n(q, \varepsilon)) \leq \exp\{-n(h^* - 4\gamma)\}$. So for any $q \in F$,

$$\underline{h}_\mu(T, q) \geq h^* - 5\gamma.$$

By (4.3), $h_{top}(T, F) = \sup\{\underline{h}_\mu(T) : \mu \in \mathcal{M}(X), \mu(F) = 1\} \geq h^* - 5\gamma$. This finishes the proof of the first part of the theorem. \square

Proof of Theorem A (2). From (2.2), we know that

$$h_{top}(T, G_{\mu_1, \mu_2}) \leq \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}.$$

Thus in order to prove (2.4), we need to prove

$$h_{top}(T, G_{\mu_1, \mu_2}) \geq \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}.$$

We assume without loss of generality that the measures μ_1, μ_2 are ergodic. In general, since one can use the ergodic decomposition of invariant measures exactly as in [23, Lemma 6.2] we shall omit the details here. Let $h^* = \min\{h_{\mu_1}(T), h_{\mu_2}(T)\}$. We need to show that for any $\gamma > 0$,

$$h_{top}(T, G_{\mu_1, \mu_2}) \geq h^* - 5\gamma. \quad (5.13)$$

Then (2.4) holds. Now we fix $\gamma > 0$ and start to prove (5.13).

Step 1. Choice of separated sets.

Fix a number $\rho_0 \in (0, 1)$. For the measures μ_1 and μ_2 , by Katok's definition of metric entropy, take small number $\varepsilon > 0$ such that for $i = 1, 2$,

$$h_{\mu_i}(T, 4\varepsilon) = \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_n^{\mu_i}(4\varepsilon, \rho_0) > h_{\mu_i}(T) - \gamma. \quad (5.14)$$

Fix a rational number $\theta_0 > 0$ such that

$$\frac{1}{1 + \theta_0}(h^* - 3\gamma) > h^* - 4\gamma. \quad (5.15)$$

Take a strictly decreasing sequence of positive rational numbers $\theta_n \downarrow 0$ with $\theta_1 < \theta_0$.

Let $g : \mathbb{N} \rightarrow \{1, 2\}$ be given by $g(k) = (k + 1)(\bmod 2) + 1$. Choose a strictly decreasing sequence $\delta_k \rightarrow 0$ and a strictly increasing sequence $L_k \rightarrow \infty$ so that the set

$$J_k := \{x \in \Delta : \frac{p(x, n, \varepsilon)}{n} < \theta_k, \rho(\mathcal{E}_n(x), \mu_{g(k)}) < \delta_k \text{ for all } n \geq L_k\}$$

of points whose empirical measures are δ_k -close to $\mu_{g(k)}$ satisfies $\mu_{g(k)}(J_k) > 1 - \rho_0$ for all $k \geq 1$. This is a simple consequence of the fact that the ergodic basin of attraction of μ_i is a full μ_i -measure set. Let E_n^k be a maximal $(n, 4\varepsilon)$ -separated set for J_k . Notice that a maximal $(n, 4\varepsilon)$ -separated set for J_k is also a $(n, 4\varepsilon)$ -spanning set for J_k and thus E_n^k is also a $(n, 4\varepsilon)$ -spanning set for J_k . Since $\mu_{g(k)}(J_k) > 1 - \rho_0$, then by (5.14)

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \#E_n^k \geq \liminf_{n \rightarrow +\infty} \frac{1}{n} \log N_n^{\mu_{g(k)}}(4\varepsilon, \rho_0) > h_{\mu_{g(k)}}(T) - \gamma. \quad (5.16)$$

Take $(n_k)_k$ a strictly increasing sequence of positive integers such that $n_k \geq L_k$, $\frac{1}{n_k} \log \#E_{n_k}^k > h_{\mu_{g(k)}}(T) - 3\gamma$ and $\frac{n_k}{n_{k+1}} \rightarrow 0$. We may assume in addition that each $n_k \theta_k$ is always integer. Let

$$\mathcal{S}_k := E_{n_k}^k, \mu_k = \mu_{g(k)}, m_k = n_k \theta_k.$$

From above construction, we have

$$\#\mathcal{S}_k \geq \exp(h_{\mu_k}(T) - 3\gamma)n_k. \quad (5.17)$$

Step 2. Construction of the fractal F .

We mimic the construction of F as in the proof of Theorem A (1), by considering

$$B(\underline{x}_1, \dots, \underline{x}_k) := \bigcap_{i=1}^k \bigcap_{j=1}^{N_i} T^{-\sum_{l=0}^{i-1} N_l(n_l+m_l) - (i-1)N - (j-1)(n_i+m_i)} \hat{B}_{n_i}(x_j^i, \varepsilon) \neq \emptyset,$$

where $\hat{B}_n(x, \varepsilon) = \{y : \sum_{i=0}^{n-1} d(T^i(x), T^i(y)) < \varepsilon\}$, instead of the sets defined in (5.7). Here the non-emptiness of $B(\underline{x}_1, \dots, \underline{x}_k)$ follows from the strong non-uniform specification property. After the construction of F , we need to replace Lemma 5.1 by following lemma.

Lemma 5.5. $F \subseteq G_{\mu_1, \mu_2}$.

Proof. The proof is analogous to the one of Lemma 5.1, replacing (5.8) by the following estimate

$$\begin{aligned} & \rho\left(\frac{1}{N_k} \sum_{j=1}^{N_k} \mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{t_k - t_{k-1}}(p_k)\right) \\ & \leq \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k+m_k}(T^{(j-1)(n_k+m_k)} p_k)) \\ & \leq \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k)) \\ & \quad + \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k), \mathcal{E}_{n_k+m_k}(T^{(j-1)(n_k+m_k)} p_k)) \\ & \leq \frac{1}{n_k} \varepsilon + 2 \frac{m_k}{n_k + m_k} \leq \frac{1}{n_k} \varepsilon + 2\theta_k. \end{aligned} \quad (5.18)$$

Here the part $\frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k)) \leq \frac{1}{n_k} \varepsilon$ follows from strong non-uniform specification:

$$\begin{aligned} & \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{n_k}(T^{(j-1)(n_k+m_k)} p_k)) \\ & = \frac{1}{N_k} \sum_{j=1}^{N_k} \frac{1}{n_k} \sum_{l=1}^{n_k} d(f^l(x_j^k), f^l(T^{(j-1)(n_k+m_k)} p_k)) \\ & \leq \frac{1}{N_k} \sum_{j=1}^{N_k} \frac{1}{n_k} \varepsilon = \frac{1}{n_k} \varepsilon. \end{aligned}$$

Using Lemma 4.1 (1) we deduce that

$$\begin{aligned} & \rho(\mathcal{E}_{t_k}(p), \mu_k) \\ & \leq \rho(\mathcal{E}_{t_k}(p), \mathcal{E}_{t_k - t_{k-1}}(p_k)) + \rho\left(\frac{1}{N_k} \sum_{j=1}^{N_k} \mathcal{E}_{n_k}(x_j^k), \mathcal{E}_{t_k - t_{k-1}}(p_k)\right) + \frac{1}{N_k} \sum_{j=1}^{N_k} \rho(\mathcal{E}_{n_k}(x_j^k), \mu_k) \\ & \leq 2 \frac{t_k - (t_k - t_{k-1}) + t_{k-1}}{t_k} + \left(\frac{1}{n_k} \varepsilon + 2\theta_k\right) + \delta_k. \end{aligned}$$

Then $\limsup_{k \rightarrow \infty} \rho(\mathcal{E}_{t_k}(p), \mu_k) = 0$. For other subsequences the strategy of the proof follows well known ideas by constructing an appropriate convex sum of the measures μ_1 and μ_2 . Here we omit the details. \square

Then one can follow Step 3 and 4 same as the proof of Theorem A (1) to end the proof of Theorem A (2), just in Step 4 replacing $QG_{\mu_1, \mu_2}(\varepsilon)$ by G_{μ_1, μ_2} , replacing $m_i = n_i\theta$ by $m_i = n_i\theta_i$, and replacing (5.10) by $\geq \exp\left\{\frac{(h^* - 3\gamma)}{1 + \theta_0} n\right\}$. \square

6. MULTIFRACTAL ANALYSIS

In this section we prove the corollaries of Theorem A to the multifractal analysis of Birkhoff averages for continuous observables.

Proof of Corollary A. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X and assume that T satisfies the strong non-uniform specification property on $\Delta \subseteq X$. Let $\phi : X \rightarrow \mathbb{R}$ be a continuous function.

Given $a \in \mathbb{R}$ let $D = \{\mu \in \mathcal{M}_T(\Delta) \mid \int \phi d\mu = a\}$. To prove the Corollary A it is enough to prove that $h_{top}(T, R_{\phi, a}(T)) \geq h_\mu(T)$ for every $\mu \in D$. Notice that $G_{\mu, \mu} \subseteq R_{\phi, a}(T)$. Together with relation (2.4) in Theorem A this implies that

$$h_{top}(T, R_{\phi, a}(T)) \geq h_{top}(T, G_{\mu, \mu}) \geq h_\mu(T).$$

This finishes the proof. □

Proof of Corollary B. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X so that T has non-uniform specification on $\Delta \subset X$, and let $\phi : X \rightarrow \mathbb{R}$ be a continuous function. Denote $h = \sup \{h_\mu(T) \mid \mu \in \mathcal{M}_T(\Delta)\}$.

For any $\tau > 0$, choose an invariant measure $\mu \in \mathcal{M}_T(\Delta)$ such that $h_\mu(T) > h - \tau$. Then we select a number $0 < \theta < 1$ close to 1 such that $\theta h_\mu(T) > h - \tau$. By assumption, there exists $\omega \in \mathcal{M}_T(\Delta)$ such that $\int \phi d\mu \neq \int \phi d\omega$. If $\nu = \theta\mu + (1 - \theta)\omega$ then we observe that $h_\nu(T) \geq \theta h_\mu(T) > h - \tau$ and $\int \phi d\mu \neq \int \phi d\nu$. Let $K = \{t\mu + (1 - t)\nu \mid t \in [0, 1]\}$. Choose $\epsilon > 0$ small enough such that if $x \in X$ satisfies $d_{Haus}(V_T(x), K) < \epsilon$ then $V_T(x)$ is not a singleton and

$$\inf_{m \in V_T(x)} \int \phi dm < \sup_{m \in V_T(x)} \int \phi dm. \quad (6.1)$$

By relation (2.3) in Theorem A, reducing ϵ if necessary,

$$h_{top}(T, QG_{\mu, \nu}(\epsilon)) \geq \min\{h_\mu(T), h_\nu(T)\} - \tau > h - 2\tau,$$

where $QG_{\mu, \nu}(\epsilon) = \{x \in X \mid d_{Haus}(V_T(x), K) < \epsilon\}$. By (6.1), every $x \in QG_{\mu, \nu}(\epsilon)$ belongs to $I_\phi(T)$. Therefore we conclude that

$$h_{top}(T, I_\phi(T)) \geq h_{top}(T, QG_{\mu, \nu}(\epsilon)) > h - 2\tau.$$

Since τ was chosen arbitrary this completes the proof of the corollary. □

Proof of Corollary C. Let $T : X \rightarrow X$ be a continuous map of a compact metric space X and $\Delta \subseteq X$ so that T has non-uniform specification on Δ . Let $\phi : X \rightarrow \mathbb{R}$ be a continuous function. Since the item (2) is a direct consequence of item (1) we only need to prove the later. Given $a \in \mathbb{R}$ and $\sigma > 0$ let

$$D_\sigma = \{\mu \in \mathcal{M}_T(\Delta) \mid \int \phi d\mu \in (a - \sigma, a + \sigma)\}.$$

For any $\mu \in D_\sigma$ take $\epsilon > 0$ small enough such that $\int \phi d\nu \in (a - \sigma, a + \sigma)$ whenever $\nu \in B(\mu, \epsilon)$ (here the ball is determined by the metric ρ). To prove that $h_{top}(T, R_{\phi, a, \sigma}(T)) \geq h_\mu(T)$ first notice that $QG_{\mu, \mu}(\epsilon) \subseteq R_{\phi, a, \sigma}(T)$. This property together with the estimate (2.3) of Theorem A yields

$$h_{top}(T, R_{\phi, a, \sigma}(T)) \geq h_{top}(T, QG_{\mu, \mu}(\epsilon)) \geq h_\mu(T).$$

The proof is now complete. □

7. NON-UNIFORMLY EXPANDING MAPS

7.1. Hyperbolic times and exponential non-uniform specification. We first recall some properties of hyperbolic times in the case of C^1 -maps on a compact Riemannian manifold X . A sufficiency criterium for the existence of hyperbolic times is given as application of Pliss' lemma.

Lemma 7.1. [2, Lemma 5.4] *There exists constants $\theta > 0$ and $\delta > 0$ (depending only on T and c) such that if $x \in X \setminus \cup_n T^n(\mathcal{C})$ satisfies (2.6) and (2.7) then the following holds: for every large $N \geq 1$ there exist a sequence of integers $1 \leq n_1(x) < n_2(x) < \dots < n_l(x) \leq N$, with $l \geq \theta N$ so that*

$$\prod_{j=N-k}^{N-1} \|DT(T^j(x))^{-1}\| \leq \sigma^{-k} \quad \text{and} \quad \text{dist}_\delta(T^{N-k}(x), \mathcal{C}) > \sigma^{bk}. \quad (7.1)$$

One of the main features of hyperbolic times is the following backward contraction property.

Lemma 7.2. [2, Lemma 2.7] *Given $\sigma > 1$ and $\delta > 0$ there exists a constant $\delta_1 = \delta_1(\sigma, \delta, T) > 0$ such that if n is a hyperbolic time for a point x then T^n maps diffeomorphically the dynamical ball $B_n(x, \delta_1)$ onto the ball $B(T^n(x), \delta_1)$ around $T^n(x)$ and radius δ_1 and*

$$d(T^{n-j}(y), T^{n-j}(z)) \leq \sigma^{-\frac{j}{2}} d(T^n(y), T^n(z)) \quad (7.2)$$

for every $1 \leq j \leq n$ and every $y, z \in B_n(x, \delta_1)$. In particular

$$\text{diam } B_n(x, \delta_1) \leq \sigma^{-\frac{n}{2}} \delta_1 < \sigma^{-\frac{n}{2}} \text{diam}(X).$$

Given $\sigma > 1, \delta > 0$, we say that $x \in M$ admits a *non-lacunar sequence of (σ, δ) -hyperbolic times* if $(n_k(x))_{k \geq 1}$ is a sequence of (σ, δ) -hyperbolic times and

$$\lim_{k \rightarrow \infty} \frac{n_{k+1}(x) - n_k(x)}{n_k(x)} = 0.$$

If the first hyperbolic time map is integrable then the sequence of hyperbolic times is almost everywhere well defined and non-lacunar (see e.g. [29]).

7.2. Proof of Theorem B. The topological exactness assumption implies that for every $\delta > 0$ there exists $N_\delta \geq 1$ such that $T^{N_\delta}(B) = X$ for every ball B of radius δ . This fact plays a key role in the proof of the exponential non-uniform specification property for a full measure set with respect to all probability measures in \mathbb{M} . Given $\sigma > 1, \delta > 0$, let $\Delta_{\delta, \sigma}$ denote the set of points $x \in M$ that admit a non-lacunar sequences sequence $(n_k(x))_{k \geq 1}$ of (σ, δ) -hyperbolic times. Since T is non-uniformly expanding then the set $\Delta = \bigcup_{\delta, \sigma > 0} \Delta_{\delta, \sigma}$ is non-empty. Moreover, if $x \in \Delta_{\delta, \sigma}$, $0 < \varepsilon < \delta$, $n \geq 1$ is large and $n_k(x) < n < n_{k+1}(x)$ are consecutive (σ, δ) -hyperbolic times for x then $B_{n_{k+1}}(x, \varepsilon) \subset B_n(x, \varepsilon)$ and

$$T^{N_\varepsilon + n_{k+1}}(B_{n_{k+1}}(x, \varepsilon)) = T^{N_\varepsilon}(B(T^{n_{k+1}}(x), \varepsilon)) = X.$$

Then the map

$$p = p_{\varepsilon, \sigma} : \begin{array}{ccc} \Delta_{\varepsilon, \sigma} & \rightarrow & \mathbb{N} \\ x & \mapsto & p(x, n, \varepsilon) \end{array}$$

defined as $p(x, n, \varepsilon) := N_\varepsilon + n_{k+1}(x) - n$ verifies

$$\limsup_{n \rightarrow \infty} \frac{p(x, n, \varepsilon)}{n} \leq \limsup_{k \rightarrow \infty} \frac{N_\varepsilon + n_{k+1}(x) - n_k(x)}{n_k(x)} = 0$$

for every $x \in \Delta_{\delta, \sigma}$ (this is well defined at scale δ for all measures $\eta \in \mathbb{M}_{\sigma, \delta}$).

Since the previous limit is zero independently of the scale $0 < \varepsilon < \delta$, in order to complete the proof of the first part of the proposition it remains to prove the strong non-uniform specification property holds with exponential shadowing estimates. Given any x_1, \dots, x_m in $\Delta_{\delta, \sigma}$, $0 < \varepsilon < \delta$, any positive integers $n_1, \dots, m_m \geq 1$ and $p_i \geq p(x_i, n_i, \varepsilon)$ we get that $T^{p_i + n_i}(B_{n_i}(x_i, \varepsilon)) = X$ for all $1 \leq i \leq m$. Therefore there exists $z \in B_{n_1}(x_1, \varepsilon)$ such that

$$T^{n_1 + p_1 + \dots + n_{i-1} + p_{i-1}}(z) \in B_{n_i}(x_i, \varepsilon)$$

for every $2 \leq i \leq k$. In particular, relation (7.2) yields

$$d(T^j(x_i), T^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(y)) < \varepsilon \sigma^{-\frac{1}{2}(n_i-j)}$$

for every $0 \leq j \leq n_i$ and $1 \leq i \leq k$. This proves the theorem. Indeed, given $x_1, \dots, x_k \in \Delta$ let η_1, \dots, η_m be probability measures on \mathbb{M} so that $x_i \in B(\eta_i)$ for every i , and take $0 < \delta < \min_{1 \leq i \leq m} \delta_{\eta_i}$. \square

Remark 7.1. *While, in general, one cannot expect the scale δ in the proof of Theorem B to be taken uniform for all points with some non-uniform expansion, the proof yields the following (stronger) exponential non-uniform specification for all points in the basin of a finite number of ergodic expanding measures. More precisely, if the T -invariant and ergodic probability measures $(\mu_i)_{i=1}^k$ are expanding and $\Delta = \bigcup_{1 \leq i \leq k} B(\mu_i)$ then there exists $\Lambda > 0$ so that T satisfies the strong exponential non-uniform specification with exponential λ .*

7.3. Cohomology criterium and closing lemma. In the present subsection, of independent interest, we address the problem of determining criteria equivalent conditions for the relation

$$\inf_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu < \sup_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu$$

can be derived for non-uniformly expanding maps. Let $\Delta \subset X$ be T -invariant and let $V_T(\Delta)$ denote the accumulation points of the empirical measures $(\mathcal{E}_n(x))_{n \geq 1}$ for points $x \in \Delta$. First we observe that ϕ is a coboundary if the previous relation holds for all measures that are accumulation points of empirical measures associated to points in Δ . More precisely,

Lemma 7.3. *The following properties are equivalent:*

- (1) $\inf_{\mu \in V_T(\Delta)} \int \phi d\mu < \sup_{\mu \in V_T(\Delta)} \int \phi d\mu$
- (2) $\phi \notin \overline{Cob(\Delta)}$
- (3) the sequence $\frac{1}{n} S_n \phi$ is not (uniformly) convergent to $c \in \mathbb{R}$ on Δ .

Proof. Since the lemma mimics the ideas from Lemma 1.9 in [31] we shall omit the details. \square

Observe that if Δ is compact then $V_T(\Delta) = \mathcal{M}_T(\Delta)$ and relation (2.5) is equivalent to say that ϕ is not a coboundary nor C^0 -accumulated by coboundaries. In general, the later conditions could be distinct. For that reason we prove the denseness of periodic measures on $\mathcal{M}_T(\Delta)$.

Proposition 7.1. *Let $T : X \rightarrow X$ be a C^1 local diffeomorphism and let $\Delta \subset X$ be a T -invariant and dense subset. Assume that there exist $\sigma > 1$ and $\delta > 0$ so that every $x \in \Delta$ has infinitely many (σ, δ) -hyperbolic times and that T is topologically exact. Then every $\mu \in \mathcal{M}_T^e(\Delta)$ is accumulated (in the weak* topology) by periodic measures.*

Proof. By ergodic decomposition it is enough to prove that every $\mu \in \mathcal{M}_T^e(X)$ is approximated by periodic measures in the weak* topology. Set $L = \sup_{x \in X} \|DT(x)\|$. Let $\sigma > 1$ and $\delta > 0$ be so that every $x \in \Delta$ has infinitely many (σ, δ) -hyperbolic times.

Fix $0 < \varepsilon < \delta$ arbitrary. By topological exactness, there exists $K_\varepsilon \geq 1$ so that $T^{K_\varepsilon}(B(z, \varepsilon)) = X$ for every $z \in X$. Let $N_1(\varepsilon) \geq 1$ be so that

$$\sigma^{-N_1(\varepsilon)} L^{K_\varepsilon} < 1. \quad (7.3)$$

For any $\mu \in \mathcal{M}_T^e(\Delta)$ pick $x_* \in B(\mu) \subset \Delta$ and let $N_2(x_*, \varepsilon) \geq 1$ be so that

$$\rho(\mathcal{E}_n(x_*), \mu) < \varepsilon \quad \forall n \geq N_2(x_*, \varepsilon).$$

Set $N = \max\{N_1(\varepsilon), N_2(x_*, \varepsilon)\}$. Using that $x_* \in \Delta$ has infinitely many hyperbolic times one can choose a (σ, δ) -hyperbolic time n_{x_*} for x_* so that $n_{x_*} \geq N$. In particular, $T^{n_{x_*} + K_\varepsilon}(B_{n_{x_*}}(x_*, \varepsilon)) = X$ and there exists $B \subset B_{n_{x_*}}(x_*, \varepsilon)$ a domain of injectivity for $T^{n_{x_*} + K_\varepsilon}$ in such a way that $T^{n_{x_*} + K_\varepsilon} : B \rightarrow B_{n_{x_*}}(x_*, \varepsilon)$ is a C^1 diffeomorphism. It is not hard to check that

$$\begin{aligned} d(y, z) &\leq \sigma^{-n_{x_*}} L^{K_\varepsilon} d(T^{n_{x_*} + K_\varepsilon}(y), T^{n_{x_*} + K_\varepsilon}(z)) \\ &\leq \sigma^{-N} L^{K_\varepsilon} d(T^{n_{x_*} + K_\varepsilon}(y), T^{n_{x_*} + K_\varepsilon}(z)) \end{aligned}$$

for every $y, z \in B$. Thus, the inverse branch $G : \overline{B_{n_{x_*}}(x_*, \varepsilon)} \rightarrow \overline{B}$ of $T^{n_{x_*} + K_\varepsilon}$ is a contraction on a complete metric space. By Banach's fixed point theorem there exists a unique $p \in \overline{B_{n_{x_*}}(x_*, \varepsilon)}$ so that $T^{n_{x_*} + K_\varepsilon}(p) = p$. Using that $p \in \overline{B_{n_{x_*}}(x_*, \varepsilon)}$, it follows that $\rho(\mathcal{E}_{n_{x_*}}(x_*), \mathcal{E}_{n_{x_*} + K_\varepsilon}(p)) \leq \varepsilon$ and, by Lemma 4.1,

$$\rho(\mathcal{E}_{n_{x_*}}(x_*), \mathcal{E}_{n_{x_*} + K_\varepsilon}(p)) \leq 2\varepsilon.$$

This proves that $\rho(\mu, \mathcal{E}_{n_{x_*} + K_\varepsilon}(p)) \leq 3\varepsilon$. Since $0 < \varepsilon < \delta$ was taken arbitrary this proves that μ is accumulated by periodic measures. This finishes the proof of the proposition. \square

We observe that the periodic measures obtained in the previous proposition may be associated to periodic points that could not be uniformly expanding. As a direct consequence of the previous proposition we obtain the following:

Corollary 7.1. *If $\inf_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu < \sup_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu$ then there are periodic points $p_1, p_2 \in X$ of period $\pi_1, \pi_2 \geq 1$, respectively, so that*

$$\frac{1}{\pi_1} S_{\pi_1} \phi(p_1) < \frac{1}{\pi_2} S_{\pi_2} \phi(p_2). \quad (7.4)$$

Conversely, if (7.4) holds for a pair of expanding periodic points $p_1, p_2 \in \Delta$ for T then $\inf_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu < \sup_{\mu \in \mathcal{M}_T(\Delta)} \int \phi d\mu$

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