Specification and thermodynamical properties of semigroup actions

Fagner B. Rodrigues[a)] and Paulo Varandas[b)]

(Dated: 24 September 2015)

In the present paper we study the thermodynamical properties of finitely generated continuous subgroup actions. We propose a notion of topological entropy and pressure functions that does not depend on the growth rate of the semigroup and introduce strong and orbital specification properties, under which, the semigroup actions have positive topological entropy and all points are entropy points. Moreover, we study the convergence and Lipschitz regularity of the pressure function and obtain relations between topological entropy and exponential growth rate of periodic points in the context of semigroups of expanding maps, obtaining a partial extension of the results obtained by Ruelle for $\mathbb{Z}^d$-actions[33] . The specification properties for semigroup actions and the corresponding one for its generators and the action of push-forward maps is also discussed.

Keywords: Group actions, specification properties, thermodynamical formalism, topological entropy, semigroups of expanding maps

I. INTRODUCTION

The thermodynamical formalism was brought from statistical mechanics to dynamical systems by the pioneering works of Sinai, Ruelle and Bowen[9,10,32,37] in the mid seventies. The correspondance between one-dimensional lattices and uniformly hyperbolic maps allowed to translate and introduce several notions of Gibbs measures and equilibrium states in the realm of dynamical systems. The present study of the thermodynamical formalism for non-uniformly hyperbolic dynamical systems is now paralel to the development of a thermodynamical formalism of gases with infinitely many states, a hard subject not yet completely understood. Moreover, the notion of entropy constitutes one of the most important in the study of dynamical systems (we refer the reader to Katok[25] and references therein for a survey on the state of the art).

An extension of the thermodynamical formalism for continuous finitely generated group actions has revealed fundamental difficulties and the global description of the theory is still incomplete. A first attempt was to consider continuous actions associated to finitely generated abelian groups. The statistical mechanics of expansive $\mathbb{Z}^d$-actions satisfying a specification property was studied by Ruelle[33], where he introduced a pressure function, defined on the space of continuous functions, and discussed its relations with measure theoretical entropy and free energy. The notion of specification was introduced in the seventies as a property of uniformly hyperbolic basic pieces and became a characterization of complexity in dynamical systems. The crucial fact that continuous $\mathbb{Z}^d$-actions on compact spaces admit probability measures invariant by every continuous maps associated to the group action, allowed Ruelle to prove a variational principle for the topological pressure and to build equilibrium states as the class of pressure maximizing invariant probability measures. This duality between topological and measure theoretical complexity of the dynamical system is very fruitfull, e.g. was used later by Eizenberg, Kifer and Weiss[18] to establish large deviations principles to $\mathbb{Z}^d$-actions satisfying the specification property. Other specification properties of interest have been introduced recently (see e.g.[14,39]).

---

a) Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Brazil. & CMUP, University of Porto, Portugal; Electronic mail: fagnerbernardini@gmail.com
b) Departamento de Matemática, Universidade Federal da Bahia
Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil. & CMUP, University of Porto, Portugal; Electronic mail: paulo.varandas@ufba.br

A unified approach to the thermodynamical formalism of continuous group actions is still unavailable, while still few definitions of topological pressure exists and most of them unrelated. Moreover the connection between topological and ergodic properties of group actions still fails to provide a complete description the complexity of the dynamical system. In many cases the existent definitions for topological entropy take into account either abelianity, amenability or growth rate of the corresponding group. A non-extensive list of contributions by many authors include important contributions by Ghys, Langevin, Walczak, Friedland, Lind, Schmidt, Bufetov, Biś, Urbanski, Ma, Wu, Miles, Ward, Chen, Zheng and Schneider among others (see e.g.[3,4,5,6,11,20,21,27,28,29,35,43] and references therein).

Our main goal here is to describe the topological aspects of the thermodynamical formalism for semigroup actions for general finitely generated semigroup actions, where no commutativity or conditions on the semigroup growth rate are required. Inspired by a notion of topological entropy of free semigroups by Bufetov[11], given a finitely generated semigroup (G, G_1) where $G_1 = \{id, g_1, \dots, g_m\}$ is a set of generators we consider the coding

$$\begin{array}{rccc} \iota: & F_m & \to & G \\ & i_n \dots i_1 & \mapsto & g_{i_n} \circ \dots \circ g_{i_1} \end{array} \tag{1}$$

where F_m denotes the free semigroup with m elements. This coding is injective if and only if G is a free semigroup. Our thermodynamical approach for the semigroup action is to average the complexity of each dynamics $g \in G$ with a weight corresponding to the size of $\iota^{-1}(g)$, that is, how often a particular semigroup element g arises by concatenation of the generators.

E.g. if all generators commute and do not have finite order then $G \simeq \mathbb{Z}^m$ and every element in G has the same weight, a property that will change substantially in the case of semigroups of exponential growth with a non-trivial abelian subgroup. This approach has the advantage of being independent of the semigroup growth rate, hence to propose a unified approach to the study of semigroups with substantially different growth rates (see Section V for examples) and the disadvantage to depend *a priori* on the set of generators for the semigroup. Inspired by several forms of the specification property for discrete time transformations with some hyperbolicity (see e.g.[30,31,34,36,39]), we also introduce some notions of strong and orbital specification properties for continuous actions associated to finitely generated (not necessarily abelian) groups which are of independent interest. In the particular case of semigroups (G, G_1) of expanding maps our main contributions can be summarized as follows:

(a) we introduce a notion of topological pressure $P_{top}((G, G_1), \varphi, X)$ which in independent of the semigroup growth rate;

(b) we prove that the orbital specification properties hold and, consequently, the local complexity at every neighborhood of any point coincides with the topological pressure of the dynamical system (see the notions of 'entropy point' in Subsection III A);

(c) using expansiveness, we prove that topological pressure can be computed at a finite scale (omitting a limit in the original definition)

(d) we prove that the topological pressure function $t \mapsto P_{top}((G, G_1), t\varphi, X)$ for Hölder continuous observables φ is a uniform limit of C^1 functions, hence it is Lipschitz and differentiable Lebesgue almost everywhere; and

(d) the exponential mean growth of periodic points is bounded from below by topological entropy $P_{top}((G, G_1), \varphi, X)$.

In[33], Ruelle studied expansive $\mathbb{Z}^d$-actions with specification property and obtained that the topological pressure function is smooth, existence and uniqueness of equilibrium states. Here we obtained the Lebesgue almost everywhere differentiability of the pressure function for semigroups of expanding maps that may have exponential growth. To the best of our knowledge these are the first results after[33] (that considered $\mathbb{Z}^d$-actions) where there are partial results on the the differentiability of the topological pressure function for group or semigroup actions.

Finally we observe that this is the first part of a program to describe the thermodynamical properties of semigroup actions following the program of Ruelle[33], and the construction of relevant stationary measures that describe the ergodic theory of finitely generated semigroup actions of expanding maps will appear elsewhere[15]. The relation between orbital specification properties for the group action is also discussed and a class of examples of group actions is given where orbital specification properties present a flavor of the non-uniform versions arising in non-uniformly hyperbolic dynamics. In fact, we also study semigroups with non-expanding elements and compare these with the notions of entropy introduced by Ruelle[33] and Ghys, Langevin, Walczak[21]. For the convenience of the reader, we describe briefly the beginning of each section the main results to be proved there. Except when we mention explicit otherwise, we shall consider the context of semigroup actions and, in case the existence of inverse elements is needed, we shall make precise mention to that fact. We refer the reader to the statement of the main results and to Section V for some examples.

This paper is organized as follows. In Section II we introduce both the strong specification property and some orbital specification properties for finitely generated semigroups actions and discuss the relation between these notions and the specification property for the generators. The connections between specification properties for group actions, for the push-forward group actions and hyperbolicity are also discussed.

In Section III we introduce a notion of topological entropy and pressure for continuous semigroup actions and study group actions that exhibit some forms of specification. In particular, we prove that these have positive topological entropy and every point is an entropy point.

In Section IV we study the semigroup action induced by expanding maps. We prove that these semigroups satisfy the previous notions of specification and that topological entropy is a lower bound for the exponential growth rate of periodic orbits. We also deduce that the pressure function acting on the space of Hölder continuous potentials is Lipschitz, hence almost everywhere differentiable along families $t\varphi$ with $t \in \mathbb{R}$ and φ Hölder continuous.

Finally, in Section V we provide several examples where we discuss the specification properties and establish a comparison between some notions of topological entropy.

II. SPECIFICATION FOR A FINITELY GENERATED SEMIGROUP ACTIONS

In this section we introduce the notions of specification and orbital specification properties for the context of group and semigroup actions. The specification property for the group action implies that all generators satisfy the specification property (Lemma 2) and also that the push-forward group action satisfies the specification property (Theorem 3). Moreover, C^1-robust specification implies structural stability (Corollary 10).

A. Strong specification property

The specification property for a continuous map on a compact metric space X was introduced by Bowen[8]. A continuous map $f : X \to X$ satisfies the *specification property* if for any $\delta > 0$ there exists an integer $p(\delta) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any sequence of positive integers $n_1, \dots, n_k$ and $p_1, \dots, p_k$ with $p_i \geq p(\delta)$ there exists a point x in X such that

$$d\Big(f^j(x), f^j(x_1)\Big) \leq \delta, \ \forall\, 0 \leq j \leq n_1$$

and

$$d\Big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\Big) \leq \delta$$

for every $2 \le i \le k$ and $0 \le j \le n_i$. This property means that pieces of orbits of f can be δ-shadowed by a individual orbit provided that the time lag between each shadowing is larger than some prefixed time $p(\delta)$.

The notion of specification was extended to the context of continuous $\mathbb{Z}^d$-actions on a compact metric space X by Ruelle motivated by statistical mechanics. Let $(\mathbb{Z}^d, +)$ be endowed with the distance $d_{\mathbb{Z}^d}(a,b) = \sum_{i=1}^p |a_i - b_i|$. Following [33], the group action $\mathbb{Z}^d \times X \to X$ satisfies the *specification property* if for any $\delta > 0$ there exists $p(\delta) > 0$ such that for any finite families $(\Lambda_i)_{i\in\mathcal{I}}$, $(x_i)_{i\in\mathcal{I}}$ satisfying if $i \ne j$, the distance of Λ_i, Λ_j (as subsets of $\mathbb{Z}^d$) is $> p(\delta)$, there is $x \in X$ such that $d(m_i x, m_i x_i) < \delta$, for all $i \in \mathcal{I}$, and all $m_i \in \Lambda_i$. This notion clearly extends to group actions associated to finitely generated abelian groups.

Specification property for groups and its generators

In this article we shall address the specification properties and thermodynamical formalism to deal both with finitely generated group and semigroup actions. For simplicity, we shall state our results in the more general context of semigroup actions whenever the results do not require the existence of inverse elements. More precisely, given a finitely generated semigroup $(G, \circ)$ with a finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_m\}$ one can write $G = \bigcup_{n\in\mathbb{N}_0} G_n$ where $G_0 = id$ and

$$\underline{g} \in G_n \text{ if and only if } \underline{g} = g_{i_n} \dots g_{i_2} g_{i_1} \text{ with } g_{i_j} \in G_1 \tag{2}$$

(where we use $g_j\, g_i$ instead of $g_j \circ g_i$ for notational simplicity). If, in addition, the elements of G_1 are invertible, the finitely generated group $(G, \circ)$ is defined by $G = \bigcup_{n\in\mathbb{N}_0} G_n$ where $G_0 = id$, $G_1 = \{id, g_1^{\pm}, g_2^{\pm}, \dots, g_m^{\pm}\}$ and the elements $\underline{g} \in G_n$ are defined by (2). In both settings, G_n consists of those group elements which are concatenations of at most n elements of G_1. Since $id \in G_n$ then $(G_n)_{n\in\mathbb{N}}$ defines an increasing family of subsets of G. Moreover, G is a finite semigroup if and only if G_n is empty for every n larger than the cardinality of the group. Given a semigroup G we say $g \in G$ has *finite order* if there exists $n \ge 1$ so that $g^n = id$. If the later property does not hold then an element $g \in G$ is said to have *infinite order*. We say that $\underline{g} = g_{i_n} \dots g_{i_1}$ is reduced if it is the smaller concatenations of elements of G_1 which generates $\underline{g}$. Denote by $G_1^* = G_1 \setminus \{id\}$ and $G_n^* = \{\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1} : g_{i_j} \in G_1^*\}$. Using the coding function ι (recall (1)) observe $G_n^* = \iota(\{i_n \dots i_1 : i_j \in \{1, \dots, k\}\})$.

Motivated by applications by actions of semigroups we first introduce some generalizations of the previous specification property for group actions. Let $(G, \circ)$ be a finitely generated group of maps on a compact metric space X endowed with the distance $d_G(h, g) = |h^{-1} g|$, for $h, g \in G$. It is not difficult to check that it is a metric in the group G and that $d_G(h, g) = n$ if and only if there exists $\underline{g}_n \in G_n$ so that $g = h\, \underline{g}_n$. We are unaware of a natural notion of metric for semigroups. The following notion extends of the specification property introduced by[33] to more general group actions.

Definition 1. *Let G be a finitely generated group, X be a compact metric space and let $T : G \times X \to X$ be a continuous action. We say that the group action T has the* specification property *if for any $\delta > 0$ there exists $p(\delta) > 0$ such that for any finite families $(\Lambda_i)_{i\in\mathcal{I}}$, $(x_i)_{i\in\mathcal{I}}$ so that the $d_G(\Lambda_i, \Lambda_j) > p(\delta)$ for every $i \ne j$, then there is $x \in X$ such that $d(g_i x, g_i x_i) < \delta$ for every $i \in \mathcal{I}$ and $g_i \in \Lambda_i$.*

The later notion implies on a strong topological indecomposability of the group action. Given a continuous action $T : G \times X \to X$ we say that T is *topologically transitive* if there exists a point $x \in X$ such that the orbit $O_G(x) := \{\underline{g}(x) : \underline{g} \in G\}$ is dense in X. We say that T is *topologically mixing* if for any open sets A, B in X there exists $N \ge 1$ such that for any $n \ge N$ there is $\underline{g} \in G$ with $\underline{g} \in G_n^*$ satisfying $\underline{g}(A) \cap B \ne \emptyset$. It is easy to check that any continuous action with the specification property is topologically mixing, hence topologically transitive. For a survey on several mixing properties for group actions we refer the reader to the survey[13] and references therein.

Given a continuous action $T : G \times X \to X$ of a group G on a compact metric space X we denote, by some abuse of notation, $g : X \to X$ to be the continuous map $x \mapsto T(g, x)$. Given $\underline{g} \in G$ we say that $x \in X$ is a *fixed point* for $\underline{g}$ if $\underline{g}(x) = x$ and use the notation $x \in Fix(\underline{g})$. We say that $x \in M$ is a *periodic point of period* n if there exists $\underline{g}_n \in G_n$ so that $\underline{g}_n(x) = x$. In other words, $x \in \bigcup_{\underline{g}_n \in G_n} Fix(\underline{g}_n)$. We let $Per(G_n)$ denote the set of periodic points of period n and set $Per(G) = \bigcup_{n\geq 1} Per(G_n)$. If the tracing orbit in the specification property can be chosen periodic we will say that the action satisfies the *periodic specification* property. It is not hard to check that an invertible transformation $f : X \to X$ satisfies the specification property if and only if the group action on X associated to the group $G = \{f^n : n \in \mathbb{Z}\}$ (isomorphic to $\mathbb{Z}$) satisfies the specification property.

The next lemma asserts that this specification property for group actions implies all generators to satisfy the corresponding property.

Lemma 2. *Let G be a finitely generated with generators $G_1 = \{g_1^{\pm}, g_2^{\pm}, \dots, g_k^{\pm}\}$. If the group action $T : G \times X \to X$ satisfies the specification property then every $g \in G_1$ with infinite order has the specification property.*

Proof. Let $\delta > 0$ be fixed and let $p(\delta) > 0$ be given by the specification property for the group action T. Take arbitrary $k \geq 1$, points $x_1, \dots, x_k$, and positive integers $n_1, \dots, n_k$ and $p_1, \dots, p_k$ with $p_i \geq p(\delta)$. Since $g \in G_1$ is a generator then for any $i = 1 \dots k$ the set

$$\Lambda_i = \Big\{ g^j : \ \sum_{s=0}^{i-1} (p_s + n_s) \leq j \leq n_i + \sum_{s=0}^{i-1} (p_s + n_s) \Big\}$$

is finite and connected (assume $n_0 = p_0 = 0$). Moreover, since g has infinite order it is not hard to check that $d_G(\Lambda_i, \Lambda_j) \geq p(\delta)$ for any $i \neq j$. Let $\bar{x}_j = g^{-\sum_{s=0}^{j-1} p_s + n_s}(x_j)$, for $1 \leq j \leq k$. Thus, by the specification property there exists a point $x \in X$ such that $d(hx, h\bar{x}_i) < \delta$, for all $i = 1 \dots k$ and all $h \in \Lambda_i$ which are reduced in this case to

$$d\Big(g^j(x), g^j(x_1)\Big) \leq \delta, \ \ \forall \, 0 \leq j \leq n_1$$

and

$$d\Big(g^{j+n_1+p_1+\dots+n_{i-1}+p_{i-1}}(x) \ , \ g^j(x_i)\Big) \leq \delta$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. This proves that the map g has the specification property and finishes the proof of the lemma. □

Let us mention that the existence of elements of generators of finite order is not an obstruction for the group action to have the specification (e.g. the $\mathbb{Z}^2$-action on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose generators are a hyperbolic automorphism and the reflection on the real axis). We refer the reader to Section V for a simple example of a $\mathbb{Z}^2$-action for which the converse implication is not necessarily true.

The push-forward group action

Given a compact metric space X let $\mathcal{P}(X)$ denote the space of probability measures on X, endowed with the weak*-topology. It is well known that $\mathcal{P}(X)$ with the weak* topology is a compact set. We recall that the weak*-topology in $\mathcal{P}(X)$ is metrizable and a metric that generates the topology can be defined as follows. Given a countable dense set of continuous functions $(\phi_k)_{k\geq 1}$ in $C(X)$ and $\mu, \nu \in \mathcal{P}(X)$ define

$$d_{\mathcal{P}}(\mu, \nu) = \sum_{k\geq 1} \frac{1}{2^k \|\phi_k\|} \left| \int \phi_k \, d\mu - \int \phi_k \, d\nu \right|.$$

For a continuous map $f : X \to X$, the space of f-invariant probability measures correspond to the fixed points of the *push-forward map* $f_\sharp : \mathcal{P}(X) \to \mathcal{P}(X)$, which is a continuous map. For that reason the push-forward $f_\sharp$ reflects the ergodic theoretical aspects of f. Moreover, the dynamics of f is embedded in the one of $f_\sharp$ since it corresponds to the restriction of $f_\sharp$ to the space $\{\delta_x : x \in X\} \subset \mathcal{P}(X)$ of Dirac measures on X. This motivates the study of specification properties for the group action of the push-forward maps.

Given a finitely generated group G and a continuous group action $T : G \times X \to X$ let us denote by $T_\sharp : G \times \mathcal{P}(X) \to \mathcal{P}(X)$ denote the group action defined by $g \cdot \nu = T(g, \cdot)_\sharp \, \nu$. It is natural to ask wether the specification property can be inherited from this duality relation.

Theorem 3. *Let G be a finitely generated group and $T : G \times X \to X$ be a continuous group action satisfying the specification property. Then the group action $T_\sharp : G \times \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the specification property.*

The following lemma will play an instrumental role in the proof of the theorem.

Lemma 4. *Given probability measures $\mu_1, ..., \mu_k \in \mathcal{P}(X)$ and $\delta > 0$, there are $N \in \mathbb{N}$ and points $(x^i_1, ..., x^i_N) \in X^N$ such that the probabilities $\mu'_i = \frac{1}{N}\sum_{j=1}^N \delta_{x^i_j}$ satisfy $d(\mu_i, \mu'_i) < \delta$ for $1 \le i \le k$.*

Proof. It is well known that the finitely supported atomic measures are dense in $\mathcal{P}(X)$. Then, for $\delta > 0$, there are $\bar\mu_1, ..., \bar\mu_k$, with $\bar\mu_j = \sum_{j=1}^M \alpha^j_i \delta_{x^j_i} \in \mathcal{P}(X)$, so that $d(\mu_k, \bar\mu_k) < \delta/2$. Let p^j_i/q^j_i be a positive rational such that $|\alpha^j_i - p^j_i/q^j_i| < \delta/10$. Let $N = \prod_{i,j=1}^M q^j_i$ and $N^j_k = p^j_k \prod_{i,j=1, i\neq k}^M q^j_i$. Notice that $|N^j_k/N - \alpha^j_k| < \delta/10$ and

$$\mu'_j = \frac{1}{N}\left(\sum_{i=1}^{N^j_1} \delta_{x^j_1} + \sum_{i=1}^{N^j_2} \delta_{x^j_2} + ... + \sum_{i=1}^{N^j_k} \delta_{x^j_k}\right),$$

satisfies $d(\mu'_j, \bar\mu_j) < \delta/2$, and by triangular inequality, $d(\mu_j, \mu'_j) < \delta$. □

Proof of the Theorem 3. Assume that the action $T : G \times X \to X$ has the specification property. Clearly, if T satisfies the specification property then for any $N \ge 1$ the continuous action $T^{(N)} : G \times X^N \to X^N$ on the product space X^N endowed with the distance $d_N((x_i)_i, (y_i)_i) = \max_{1\le i\le N} d(x_i, y_i)$ and given by $g\cdot(x_1, \dots, x_N) = (gx_1, \dots, gx_N)$ also satisfies the specification. In fact, for any $\delta > 0$ just take $p(\delta) > 0$ as given by the specification property for T.

Let us proceed with the proof of the theorem. Take $\delta > 0$ and let $p(\delta/2)$ be given by the specification property. Take $\mu_1, ..., \mu_k \in \mathcal{P}(X)$ and $\Lambda_1, \dots, \Lambda_k$ finite subsets of G with $d(\Lambda_i, \Lambda_j) > p(\delta/2)$. Let $\mu'_i = \frac{1}{N}\sum_{j=1}^N \delta_{x^i_j}$, such that $d(g\mu'_i, g\mu_i) < \delta/2$ for all $g \in \Lambda_i$. By considering the finite sequence $(x^i_1, ..., x^i_N)_{i=1}^k \subset X^N$ and the sets $\Lambda_1, ..\Lambda_k$, there exists a point $(x_1, ..., x_N) \in X^N$ in the product space such that

$$d(g \cdot (x_1, ..., x_N), g \cdot (x^i_1, ..., x^i_N)) < \frac{\delta}{2} \text{ for all } g \in \Lambda_i.$$

It implies that the probability measure $\mu = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ satisfies

$$d(g \cdot \mu_i, g \cdot \mu) \le d(g \cdot \mu'_i, g \cdot \mu) + d(g \cdot \mu'_i, g \cdot \mu_i) < \delta, \text{ for all } g \in \Lambda_i.$$

This completes the proof of the theorem. □

The converse implication in the previous theorem is not immediate. In fact, given the specification property for $T_\sharp$ and any specified pieces of orbit by $T_\sharp$ it is not clear that this can be shadowed by the $T_\sharp$-orbit of a Dirac probability measure δ_x. Nevertheless this is indeed the case for the dynamics of continuous interval maps.

Corollary 5. *Let f be a continuous interval map. Then f satisfies the specification property if and only if $f_\sharp$ satisfies the specification property.*

Proof. It follows from Theorem 3 that the specification property for f implies the specification property for $f_\sharp$, and so we are reduced to prove the other implication. First we observe that the specification property implies the topologically mixing one. By[2], f is topologically mixing if and only if $f_\sharp$ is topologically mixing. Moreover, Blokh[7] proved that any continuous topologically mixing interval map satisfies the specification property, thus these are equivalent properties for continuous interval maps. This proves the corollary. $\square$

It is not clear to us if[7] can be extended to group actions, and so the previous equivalence does not have immediate counterpart for group actions of continuous interval maps.

B. Orbital specification properties

In this subsection we introduce weaker notions of specification. In opposition to the notion introduced in Definition 1, which takes into account the existence of a metric in the group, the following orbital specification properties are most suitable for semigroups actions. A first problem to define orbital specification properties is that group elements $g \in G$ may have different representations as concatenation of the generators. For that reason one should explicitly mention what is the 'path', or concatenation of elements, that one is interested in tracing.

Definition 6. *We say that the continuous semigroup action $T : G \times X \to X$ associated to the finitely generated semigroup G satisfies the* strong orbital specification property *if for any $\delta > 0$ there exists $p(\delta) > 0$ such that for any $\underline{h}_{p_j} \in G^*_{p_j}$ (with $p_j \geq p(\delta)$ for $1 \leq j \leq k$) any points $x_1, \dots, x_k \in X$ and any natural numbers $n_1, \dots, n_k$, any semigroup elements $\underline{g}_{n_j,j} = g_{i_{n_j},j} \dots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ $(j = 1 \dots k)$ there exists $x \in X$ so that $d(\underline{g}_{\ell,1}(x)\ ,\ \underline{g}_{\ell,1}(x_1)) < \delta$ for every $\ell = 1 \dots n_1$ and*

$$d(\ \underline{g}_{\ell,j}\,\underline{h}_{p_{j-1}} \cdots \underline{g}_{n_2,2}\,\underline{h}_{p_1}\,\underline{g}_{n_1,1}(x)\ ,\ \underline{g}_{\ell,j}(x_j)\ ) < \delta$$

for every $j = 2 \dots k$ and $\ell = 1 \dots n_j$ (here $\underline{g}_{\ell,j} := g_{i_\ell,j} \dots g_{i_1,j}$).

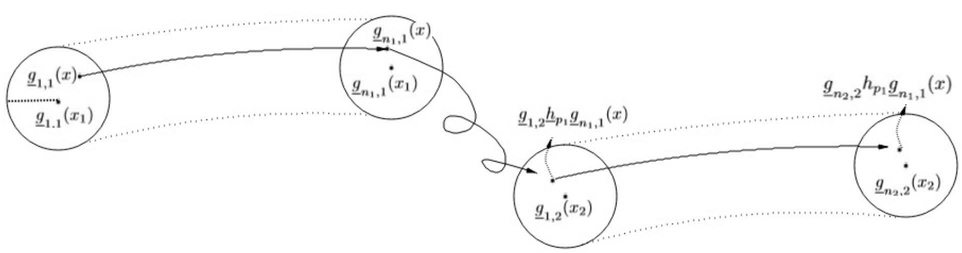

FIG. 1.

Remark 7. *The previous notion demands that every 'long word' semigroup element h_{p_j} can be used to shadow the pieces of orbits. Here, 'long word' means that the element has at least one representation that is obtained by concatenation of a large number ($\geq p_j$) of generators, the identity not included. In the case of finitely generated free semigroups the representation of every element as a concatenation of generators is unique and it makes sense to notice that the size $|h_{p_j}|$ of an element h_{p_j} is well defined and coincides with p_j. However, the later property holds for group actions if and only if X is a unique point, since in the case that G is a group then $id \in G_n$ for every $n \geq 2$. This is one of the reasons to choose G^*_n instead of G_n.*

We also introduce a weaker notion of orbital specification for semigroups inspired by some nonuniform versions for maps.

Definition 8. *We say that the continuous semigroup action $T : G \times X \to X$ associated to the finitely generated semigroup G satisfies the* weak orbital specification property *if for any $\delta > 0$ there exists $p(\delta) > 0$ so that for any $p \geq p(\delta)$, there exists a set $\tilde{G}_p \subset G_p^*$ satisfying $\lim_{p\to\infty} \frac{\sharp \tilde{G}_p}{\sharp G_p^*} = 1$ and for which the following holds: for any $h_{p_j} \in \tilde{G}_{p_j}$ with $p_j \geq p(\delta)$, any points $x_1, \dots, x_k \in X$, any natural numbers $n_1, \dots, n_k$ and any concatenations $\underline{g}_{n_j,j} = g_{i_{n_j},j} \dots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ with $1 \leq j \leq k$ there exists $x \in X$ so that $d(\underline{g}_{\ell,1}(x)\ ,\ \underline{g}_{\ell,1}(x_1)) < \delta$ for every $\ell = 1 \dots n_1$ and*

$$d(\ \underline{g}_{\ell,j}\, \underline{h}_{p_{j-1}} \cdots \underline{g}_{n_2,2}\, \underline{h}_{p_1}\, \underline{g}_{n_1,1}(x)\ ,\ \underline{g}_{\ell,j}(x_j)\ ) < \delta$$

for every $j = 2 \dots k$ and $\ell = 1 \dots n_j$.

We emphasize that the previous definitions are independent of the set of generators for G, hence these are properties intrinsic to the semigroup. This definition weakens the later one by allowing a set of admissible elements (whose proportion increases among all possible semigroup elements) for the shadowing. It is not hard to check that the later notions do not depend on the set of generators for the semigroup. Non-uniform versions of the previous orbital specification properties can be defined in the same spirit as[30,31,38,39,42], but we shall not need or use this fact here. In Section V we provide examples satisfying the orbital specification property but not the usual specification property. The following proposition is the counterpart of Theorem 3 for orbital specification properties.

Proposition 9. *Let G be a finitely generated group. If a continuous group action $T : G \times X \to X$ satisfies the strong (resp. weak) orbital specification property then the push-forward group action $T_\sharp : G \times \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the strong (resp. weak) orbital specification property.*

Proof. Since the proofs of the two claims in the proposition are similar we shall prove the first one with detail and omit the other. By Lemma 4, it is enough to prove the proposition for probabilities that lie on the set $\mathcal{M}_N(X) = \{\frac{1}{N}\sum_{\ell=1}^N \delta x_\ell : x_\ell \in X\}$, for any $N \in \mathbb{N}$. Observe that if T satisfies the strong orbital specification property then the same property holds for the induced action $T^{(N)}$ on the product space X^N. Let $\delta > 0$ and take $p(\delta) \in \mathbb{N}$ given by the strong orbital specification property of the induced action on X^N. Let $\mu_1, ..., \mu_k \in \mathcal{M}_N(X)$ with $\mu_j = \frac{1}{N}\sum_{l=1}^N \delta_{x_l^j}$ and $\underline{g}_{n_j,j} \in G_{n_j}$ $(1 \leq j \leq k)$ be given. If we consider $\bar{x}_j = (x_j^1, ..., x_j^N)$, for any $|\underline{h}_{p_j}| = p_j \geq p(\delta)$ there exists $\bar{x} = (x_1, ..., x_N) \in X^N$ such that $d(\underline{g}_{\ell,1}(x), \underline{g}_{\ell,1}(\bar{x}_1)) < \delta$ for every $\ell = 1, ..., n_1$ and

$$d(\underline{g}_{\ell,j}\underline{h}_{p_{j-1}} \cdots \underline{g}_{n_2,2}\underline{h}_{p_1}\underline{g}_{n_1,1}(\bar{x}), \underline{g}_{\ell,j}(\bar{x}_j)) < \delta$$

for every $j = 2, ..., k$ and $\ell = 1, ..., n_j$. Let $\mu = \frac{1}{N}\sum_{l=1}^N \delta_{x_l}$. In particular μ satisfies $d(\underline{g}_{\ell,1} \cdot \mu\ ,\ \underline{g}_{\ell,1} \cdot \mu_1) < \delta$ for every $\ell = 1, ..., n_1$ and

$$d(\underline{g}_{\ell,j}\underline{h}_{p_{j-1}} \cdots \underline{g}_{n_2,2}\underline{h}_{p_1}\underline{g}_{n_1,1} \cdot \mu\ ,\ \underline{g}_{\ell,j} \cdot \mu_j) < \delta,$$

for every $j = 2, ..., k$ and $\ell = 1, ..., n_j$, which finishes the proof of the proposition. □

C. Specification and hyperbolicity

The relation between specification properties, uniform hyperbolicity and structural stability has been much studied in the last decades, a concept that we will recall briefly. The content of this subsection is of independent interest and will not be used later on along the paper. Given a C^1 diffeomorphism f on a compact Riemannian manifold M and an f-invariant compact set $\Lambda \subset M$ (that is $f(\Lambda) = \Lambda$) we say that Λ is *uniformly hyperbolic* if there exists a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$, $0 < \lambda < 1$ so that $\|Df^n(x) \mid_{E^s_x} \| \le C\lambda^n$ and $\|(Df^n(x) \mid_{E^u_x})^{-1}\| \le C\lambda^n$ for every $x \in \Lambda$ and $n \ge 1$. If $\Lambda = M$ is a hyperbolic set for f then f is called an *Anosov* diffeomorphism.

Originally the notion of specification was introduced by Bowen[8] for uniformly hyperbolic dynamics but fails dramatically in the complement of uniform hyperbolicity (even partially hyperbolic dynamical systems with period points of different index do not satisfy the specification property, see[40,41] for more details). On the other hand Sakai, Sumi and Yamamoto[34] proved that if the specification property holds in a C^1-open set of diffeomorphisms then the dynamical systems are Anosov. It is well know that every C^1 Anosov diffeomorphism f is *structurally stable*, that is, there exists a C^1-open neighborhood $\mathcal{U}$ of f in $\text{Diff}^1(M)$ so that for every $g \in \mathcal{U}$ there is an homeomorphism $h_g : M \to M$ satisfying $g \circ h_g = h_g \circ f$. Thus the C^1-robust specification implies rigidity of the underlying dynamical systems.

The previous results can be extended for finitely generated group actions acting on a compact Riemannian manifold M in a more or less direct way as we now describe. Let G be a finitely generated subgroup of $\text{Diff}^1(M)$ with generators $G_1 = \{g_1^\pm, \dots, g_k^\pm\}$. We will say that the group action $G \times M \to M$ is *structurally stable* if all the generators are structurally stable. In other words, there are C^1-neighborhoods $\mathcal{U}_i$ of the generators g_i $(1 \le i \le k)$ such that for any choice $\tilde g_i \in \mathcal{U}_i$ there exists a homeomorphism h_i such that $\tilde g_i \circ h_i = h_i \circ g_i$. In the case that G is abelian one can require the conjugacies to coincide (c.f. definition of structural stability by Sad[24]). We say that the group action $T : G \times M \to M$ satisfies the *C^1-robust specification property* if there exists a C^1-neighborhood $\mathcal{V}$ of T such that any C^1-action $\tilde T \in \mathcal{V}$ satisfies the specification property. As a byproduct of the previous results we deduce the following consequence:

Corollary 10. *Let G be a finitely generated subgroup of Diff$^1(M)$ such that group action $T : G \times M \to M$ satisfies the C^1-robust specification property. Then every generator is an Anosov diffeomorphism and the group action is structurally stable.*

Proof. Since the group action $T : G \times M \to M$ satisfies the C^1-robust specification property there exists a C^1-neighborhood $\mathcal{V}$ of T such that any C^1-action $\tilde T \in \mathcal{V}$ satisfies the specification property. Moreover, from Lemma 2, any such $\tilde T$ can be identified with a group action associated to a subgroup $\tilde G$ of $\text{Diff}^1(M)$ whose generators $\tilde G_1 = \{\tilde g_1^\pm, \dots, \tilde g_k^\pm\}$ satisfy the specification property. This proves that the generators $g_i \in \text{Diff}^1(M)$ satisfy the C^1-robust specification property and, by[34], are Anosov diffeomorphisms, hence structurally stable. This proves the corollary. $\square$

The previous discussion raises the question of wether the C^1-smoothness assumption is necessary in the previous characterization. For instance, one can ask if a homeomorphism satisfying the specification property C^0-robustly has some form of hyperbolicity. In the remaining of this subsection we shall address some comments on this problem taking as a simple model the push-forward dynamics, which is continuous and acts on the compact metric space of probability measures. Roughly, we will look for some hyperbolicity of the push-forward dynamics assuming that it has the specification property. Clearly, if f is a topologically mixing subshift of finite type then it satisfies the specification property and so does $f_\sharp$. On the other hand, the set of f-invariant measures are (non-hyperbolic) fixed points for $f_\sharp$ and, consequently, this map does not present global hyperbolicity. For that reason we will focus on the fixed points for the continuous map $f_\sharp$ acting on the compact metric space $\mathcal{P}(X)$. Given $\mu \in \mathcal{P}(X)$ and $\delta > 0$ we define the *local stable set* $W^s_\delta(\mu)$ by

$$W^s_\delta(\mu) := \{\eta \in \mathcal{U} : d_{\mathcal{P}}(f^j_\sharp(\mu), f^j_\sharp(\eta)) < \delta \text{ for every } j \ge 0\}$$

(the *local unstable set* $W^u_\delta(\mu)$ is defined analogously with $f_\sharp$ above replaced by $f_\sharp^{-1}$). We say that $\mu \in \mathcal{P}(X)$ is a *hyperbolic fixed point* for $f_\sharp$ if it is a fixed point and there exists $\delta > 0$ and constants $C > 0$ and $0 < \lambda < 1$ so that:

(i) $d_{\mathcal{P}}(f_\sharp^j(\mu), f_\sharp^j(\eta)) < C\lambda^j$ for every $j \geq 1$ and $\eta \in W^s_\delta(\mu)$

(ii) $d_{\mathcal{P}}(f_\sharp^{-j}(\mu), f_\sharp^{-j}(\eta)) < C\lambda^j$ for every $j \geq 1$ and $\eta \in W^u_\delta(\mu)$

We say that the hyperbolic fixed point is of saddle type if both stable and unstable sets are non-trivial. Since the specification implies the topologically mixing property then we will mostly be interested in hyperbolic fixed points of saddle type for $f_\sharp$. It follows from the definition that hyperbolic fixed points for $f_\sharp$ are isolated. The following properties follow from the definitions and Lemma 4:

1. $f_\sharp$ is an affine map, that is, $f_\sharp(t\eta + s\mu) = tf_\sharp(\eta) + sf_\sharp(\mu)$ for every $t, s \geq 0$ with $t + s = 1$ and $\eta, \mu \in \mathcal{P}(X)$
2. μ is a isolated fixed point for $f_\sharp$ if and only if the set of f-invariant probability measures satisfies $\mathcal{M}_f(X) = \{\mu\}$ (i.e. f is uniquely ergodic),
3. $\mathcal{M}_n(X) = \{\frac{1}{n}\sum_{i=1}^n \delta_{x_i} : x_i \in X\} \subset \mathcal{P}(X)$ is a closed $f_\sharp$-invariant set, and
4. $\bigcup_{n\geq 1} \mathcal{M}_n(X)$ is a dense subset of $\mathcal{P}(X)$.

Therefore, to analyze the existence of hyperbolic fixed points of saddle type for $f_\sharp$ that satisfies the specification property we are reduced to the case that f is uniquely ergodic. If f is a contraction on a compact metric space then Banach's fixed point theorem implies the existence of a unique fixed point that is a global attractor and, consequently, the Dirac measure at the attractor is the unique hyperbolic (attractor) fixed point for $f_\sharp$, which is incompatible with transitivity. However, it is nowadays well known that C^0-generic maps have a dense set of periodic points (see e.g.[26]) and, in particular, C^0-generic homeomorphisms f are not uniquely ergodic. In conclusion, there is no open set of homeomorphisms f so that $f_\sharp$ has a unique hyperbolic fixed point of saddle type.

III. SPECIFICATION PROPERTIES AND THE ENTROPY OF SEMIGROUP ACTIONS

The notion of entropy is one of the most important in dynamical systems, either as a topological invariant or as a measure of the chaoticity of the dynamical system. For that reason several notions of entropy and topological pressure have been introduced for group actions in an attempt to describe its dynamical characteristics. As discussed in the introduction, some of the previously introduced definitions take into account the growth rate of the (semi)group, that is, the growth of $|G_n|$ as n increases (see e.g.[3] and references therein). We refer the reader to [16,23] for a detailed description about growth rates for groups and geometric group theory. In this section we characterize entropy points of semigroup actions with specification (Theorem 11) and prove that these actions have positive topological entropy (Theorems 14 and 15).

A. Entropy points

Let X be a compact metric space and G be a semigroup. First we shall introduce the notion of dynamical balls. Given $\varepsilon > 0$ and $\underline{g} := g_{i_n} \dots g_{i_2}\, g_{i_1} \in G_n$ we define the *dynamical ball* $B(x, \underline{g}, \varepsilon)$ by

$$\begin{aligned} B(x, \underline{g}, \varepsilon) &:= B(x, g_{i_n} \dots g_{i_2}\, g_{i_1}, \varepsilon) \\ &= \Big\{ y \in X : d(\underline{g}_j(y), \underline{g}_j(x)) \leq \varepsilon, \ \text{ for every } 0 \leq j \leq n \Big\} \end{aligned} \tag{3}$$

where, by some abuse of notation, we set $\underline{g}_j := g_{i_j} \dots g_{i_2}\, g_{i_1} \in G_n$ for every $1 \leq j \leq n-1$ and $\underline{g}_0 = id$. We also assign a metric $d_{\underline{g}}$ on X by setting

$$d_{\underline{g}}(x_1, x_2) := d_{g_{i_n} \dots g_{i_2}\, g_{i_1}}(x_1, x_2) = \max_{0 \leq j \leq n} d(\underline{g}_j(x_1), \underline{g}_j(x_2)). \tag{4}$$

It is important to notice that here both the dynamical ball and metric are adapted to the underlying concatenation of generators $g_{i_n} \dots g_{i_1}$ instead of the group element $\underline{g}$, since the later one may have distinct representations. For notational simplicity we shall use the condensed notations $B(x, \underline{g}, \varepsilon)$ and $d_{\underline{g}}(\cdot, \cdot)$ when no confusion is possible. In the case that $\underline{g} = f^n$ the later notions coincide with the usual notion of dynamical ball $B_f(x, n, \delta)$ and dynamical distance $d_n(\cdot, \cdot)$ with respect to the dynamical system f, respectively.

Now, we recall a notion of topological entropy introduced by Ghys, Langevin, Walczak[21] and the notion of entropy point introduced by Biś[4]. Two points x, y in X are (n, ε)*-separated* by G if there exists $g \in G_n$ such that $d(g(x), g(y)) \geq \varepsilon$. Given $E \subset X$, let us denote by $s(n, \varepsilon, E)$ the maximal cardinality of (n, ε)-separated set in E. The limit

$$h((G, G_1), E) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, E) \tag{5}$$

is well defined by monotonicity on ε. The *entropy of* (G, G_1) is defined by the previous expression with $E = X$. This definition depends on the generators of G. In this setting of a semigroup G we define by $B_G(x, n, \varepsilon) := \bigcap_{\underline{g} = g_{i_n} \cdots g_{i_1} \in G_n} B(x, \underline{g}, \varepsilon)$ the *dynamical ball for the semigroup* G associated to x, length n and size ε centered at x, where the intersection is over all concatenations that lead to elements in G_n. This corresponds to consider points that are ε-close along the orbit of x by all the trajectories arising from concatenations of generators. We say that the finitely generated semigroup (G, G_1) acting on a compact metric space X admits an *entropy point* x_0 if for any open neighbourhood U of x_0 the equality

$$h((G, G_1), \overline{U}) = h((G, G_1), X)$$

holds. Entropy points are those for which local neighborhoods reflect the complexity of the entire dynamical system. In[4], Biś proved remarkably that any finitely generated group (G, G_1) acting on a compact metric space X admits an entropy point x_0. We prove that the orbital specification property for continuous semigroup actions is enough to prove that all points are entropy points. More precisely,

Theorem 11. *Let $G \times X \to X$ be a continuous finitely generated semigroup action on a compact Riemanian manifold X so that every element $g \in G_1$ is a local homeomorphism. If the semigroup action satisfies the weak orbital specification property then every point of X is an entropy point.*

Proof. First we notice that following the proof of[4] (Theorem 2.5) *ipsis literis* we get the existence of an entropy point $x_0 \in X$ for any finitely generated semigroup of continuous maps on X (the proof does not require invertibility). Hence, for any open neighborhood U of x_0 it holds that $h((G, G_1), X) = h((G, G_1), \overline{U})$. Let $\zeta > 0$ be arbitrary and take $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \overline{U}) \geq h((G, G_1), X) - \zeta$$

for every $0 < \varepsilon \leq \varepsilon_0$.

Given any $z \in X$ and V any open neighborhood of z we claim that $h((G, G_1), \overline{V}) = h((G, G_1), X)$. Fix $0 < \varepsilon \leq \varepsilon_0$ let $p(\varepsilon) \geq 1$ be given by the strong orbital specification property. Since there are finitely many elements in $G_{p(\varepsilon)}$, finitely many of its concatenations and the local inverse branches of elements $\underline{g} : X \to X$ are uniformly continuous there exists a uniform constant $C_\varepsilon > 0$ (that tends to zero as $\varepsilon \to 0$) so that $\operatorname{diam}(\underline{h}^{-1}(B(y, \varepsilon))) \leq C_\varepsilon$ for every $\underline{h} \in G_{p(\varepsilon)}$ and $y \in X$. Take $n \geq 1$ arbitrary, let $E = \{x_1, ..., x_l\} \subset \overline{U}$ be a maximal $(n, \varepsilon, \overline{U})$-separated set and consider the open set $W \subset V$ defined by the set of points $y \in V$ so that $d(y, \partial V) > C_{\varepsilon_0}$. Assume that $0 < \varepsilon \ll \varepsilon_0$ satisfies $\varepsilon + C_\varepsilon < C_{\varepsilon_0}$.

Let $\underline{g} := g_{i_n} \dots g_{i_1} \in G_n$ be fixed. Given a maximal $(\varepsilon, \overline{W})$-separated set $F = \{z_1, ..., z_m\}$, by the weak specification property there exists $\underline{h} = h_{i_{p(\varepsilon)}} \dots h_{i_1} \in G^*_{p(\frac{\varepsilon}{4})}$ so that for any $x_i \in E$ and $z_j \in F$, there exists $y_i^j \in B(z_j, \frac{\varepsilon}{4}) \cap \underline{h}^{-1}(B(x_i, \underline{g}, \frac{\varepsilon}{4}))$. Since $\operatorname{diam}(\underline{h}^{-1}(B(x_i, \frac{\varepsilon}{4}))) \leq$

$C_{\frac{\varepsilon}{4}}$, this implies that $d(\underline{h}^{-1}(B(x_i,\underline{g},\frac{\varepsilon}{4})),\partial V) \geq C_{\varepsilon_0} - \frac{\varepsilon}{4} - C_{\frac{\varepsilon}{4}} > 0$, provided that $\varepsilon \ll \varepsilon_0$. Thus

$$\underline{h}^{-1}\big(B_G(x_i,n,\frac{\varepsilon}{4})\big) \subset \underline{h}^{-1}\big(B(x_i,\underline{g},\frac{\varepsilon}{4})\big) \subset V \quad \text{for every } i.$$

By construction, the dynamical balls $(B_G(x_i,n,\frac{\varepsilon}{4}))_{i=1\dots l}$ are pairwise disjoint and consequently the number of $(n+p(\frac{\varepsilon}{4}),\frac{\varepsilon}{4})$-separated points in $\overline{V}$ is at least $s(n,\varepsilon,\overline{U})$. In other words, $s\big(n+p\left(\frac{\varepsilon}{4}\right),\frac{\varepsilon}{4},\overline{V}\big) \geq s(n,\varepsilon,\overline{U})$ and, consequently,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n+p\left(\frac{\varepsilon}{4}\right)} \log s\big(n+p\left(\frac{\varepsilon}{4}\right),\frac{\varepsilon}{4},\overline{V}\big) &\geq \limsup_{n\to\infty} \frac{1}{n+p\left(\frac{\varepsilon}{4}\right)} \log s(n,\varepsilon,\overline{U}) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon,\overline{U}).\end{aligned}$$

The last inequalities show that $h((G,G_1),X) \geq h((G,G_1),\overline{V}) \geq h((G,G_1),X) - \zeta$. Since ζ was chosen arbitrary this completes the proof of the theorem. $\square$

The previous result indicates that the specification properties are powerfull tools to prove the local complexity of semigroup actions. Observe that the previous result clearly applies for individual transformations.

We now use the notion of topological entropy introduced in[11], which measures the mean cardinality of separated points among possible trajectories generated by the semigroup. Although one can expect that most finitely generated semigroups are free and so to have exponential growth (c.f. proof of Proposition 4.5 by Ghys[22] implying that for a Baire generic set of pairs of homeomorphisms the generated group is a free group on two elements) the notion of average entropy that we consider seems suitable for wider range of semigroups.

Let $E \subset X$ be a compact set. Given $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$, we say a set $K \subset E$ is $(\underline{g},n,\varepsilon)$-*separated set* if $d_{\underline{g}}(x_1,x_2) > \varepsilon$ for any distinct $x_1,x_2 \in K$. When no confusion is possible with the notation for the concatenation of semigroup elements, the maximum cardinality of a $(\underline{g},\varepsilon,n)$-separated sets of X will be denoted by $s(\underline{g},n,E,\varepsilon)$. We now recall the notion of topological entropy introduced by Bufetov[11].

Definition 12. *Given a compact set $E \subset X$, we define*

$$h_{top}((G,G_1),E) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Z_n((G,G_1),E,\varepsilon), \tag{6}$$

where

$$Z_n((G,G_1),E,\varepsilon) = \frac{1}{m^n} \sum_{\underline{g}\in G_n^*} s(\underline{g},n,E,\varepsilon), \tag{7}$$

where the sum is taken over all concatenation $\underline{g}$ of n-elements of $G_1 \setminus \{id\}$ and $m = |G_1 \setminus \{id\}|$. The topological entropy $h_{top}((G,G_1),X)$ *is defined for $E = X$.*

In the case that $E = X$, for simplicity reasons, we shall use simply the notations $s(\underline{g},n,\varepsilon)$ and $Z_n((G,G_1),\varepsilon)$. It is easy to check that $h_{top}((G,G_1),X) \leq h((G,G_1),X)$. Moreover, this notion of topological entropy corresponds to the exponential growth rate of the average cardinality of maximal separated sets by individual dynamical systems $\underline{g}$. This average is taken over elements that are, roughly, in the "ball of radius n in the semigroup G", corresponding to G_n. Notice that for any finite semigroup G, every element $g \in G$ has finite order. In this special case, we notice that every continuous map in the generated semigroup action has zero topological entropy, which is also coherent with the definition of entropy presented in (5).

In this context, and similarly to before, we say that $x \in X$ is an *entropy point* if for any neighborhood U of x one has $h_{top}((G,G_1),\overline{U}) = h_{top}((G,G_1),X)$. Our next theorem asserts that, under the (crucial) strong orbital specification property all points are also entropy points for this notion of entropy. More precisely,

Theorem 13. *Let $G \times X \to X$ be a continuous finitely generated semigroup action on a compact Riemanian manifold X so that every element $g \in G_1$ is a local homeomorphism. If the semigroup action satisfies the strong orbital specification then every point is an entropy point.*

Proof. Given any point $z \in X$ and V any open neighborhood of z we claim that $h_{top}((G, G_1), \overline{V}) = h_{top}((G, G_1), X)$. Let $\zeta > 0$ be arbitrary and take $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon, \overline{U}) \geq h((G, G_1), X) - \zeta$$

for every $0 < \varepsilon \leq \varepsilon_0$. Let $p(\varepsilon) \geq 1$ be given by the strong orbital specification property. Since there are finitely many elements in $G_{p(\varepsilon)}$, finitely many of its concatenations and the local inverse branches of elements $\underline{g} : X \to X$ are uniformly continuous there exists a uniform constant $C_\varepsilon > 0$ (that tends to zero as $\varepsilon \to 0$) so that $\operatorname{diam}(\underline{h}^{-1}(B(y, \varepsilon))) \leq C_\varepsilon$ for every $\underline{h} \in G_{p(\varepsilon)}$ and $y \in X$.

Fix $\underline{h} = h_{i_{p(\varepsilon)}} \dots h_{i_1} \in G^*_{p(\frac{\varepsilon}{4})}$. Take $n \geq 1$ and $\underline{g} := g_{i_n} \dots g_{i_1} \in G_n$ arbitrary, let $E = \{x_1, ..., x_l\} \subset X$ be a maximal $(\underline{g}, n, \varepsilon)$-separated set and consider the open set $W \subset V$ defined by the set of points $y \in V$ so that $d(y, \partial V) > C_{\varepsilon_0}$. Given a maximal $(\varepsilon, \overline{W})$-separated set $F = \{z_1, ..., z_m\}$, by the specification property, for any $x_i \in E$ and $z_j \in F$ there exists

$$y_i^j \in B(z_j, \frac{\varepsilon}{4}) \cap \underline{h}^{-1}(B(x_i, \underline{g}, \frac{\varepsilon}{4})).$$

Similarly as before we deduce that $\underline{h}^{-1}\big(B(x_i, \underline{g}, \frac{\varepsilon}{4})\big) \subset V$ for every i. By construction, the dynamical balls $(B(x_i, \underline{g}, \frac{\varepsilon}{4}))_{i=1\dots l}$ are pairwise disjoint and the points y_i^j are $(\underline{g}\,\underline{h}, \frac{\varepsilon}{4}, \overline{V})$-separated. This proves that

$$s\big(\underline{g}\,\underline{h}, \frac{\varepsilon}{4}, \overline{V}\big) \geq s(\underline{g}, n, X, \varepsilon)\; s(id, 0, \overline{V}, \varepsilon) \geq s(\underline{g}, n, X, \varepsilon).$$

Since the elements $\underline{g}$ and $\underline{h}$ were chosen arbitrary then, summing over all possible concatenations, we deduce

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n + p\left(\frac{\varepsilon}{4}\right)} \log \Big[\frac{1}{m^{n+p\left(\frac{\varepsilon}{4}\right)}} \sum_{\underline{g} \in G^*_n + p\left(\frac{\varepsilon}{4}\right)} s\big(\underline{g}, n + p\left(\frac{\varepsilon}{4}\right), \overline{V}, \frac{\varepsilon}{4}\big)\Big] \\
\geq \limsup_{n\to\infty} \frac{1}{n + p\left(\frac{\varepsilon}{4}\right)} \log \Big(\frac{1}{m^{n+p\left(\frac{\varepsilon}{4}\right)}} \sum_{\underline{g} \in G^*_n} s\big(\underline{g}, n, X, \varepsilon\big)\Big) \\
= \limsup_{n\to\infty} \frac{1}{n} \log \Big(\frac{1}{m^n} \sum_{\underline{g} \in G^*_n} s\big(\underline{g}, n, X, \varepsilon\big)\Big).
\end{aligned}$$

The last inequalities show that $h_{top}((G, G_1), X) \geq h_{top}((G, G_1), \overline{V}) \geq h_{top}((G, G_1), X) - \zeta$. Since both $z \in X$ and $\zeta > 0$ were chosen arbitrary this completes the proof of the theorem. □

B. Positive topological entropy

We now prove that orbital specification properties are enough to guarantee that the semigroup action has positive topological entropy.

Theorem 14. *Let G be a finitely generated semigroup with set of generators G_1 and assume that $G \times X \to X$ is a continuous semigroup action on a compact metric space X. If $G \times X \to X$ satisfies the strong orbital specification property then $h_{top}((G, G_1), X) > 0$. In consequence, $h((G, G_1), X) > 0$.*

Proof. Since the expression in the right hand side of (6) is increasing as $\varepsilon \to 0$ then it is enough to prove that there exists $\varepsilon > 0$ small so that

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{m^n} \sum_{\underline{g}\in G_n^*} s(\underline{g}, n, \varepsilon) > 0.$$

Let $\varepsilon > 0$ be small and fixed so that there are at least two distinct 2ε-separated points $x_1, x_2 \in X$. Take $p(\frac{\varepsilon}{2}) \geq 1$ given by the strong orbital specification property. Taking $\underline{g}_{n_1,1} = \underline{g}_{n_2,2} = id$ and $\underline{h} = h_{p(\frac{\varepsilon}{2})} \dots h_2\, h_1 \in G^*_{p(\frac{\varepsilon}{2})}$ there are $x_{i,j} \in B(x_i, \frac{\varepsilon}{2})$, with $i, j \in \{1, 2\}$, such that $\underline{h}(x_{i,j}) \in B(x_j, \frac{\varepsilon}{2})$. In particular it follows that $s(\underline{h}, p(\frac{\varepsilon}{2}), \varepsilon) \geq 2^2$.

By a similar argument, given $\underline{g} := g_{i_n} \dots g_{i_2} g_{i_1} \in G_n$ with $n = k.p(\frac{\varepsilon}{2})$, it can be written as a concatenation of k elements in $G_{p(\frac{\varepsilon}{2})}$. In other words, $\underline{g} = \underline{h}_k \dots \underline{h}_1$ with $\underline{h}_i \in G_{p(\frac{\varepsilon}{2})}$ and repeating the previous reasoning it follows that $s(\underline{g}, n, \varepsilon) \geq 2^k$. Thus,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log Z_n((G, G_1), \varepsilon) &\geq \limsup_{k\to\infty} \frac{1}{k\, p(\frac{\varepsilon}{2})} \log \left(\frac{1}{m^{k\, p(\frac{\varepsilon}{2})}} \sum_{|\underline{g}|=k\, p(\frac{\varepsilon}{2})} s(\underline{g}, k\, p(\frac{\varepsilon}{2}), \varepsilon) \right) \\ &\geq \frac{1}{p(\frac{\varepsilon}{2})} \log 2.\end{aligned}$$

This proves that the entropy is positive and finishes the proof of the theorem. $\square$

Let us observe that in[19] the author obtained a lower bound for the topological entropy of C^1-maps on smooth orientable manifolds. Here we require continuity of the semigroup action and a specification property (which most likely can be weakened) for deducing that topological entropy is strictly positive. One could expect that the weak orbital specification property could imply the semigroup action to have positive entropy. In fact this is the case whenever the semigroup satisfies additional conditions on the growth rate which hold e.g. for free semigroups.

Theorem 15. *Assume that G is a finitely generated semigroup and that the continuous action $G \times X \to X$ on a compact metric space X satisfies the weak orbital specification property with*

(H) $\displaystyle\limsup_{p\to\infty} \frac{|G_p^* \setminus \tilde{G}_p|}{m^{\gamma p}} < 1$ *for every* $0 < \gamma < 1$.

Then the semigroup action $G \times X \to X$ has positive topological entropy.

In Subsection V we give some examples of semigroups combining circle expanding maps and rotations that satisfies the weak orbital specification property and for which $|G_p^* \setminus \tilde{G}_p|$ is finite, hence (H) holds.

Proof of Theorem 15. Given $\varepsilon > 0$, let $p(\varepsilon) \geq 1$ be given by the specification property. For any $p \geq p(\varepsilon)$ let $\tilde{G}_p \subset G_p^*$ be given by the weak orbital specification property. Take $n = kp$ with $p \geq p(\frac{\varepsilon}{2})$ and assume that (H) holds.

For any $\underline{g} \in G_n^*$ one can write it as a concatenation of k elements in G_p^*, that is, $\underline{g} = \underline{h}_k \dots \underline{h}_1$ with $\underline{h}_i \in G_p^*$. If this is the case, given $0 < \gamma < 1$ we will say that $\underline{g} = \underline{h}_k \dots \underline{h}_1 \in G_n^*$ is γ-acceptable if $\sharp\{0 \leq j \leq k : \underline{h}_j \in \tilde{G}_p\} > \gamma k$. Notice that

$$\begin{aligned}&\sharp\{\underline{g} = \underline{h}_k \dots \underline{h}_1 \in G_{kp} : \underline{g} \text{ not } \gamma\text{-acceptable}\} \\ &\qquad\qquad \leq \sum_{l\geq[\gamma k]}^{k} \sharp\{\underline{g} \in G_{kp} : \sharp\{0 \leq j \leq k : \underline{h}_j \in G_p \backslash \tilde{G}_p\} = l\}.\end{aligned}$$

In consequence,

$$\begin{aligned}
&\sharp\{\underline{g} = \underline{h}_k \dots \underline{h}_1 \in G^*_{kp} : \underline{g} \text{ not } \gamma\text{-acceptable}\} \\
&\qquad\qquad \leq \sum_{l \geq [\gamma k]}^{k} \sharp\{\underline{g} \in G^*_{kp} : \sharp\{0 \leq j \leq k : \underline{h}_j \in G^*_p \backslash \tilde{G}_p\} = l\}.
\end{aligned}$$

In consequence,

$$\begin{aligned}
\frac{\sharp\{\underline{g} \in G^*_{kp} : \underline{g} \text{ is not } \gamma\text{-acceptable}\}}{m^{kp}} &\leq \frac{\sum_{l \geq [\gamma k]}^{k} \begin{pmatrix} k \\ l \end{pmatrix} |G^*_p|^{k-l} |G^*_p \backslash \tilde{G}_p|^l}{m^{kp}} \\
&\leq k \frac{\begin{pmatrix} k \\ [\gamma k] \end{pmatrix} m^{(1-\gamma)kp} |G^*_p \backslash \tilde{G}_p|^k}{m^{kp}} \\
&= k \binom{k}{[\gamma k]} \Big(\frac{|G^*_p \backslash \tilde{G}_p|}{m^{\gamma p}} \Big)^k. \qquad (8)
\end{aligned}$$

By assumption (H), given $0 < \gamma_0 < 1$ let $0 < \delta \ll \log 2$ be small so that $\limsup_{p\to\infty} \frac{|G^*_p \backslash \tilde{G}_p|}{m^{\gamma_0 p}} < e^{-2\delta} < 1$. Then by monotonicity of the later limsup in γ, it is clear that

$$\limsup_{p\to\infty} \frac{|G^*_p \backslash \tilde{G}_p|}{m^{\gamma p}} < e^{-2\delta} < 1$$

for every $\gamma \in (\gamma_0, 1)$. Up to consider larger γ sufficiently close to 1 so that $k \begin{pmatrix} k \\ [\gamma k] \end{pmatrix} \leq e^{\delta k}$ for every k large. The later implies that

$$\frac{\sharp\{\underline{g} \in G^*_{kp} : \underline{g} \text{ is not } \gamma\text{-acceptable}\}}{m^{kp}} \lesssim e^{\delta k} \Big(\frac{|G^*_p \backslash \tilde{G}_p|}{m^{\gamma p}} \Big)^k \lesssim e^{-\delta k}$$

which decreases exponentially fast in k (provided that p is large enough). Moreover, given $p \gg p(\frac{\varepsilon}{2})$ one can proceed as in the proof of the previous theorem and prove that $s(\underline{g}, kp, \varepsilon) \geq 2^{\gamma k}$ for any γ-admissible $\underline{g} \in G^*_{kp}$. Consequently,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log Z_n((G, G_1), \varepsilon) &\geq \limsup_{k\to\infty} \frac{1}{k\, p(\frac{\varepsilon}{2})} \log \Big(\frac{\sharp\{\underline{g} \in G^*_{kp} : \underline{g} \text{ is } \gamma\text{-acceptable}\}}{m^{kp(\frac{\varepsilon}{2})}} 2^{\gamma k} \Big) \\
&\geq \frac{1}{p(\frac{\varepsilon}{2})} \log 2^\gamma + \limsup_{k\to\infty} \frac{1}{k\, p(\frac{\varepsilon}{2})} \log \big(1 - e^{-\delta k}\big) \\
&\geq \frac{\gamma}{p(\frac{\varepsilon}{2})} \log 2 - \frac{\delta}{p(\frac{\varepsilon}{2})}
\end{aligned}$$

which is strictly positive, by the choice of δ and γ. This proves the theorem. □

IV. THERMODYNAMICS OF EXPANSIVE SEMIGROUP ACTIONS WITH SPECIFICATION

In this section we study thermodynamical properties of positively expansive semigroup actions satisfying specification and also semigroups of uniformly expanding maps. First we prove that semigroups of expanding maps satisfy the orbital specification properties (Theorems 16). Then we obtain conditions for the convergence of topological pressure (Theorem 25). Finally we prove a strong regularity of the topological pressure function (Theorem 27) and prove that topological entropy is a lower bound for the exponential growth rate of periodic points (Theorem 28).

A. Semigroup of expanding maps and specification

Throughout this subsection we shall assume that X is a compact Riemannian manifold. We say that a C^1-local diffeomorphism $f : M \to M$ is an *expanding map* if there are constants $C > 0$ and $0 < \lambda < 1$ such that $\|(Df^n(x))^{-1}\| \le C\lambda^n$ for every $n \ge 1$ and $x \in X$.

Theorem 16. *Let $G_1 = \{g_1, g_2, \dots, g_k\}$ be a finite set of expanding maps and let G be the generated semigroup. Then G satisfies the strong orbital specification property.*

The following two lemmas will be instrumental in the proof of Theorem 16.

Lemma 17. *Let $g_1, \dots, g_k$ be C^1-expanding maps on the compact manifold X. There exists $\delta_0 > 0$ so that $\underline{g}(B(x, \underline{g}, \delta)) = B(\underline{g}(x), \delta)$ for any $0 < \delta \le \delta_0$, any $x \in X$ and any $\underline{g} \in G$.*

Proof. Let $d_i = \deg(g_i)$ be the degree of the map g_i. Since g_i is a local diffeomorphism there exists $\delta > 0$ (depending on g_i) so that for every $x \in X$ setting $g_i^{-1}(x) = \{x_{i,1}, \dots, x_{i,d_i}\}$ there are d_i well defined inverse branches $g_{i,j}^{-1} : B(x, \delta) \to V_{x_{i,j}}$ onto an open neighborhood of $x_{i,j}$. Since there are finitely many maps g_i there exists a uniform constant $\delta_0 > 0$ so that all inverse branches for g_i are defined in balls of radius δ_0. Furthermore, since all g_i are uniformly expanding all inverse branches are λ-contracting for some uniform $0 < \lambda < 1$, meaning that $d(\ g_{i,j}^{-1}(y), g_{i,j}^{-1}(z)\ ) \le \lambda\, d(y, z)$ for any $x \in X$, any $y, z \in B(x, \delta_0)$ and $i = 1 \dots k$. In particular $g_{i,j}^{-1}(B(x, \delta_0)) \subset B(x_{i,j}, \delta_0)$ and so

$$V_{x_{i,j}} = \{y \in X : d(y, x_{i,j}) < \delta_0 \,\&\, d(g_i(y), g_i(x_{i,j})) < \delta_0\} = B_{g_i}(x_{i,j}, 1, \delta_0).$$

Using this argument recursively, every $\underline{g}_j = g_{i_j} \dots g_{i_2}\, g_{i_1} \in G_j$ is a contraction and we get that the dynamical ball $B(x, \underline{g}, \delta) = \bigcap_{j=0}^{n} \underline{g}_j^{-1}(B(\underline{g}_j(x), \delta))$ (for $0 < \delta < \delta_0$) is mapped diffeomorphically by $\underline{g}$ onto $B(\underline{g}(x), \delta)$, proving the lemma. $\square$

Lemma 18. *Let $g_1, \dots, g_k$ be C^1-expanding maps on the compact manifold X. For any $\delta > 0$ there exists $N = N(\delta) \in \mathbb{N}$ so that $\underline{g}_N(B(x, \delta)) = X$ for every $x \in X$ and every $\underline{g}_N \in G_N^*$.*

Proof. There exists a uniform $0 < \lambda < 1$ so that all inverse branches for g_i are λ-contracting for any i. Fix $\delta > 0$. Using the compactness of X it is enough to prove that for any $x \in X$ there exists $N \ge 1$ so that $\underline{g}_N(B(x, \delta)) = X$ for every $\underline{g}_N \in G_N^*$. Take $N = N(\delta) \ge 1$ be large and such that $\lambda^N(1 + \operatorname{diam} X) < \delta$. Let $\underline{g}_N \in G_N^*$ be arbitrary and assume, by contradiction, that $\underline{g}_N(B(x, \delta)) \ne X$. Then there exists a curve γ_N with diameter at most $\operatorname{diam} X + 1$ connecting the points x and $y \in X \setminus \underline{g}_N(B(x, \delta))$. Consider a covering of γ_N by balls of radius δ and consider γ the image of γ_N by the inverse branches, such that γ connects x to some point $z \notin B(x, \delta)$ so that $\underline{g}_N(z) = y$. Using that $y \notin \underline{g}_N(B(x, \delta))$ one gets that $z \notin B(x, \delta)$. Since $\underline{g}_N$ is a λ^N-contraction then $\delta < d(x, z) \le \operatorname{length}(\gamma) \le \lambda^N(1 + \operatorname{diam} X) < \delta$, which is a contradiction. Thus the lemma follows. $\square$

Proof of Theorem 16. The proof of the theorem follows from the previous lemmas. In fact, let $\delta > 0$ be fixed and consider $x_1, x_2, \dots, x_k \in X$, natural numbers $n_1, n_2, \dots, n_k$ and group elements $\underline{g}_{n_j, j} = g_{i_{n_j}, j} \dots g_{i_2, j}\, g_{i_1, j} \in G_{n_j}$ $(j = 1 \dots k)$. By Lemma 17, there exists ε_0 such that for $\varepsilon \le \varepsilon_0$

$$\underline{g}_{n_j}(B(x_j, \underline{g}_{n_j}, \varepsilon)) = B(\underline{g}_{n_j}(x_j), \varepsilon),\ \forall 1 \le j \le k.$$

We may assume without loss of generality that $\delta < \varepsilon_0$. Let $p(\delta) = N(\delta)$ be given by Lemma 18. Given $p_1, \dots, p_k \ge p(\delta)$, for $\underline{h}_{p_j} \in G_{p_j}^*$ we have that $\underline{h}_{p_i}(B(\underline{g}_{n_i}(x_i), \delta)) = X$. It implies that given $\bar{x}_k \in B(x_k, \underline{g}_{n_k}, \delta)$, one has $\bar{x}_k = \underline{h}_{p_{k-1}}(\bar{x}_{k-1})$, with $\bar{x}_{k-1} \in$

$B(\underline{g}_{n_{k-1}}(x_{k-1},\varepsilon))$, and then $\bar{x}_k = \underline{g}_{n_{k-1}}\underline{h}_{p_{k-1}}(\bar{x}_{k-2})$, for some $\bar{x}_{k-2} \in B(x_{k-1},\underline{g}_{n_{k-1}},\varepsilon)$. By induction, there exists $x \in B(x_1,\underline{g}_{n_1},\varepsilon)$, such that

$$\underline{g}_{\ell,j}\,\underline{h}_{p_{j-1}}\cdots\underline{g}_{n_2,2}\,\underline{h}_{p_1}\,\underline{g}_{n_1,1}(x) \in B(x_j,\underline{g}_{\ell,j},\varepsilon)$$

for every $j = 2\ldots k$ and $\ell = 1\ldots n_j$. This completes the proof of the theorem. □

For completeness, let us mention that the results in this subsection hold also for general topologically mixing distance expanding maps on compact metric spaces (X,d). Recall f is a distance expanding map if there are $\delta > 0$ and $0 < \lambda < 1$ so that $d(f(x),f(y)) \geq \lambda^{-1}d(x,y)$ for every $d(x,y) < \delta$. Our motivation to focus on smooth maps comes from the fact free semigroups can be constructed and shown to be robust in this context (c.f. Section V).

B. Convergence and regularity of entropy and the pressure function

In what follows we shall introduce a notion of topological pressure. For notational simplicity, given $\underline{g} \in G_n$ and $U \subset X$ we will use the notation $S_{\underline{g}}\varphi(x) = \sum_{i=0}^{n-1}\varphi(\underline{g}_i(x))$ and $S_{\underline{g}}\varphi(U) = \sup_{x\in U} S_{\underline{g}}\varphi(x)$.

Definition 19. *For any continuous observable $\varphi \in C(X)$ we define the* topological pressure of (G,G_1) with respect to φ *by*

$$P_{top}((G,G_1),\varphi,X) := \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Z_n((G,G_1),\varphi,\varepsilon), \tag{9}$$

where

$$Z_n((G,G_1),\varphi,\varepsilon) = \frac{1}{m^n}\sum_{\underline{g}\in G_n^*}\sup_E\left\{\sum_{x\in E} e^{\sum_{i=0}^{n-1}\varphi(\underline{g}_i(x))}\right\} \tag{10}$$

and the supremum is taken over all sets $E = E_{\underline{g},n,\varepsilon}$ that are $(\underline{g},n,\varepsilon)$-separated.

Observe that in the case that G has only one generator f then $|G_n| = |\{f^n\}| = 1$ and $P_{top}((G,G_1),\varphi)$ coincides with the classical pressure $P_{\mathrm{top}}(f,\varphi)$. The case that the potential is constant to zero corresponds to the notion of topological entropy introduced in Definition 12. We proceed to prove that the topological pressure of expansive semigroup actions with the specification property can be computed as a limit. For that purpose we provide an alternative formula to compute the topological pressure using open covers. Given $\varepsilon > 0$, $n \in \mathbb{N}$ and $\underline{g} \in G_n$, we say that an open cover $\mathcal{U}$ of X is an $(\underline{g},n,\varepsilon)$-cover if any open set $U \in \mathcal{U}$ has $d_{\underline{g}}$-diameter smaller than ε, where $d_{\underline{g}}$ is the metric introduced in (4). Let $cov(\underline{g},n,\varepsilon)$ be the minimum cardinality of a $(\underline{g},n,\varepsilon)$-cover of X. To obtain a characterization of the topological pressure using open covers of the space we need the continuous potential to satisfy a regularity condition. Given $\varepsilon > 0$ and $\underline{g} := g_{i_n}\ldots g_{i_1} \in G$ we define the *variation of $S_{\underline{g}}\varphi$* in dynamical balls of radius ε by

$$Var_{\underline{g}}(\varphi,\varepsilon) = \sup_{d_{\underline{g}}(x,y)<\varepsilon}|S_{\underline{g}}\varphi(x) - S_{\underline{g}}\varphi(y)|.$$

We say that φ has *bounded distortion property* (in dynamical balls of radius ε) if there exists $C > 0$ so that

$$\sup_{\underline{g}\in G}\sup_{x\in X} Var_{\underline{g}}(\varphi,\varepsilon) \leq C.$$

For short we denote by $BD(\varepsilon)$ the space of continuous potentials that have bounded distortion in dynamical balls of radius ε and we say that φ has *bounded distortion property* if there exists $\varepsilon > 0$ so that φ has bounded distortion on dynamical balls of radius ε. In what follows we prove that Hölder potentials have bounded distortion for semigroups of expanding maps.

Lemma 20. *Let G be a finitely generated semigroup of expanding maps on a compact metric space X with generators $G_1 = \{g_1, \dots, g_m\}$. Then any Hölder continuous observable $\varphi : M \to \mathbb{R}$ satisfies the bounded distortion property.*

Proof. Let $\delta_0 > 0$ and $0 < \lambda < 1$ be chosen as in the proof of the previous lemma and assume that φ is (K, α)-Hölder. Given any $0 < \varepsilon < \delta_0/2$, any $\underline{g} = g_{i_n} \dots g_{i_1} \in G_n$ and $x, y \in X$ with $d_{\underline{g}}(x, y) < \varepsilon$,

$$\begin{aligned} |S_{\underline{g}}\varphi(x) - S_{\underline{g}}\varphi(y)| &= |\sum_{i=0}^{n-1} \varphi(\underline{g}_i(x)) - \sum_{i=0}^{n-1} \varphi(\underline{g}_i(y))| \le \sum_{i=0}^{n-1} |\varphi(\underline{g}_i(x)) - \varphi(\underline{g}_i(y))| \\ &\le \sum_{i=0}^{n-1} K d(\underline{g}_i(x), \underline{g}_i(y))^\alpha \le \sum_{i=0}^{n-1} K \lambda^{(n-i)\alpha} d(\underline{g}_n(x), \underline{g}_n(y))^\alpha \\ &\le \frac{K}{1-\lambda^\alpha} \varepsilon^\alpha. \end{aligned}$$

This proves the lemma. □

Proposition 21. *Let $\varphi : X \to \mathbb{R}$ be a continuous map satisfying the bounded distortion condition. Then the topological pressure $P_{top}((G, G_1), \varphi, X)$ with respect to the potential φ satisfies*

$$P_{top}((G, G_1), \varphi, X) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left(\frac{1}{m^n} \sum_{\underline{g} \in G_n^*} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{S_{\underline{g}}\varphi(U)} \right),$$

where the infimum is taken over all open covers $\mathcal{U}$ of X such that $\mathcal{U}$ is a $(\underline{g}, n, \varepsilon)$-open cover.

Proof. Although the proof of this proposition follows a classical argument we include it here for completeness. Take $\varepsilon > 0$, $n \in \mathbb{N}$ and $\underline{g} \in G_n$. To simplify the notation we denote

$$C_n((G, G_1), \varphi, \varepsilon) = \frac{1}{m^n} \sum_{\underline{g} \in G_n^*} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{S_{\underline{g}}\varphi(U)}$$

where the infimum are taken over all $(\underline{g}, n, \varepsilon)$-open covers and let $Z_n((G, G_1), \varphi, \varepsilon)$ be given by equation (10). Given a $(\underline{g}, n, \varepsilon)$-maximal separated set E it follows that $\mathcal{U} = \{B(x, \underline{g}, \varepsilon)\}_{x \in E}$ is a $(\underline{g}, n, 2\varepsilon)$-open cover. By the bounded distortion assumption, $S_{\underline{g}}\varphi(B(x, \underline{g}, \varepsilon)) = \sup_{z \in B(x, \underline{g}, \varepsilon)} S_{\underline{g}}\varphi(z) \le S_{\underline{g}}\varphi(x) + C$ for some constant $C > 0$, depending only on ε. Consequently,

$$\limsup_{n \to \infty} \frac{1}{n} \log C_n((G, G_1), \varphi, 2\varepsilon) \le \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon). \tag{11}$$

On the other hand, if $\mathcal{U}$ is $(\underline{g}, n, \varepsilon)$-open cover, for any $(\underline{g}, n, \varepsilon)$-separated set $E \subset X$ we have that $\sharp E \le \sharp \mathcal{U}$, since the diameter of any $U \in \mathcal{U}$ in the metric $d_{\underline{g}}$ is less than ε. By the bounded distortion condition we get that

$$\limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon) \le \limsup_{n \to \infty} \frac{1}{n} \log C_n((G, G_1), \varphi, \varepsilon). \tag{12}$$

Now, combining equations (11) and (12) we get that

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon) &\le \limsup_{n \to \infty} \frac{1}{n} \log C_n((G, G_1), \varphi, \varepsilon) \\ &\le \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \frac{\varepsilon}{2}) \end{aligned} \tag{13}$$

and then the result follows. □

In the next lemma we provide a condition under which the topological pressure can be computed as a limit.

Proposition 22. *Let $\varphi : X \to \mathbb{R}$ be a continuous potential. Given $\varepsilon > 0$, the limit superior*

$$\limsup_{n\to\infty} \frac{1}{n} \log \Big(\frac{1}{m^n} \sum_{\underline{g}\in G_n^*} \inf_{\mathcal{U}} \sum_{U\in\mathcal{U}} e^{S_{\underline{g}}\varphi(U)} \Big)$$

is indeed a limit.

Proof. Since φ is continuous then it is bounded from below. Assume without loss of generality that φ is non-negative, otherwise we just consider a translation $\varphi + C$ since it will affect the lim sup by a translation of C. Given $\varepsilon > 0$, recall that the infimum is taken over all $(\underline{g}, n, \varepsilon)$-open covers $\mathcal{U}$ of X. For any element $\underline{g} = \underline{h}\,\underline{k} \in G_{\ell+n}^*$ with $\underline{h} \in G_\ell, \underline{k} \in G_n^*$, and any $(\underline{h}, n, \varepsilon)$-cover $\mathcal{U}$ and $(\underline{k}, \ell, \varepsilon)$-cover $\mathcal{V}$ then $\mathcal{W} := \underline{k}^{-1}(\mathcal{U}) \vee \mathcal{V}$ is a $(\underline{g}, \ell+n, \varepsilon)$-cover, and

$$\sum_{\substack{W\in\underline{k}^{-1}(\mathcal{U})\vee\mathcal{V} \\ W=\underline{k}^{-1}(U)\cap V}} e^{tS_{\underline{g}}\varphi(W)} \le \Big(\sum_{V\in\mathcal{V}} e^{tS_{\underline{k}}\varphi(V)} \Big) \Big(\sum_{U\in\mathcal{U}} e^{tS_{\underline{h}}\varphi(U)} \Big)$$

Taking the infimum over the open covers $\mathcal{U}$ and $\mathcal{V}$ we deduce that

$$\inf_{\mathcal{W}} \Big\{ \sum_{W\in\mathcal{W}} e^{tS_{\underline{g}}\varphi(W)} \Big\} \le \inf_{\mathcal{V}} \Big\{ \sum_{V\in\mathcal{V}} e^{tS_{\underline{k}}\varphi(V)} \Big\} \inf_{\mathcal{U}} \Big\{ \sum_{U\in\mathcal{U}} e^{tS_{\underline{h}}\varphi(U)} \Big\}.$$

where the first infimum can be taken over all $(\underline{g}, m+n, \varepsilon)$-open covers $\mathcal{W}$. Summing over every elements $\underline{g} = \underline{h}\,\underline{k} \in G_{\ell+n}^*$,

$$\sum_{|\underline{g}|=\ell+n} \inf_{\mathcal{W}} \Big\{ \sum_{W\in\mathcal{W}} e^{tS_{\underline{g}}\varphi(W)} \Big\} \le \Big(\sum_{|\underline{k}|=\ell} \inf_{\mathcal{V}} \sum_{V\in\mathcal{V}} e^{tS_{\underline{k}}\varphi(V)} \Big)\Big(\sum_{|\underline{h}|=n} \inf_{\mathcal{U}} \sum_{U\in\mathcal{U}} e^{tS_{\underline{h}}\varphi(U)} \Big).$$

Thus, the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \log \Big(\sum_{\underline{g}\in G_n^*} \inf_{\mathcal{W}} \Big\{ \sum_{W\in\mathcal{W}} e^{tS_{\underline{g}}\varphi(W)} \Big\} \Big)$$

is subaditive and $\{a_n/n\}_{n\in\mathbb{N}}$ is convergent. Since the term $\frac{1}{n}\log\frac{1}{m^n}$ is clearly constant this completes the proof of the proposition. □

From the previous results, the topological pressure can be computed as the limiting complexity of the group action as the size scale ε approaches zero. In what follows we will be mostly interested in providing conditions for the topological pressure of group actions to be computed as a limit at a definite size scale. Let us introduce the necessary notions. Let X be a compact metric space and $G \times X \to X$ be a continuous action associated to the finitely generated semigroup (G, G_1).

Definition 23. *Given $\delta^* > 0$, the semigroup action $G \times X \to X$ is* δ^*-expansive *if for every $x, y \in X$ there exists $k \ge 1$ and $\underline{g} \in G_k$ such that $d(\underline{g}(x), \underline{g}(y)) > \delta^*$. The semigroup action $G \times X \to X$ is* strongly δ^*-expansive *if for any $\gamma > 0$ and any $x, y \in X$ with $d(x, y) \ge \gamma$ there exists $k \ge 1$ (depending on γ) such that $d_{\underline{g}}(x, y) > \delta^*$ for all $\underline{g} \in G_k^*$.*

Remark 24. *By compactness of the phase space X, a continuous action is* strongly δ^*-expansive *satisfies the following equivalent formulation: given $\gamma > 0$ and $x, y \in X$ with $d(x, y) \ge \gamma$ there exists $k_0 \ge 1$ (depending on γ) such that $d_{\underline{g}}(x, y) > \delta^*$ for all $\underline{g} \in G_k^*$ and $k \ge k_0$.*

In what follows we prove that the topological entropy of expansive semigroup actions can be computed as the topological complexity that is observable at a definite scale. More precisely,

Theorem 25. *Assume the continuous action of G on the compact metric space X is strongly δ^*-expansive. Then, for every continuous potential $\varphi : X \to \mathbb{R}$ satisfying the bounded distortion condition and every $0 < \varepsilon < \delta^*$*

$$P(\varphi) := P_{top}((G, G_1), \varphi, X) = \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1}{m^n} \sum_{\underline{g}\in G_n^*} \sup_E \sum_{x\in E} e^{S_{\underline{g}}\varphi(x)} \right)$$

where the supremum is taken over all $(\underline{g}, n, \varepsilon)$-separated sets $E \subset X$.

We just observe, before the proof, that in view of the previous characterization given in Proposition 21, the same result as above also holds if we consider open covers instead of separated sets.

Proof of Theorem 25. Since X is compact and $\varphi : X \to \mathbb{R}$ is continuous we assume, without loss of generality, that φ is non negative. Fix γ and ε with $0 < \gamma < \varepsilon < \delta^*$. We want to show that

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \gamma) \le \limsup_{n\to\infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon).$$

The other inequality is clear. By strong δ^*-expansiveness and Remark 24 for any two distinct points $x, y \in X$ with $d(x, y) \ge \gamma$ there exists $k_0 \ge 1$ (depending on γ) so that $d_{\underline{g}}(x, y) \ge \delta^* > \varepsilon$ for any $\underline{g} \in G_k^*$ and $k \ge k_0$. Take $n \ge k_0$ and $\underline{g} \in G_{n+k}^*$ arbitrary and write $\underline{g} = \underline{h}_2\underline{h}_1$ with $\underline{h}_1 \in G_n^*$ and $\underline{h}_2 \in G_k^*$. Given any $(\underline{h}_1, n, \gamma)$-separated set E we claim that the set E is $(\underline{g}, n+k, \varepsilon)$-separated. In fact, given $x, y \in E$ there exists a decomposition $\underline{h}_1 = \underline{h}_{1,2}\, \underline{h}_{1,1} \in G_n^*$ so that $d(\underline{h}_{1,1}(x), \underline{h}_{1,1}(y)) > \gamma$. Using that $\underline{h}_2\, \underline{h}_{1,2} \in \bigcup_{l\ge k} G_l^*$ and Remark 24 it follows that $d_{\underline{g}}(x, y) \ge d_{\underline{h}_2\, \underline{h}_{1,2}}(\underline{h}_{1,1}(x), \underline{h}_{1,1}(y)) > \varepsilon$ proving the claim. Now, using that φ is non-negative,

$$e^{S_{\underline{g}}\varphi(x)} = e^{S_{\underline{h}_2\underline{h}_1}\varphi(x)} = e^{S_{\underline{h}_2}\varphi(\underline{h}_1(x))} e^{S_{\underline{h}_1}\varphi(x)} \ge e^{S_{\underline{h}_1}\varphi(x)},$$

which implies that $Z_n((G, G_1), \varphi, \gamma) \le m^k Z_n((G, G_1), \varphi, \varepsilon)$ because

$$\begin{aligned} Z_n((G, G_1), \varphi, \gamma) &= \frac{1}{m^n} \sum_{|\underline{h}_1|=n} \sup_E \sum_{x\in E} e^{S_{\underline{h}_1}\varphi(x)} \\ &\le \frac{m^{n+k}}{m^n} \frac{1}{m^{n+k}} \sum_{\underline{g}\in G_n^*+k} \sup_E \sum_{x\in E} e^{S_{\underline{g}}\varphi(x)} = m^k Z_{n+k}((G, G_1), \varphi, \varepsilon). \end{aligned}$$

Thus it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \gamma) \le \limsup_{n\to\infty} \frac{1}{n+k} \log Z_{n+k}((G, G_1), \varphi, \varepsilon),$$

as we wanted to prove. This completes the proof of the theorem. □

Some comments on our assumptions are in order. It is clear that if some generator for the group is an expansive map then the group is itself expansive. Clearly, expanding maps are expansive. Moreover, the semigroup G generated by $G_1 = \{g_1, ..., g_k\}$ that admits some expansive generator is clearly expansive. In Lemma 26 below we prove that semigroups of expanding maps are strongly expansive semigroups.

Lemma 26. *Let G be a finitely generated semigroup of expanding maps on a compact metric space X with generators G_1. Then there exists $\delta^* > 0$ so that G is strongly δ^*-expansive.*

Proof. Let $G_1 = \{g_1, ..., g_m\}$ be the set of generators of G. Following the proof of Lemma 17 there are uniform constants $\delta_0 > 0$ and $0 < \lambda < 1$ so that all inverse branches $g_{i,j}^{-1}$ for g_i are defined in balls of radius δ_0 and $d(\ g_{i,j}^{-1}(y), g_{i,j}^{-1}(z)\ ) \leq \lambda\, d(y,z)$. for any $x \in X$, any $y, z \in B(x, \delta_0)$ and $i = 1 \ldots m$. Take $\delta_* = \delta_0/2$. Given $\gamma > 0$ take $k \geq 1$ (depending on γ) so that $\lambda^k \delta^* < \gamma$. We claim that for any $x, y \in X$ with $d(x,y) \geq \gamma$ and $g \in G_k^*$ we have $d_{\underline{g}}(x,y) > \delta^*$. Assume, by contradiction, that there exists $\underline{g} = g_{i_k}...g_{i_1} \in G_k^*$ with $d(\underline{g}(x), \underline{g}(y)) \leq d_{\underline{g}}(x,y) \leq \delta^*$. Then $d(g_{i_j}...g_{i_1}(x), g_{i_j}...g_{i_1}(y)) \leq \lambda^{k-j} d(g_{i_k}...g_{i_1}(x), g_{i_k}...g_{i_1}(y))$ for every $1 \leq j \leq k$ and so $d(x,y) \leq \lambda^k d(\underline{g}(x), \underline{g}(y)) < \gamma$, which is a contradiction. This finishes the proof of the lemma. □

Theorem 27. *Let G be a finitely generated semigroup with generators G_1. If the semigroup action induced by G on the compact metric space X is strongly δ^*-expansive and the potentials $\varphi, \psi : X \to \mathbb{R}$ are continuous and satisfy the bounded distortion property then*

1. *$P_{top}((G, G_1), \varphi + c, X) = P_{top}((G, G_1), \varphi, X) + c$ f or every $c \in \mathbb{R}$*
2. *$|P_{top}((G, G_1), \varphi, X) - P_{top}((G, G_1), \psi, X)| \leq \|\varphi - \psi\|$, and*
3. *the pressure function $t \mapsto P_{top}((G, G_1), t\varphi, X)$ is an uniform limit of differentiable maps.*

Moreover, $t \mapsto P_{top}((G, G_1), t\varphi, X)$ is differentiable Lebesgue-almost everywhere.

Proof. We start by observing that property (1) follows directly from the definition of the topological pressure. By hypothesis let $\varepsilon_0 > 0$ be so that $\varphi, \psi \in BD(\varepsilon_0)$. On the one hand, by Theorem 25 together with equation (13) it follows that for any $0 < \varepsilon < \delta^*$,

$$P(\varphi) := P_{top}((G, G_1), t\varphi, X) = \limsup_{n \to \infty} \frac{1}{n} \log \left(\frac{1}{m^n} \sum_{\underline{g} \in G_n^*} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{tS_{\underline{g}}\varphi(U)} \right)$$

where the infimum is taken over all $(\underline{g}, n, \varepsilon)$-open covers $\mathcal{U}$. On the other hand, by Proposition 22 the right hand side above is actually a true limit. Thus, for any $t \in \mathbb{R}$ we have that

$$P_{\mathrm{top}}((G, G_1), t\varphi, X) = \lim_{n \to \infty} \frac{1}{n} \log \left(\frac{1}{m^n} \sum_{\underline{g} \in G_n^*} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{tS_{\underline{g}}\varphi(U)} \right), \tag{14}$$

where the infimum is taken over all $(\underline{g}, n, \varepsilon)$-covers $\mathcal{U}$ for any $0 < \varepsilon < \min\{\delta^*, \varepsilon_0\}$. It means that the map $t \mapsto P_{\mathrm{top}}((G, G_1), t\varphi, X)$ is a pointwise limit of real analytic functions. We claim that the convergence is indeed uniform. To prove this we will prove that the sequence of real functions $(P_n(t\varphi))_{n \geq 1}$ defined by

$$t \mapsto P_n(t\varphi) := \frac{1}{n} \log C_n((G, G_1), t\varphi, \varepsilon)$$

where

$$C_n((G, G_1), t\varphi, \varepsilon) = \frac{1}{m^n} \sum_{\underline{g} \in G_n^*} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} e^{tS_{\underline{g}}\varphi(U)}$$

is equicontinuous in compact intervals, i.e., given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then $|P_n(t_1\varphi) - P_n(t_2\varphi)| < \varepsilon$, for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ be fixed and take

$0<\delta<\varepsilon/\|\varphi\|$. Given t_1, t_2 arbitrary with $|t_1-t_2|<\delta$ it holds that

$$\begin{aligned}
|P_n(t_1\varphi)-P_n(t_2\varphi)| &= \frac{1}{n}\log\left[\frac{\sum_{\underline{g}\in G_n^*}\inf_{\mathcal{U}}\left\{\sum_{U\in\mathcal{U}} e^{t_2 S_{\underline{g}}\varphi(U)}\right\}}{\sum_{\underline{g}\in G_n^*}\inf_{\mathcal{U}}\left\{\sum_{U\in\mathcal{U}} e^{t_1 S_{\underline{g}}\varphi(U)}\right\}}\right] \\
&\le \frac{1}{n}\log\left[\frac{e^{n\delta\|\varphi\|}\sum_{\underline{g}\in G_n^*}\inf_{\mathcal{U}}\left\{\sum_{U\in\mathcal{U}} e^{t_1 S_{\underline{g}}\varphi(U)}\right\}}{\sum_{\underline{g}\in G_n^*}\inf_{\mathcal{U}}\left\{\sum_{U\in\mathcal{U}} e^{t_1 S_{\underline{g}}\varphi(U)}\right\}}\right] \\
&= \delta\|\varphi\|<\varepsilon.
\end{aligned}$$

Hence the sequence is equicontinuous. Since $(P_n(t\varphi))_{n\in\mathbb{N}}$ converges pointwise, we have that the sequence converges uniformly on compact intervals and so $t\mapsto P_{\rm top}((G,G_1),t\varphi,X)$ is a continuous function. Furthermore, for any $n\in\mathbb{N}$ the function $t\mapsto P_n(\varphi+t\psi)$ is differentiable and

$$\left|\frac{dP_n(\varphi+t\psi)}{dt}\right| = \frac{1}{C_n((G,G_1),t\varphi,\varepsilon)}\frac{1}{n}\Big(\frac{1}{m^n}\sum_{\underline{g}\in G_n^*}\inf_{\mathcal{U}}\Big\{\sum_{U\in\mathcal{U}} S_{\underline{g}}\psi(U)e^{S_{\underline{g}}(\varphi+t\psi)(U)}\Big\}\Big)$$

is bounded from above by $\|\psi\|$ (here the infimum is taken over all $(\underline{g},n,\varepsilon)$-covers $\mathcal{U}$ as in (14)). This proves property (3). Moreover, by the mean value inequality

$$|P_n(\varphi)-P_n(\psi)| \le \sup_{0\le t\le 1}\left|\frac{dP_n(\varphi+t(\psi-\varphi))}{dt}\right| \le \|\varphi-\psi\|.$$

Taking $n\to\infty$ we get that $|P_{\rm top}((G,G_1),\varphi,X)-P_{\rm top}((G,G_1),\psi,X)|\le\|\varphi-\psi\|$ and so the pressure function $P_{\rm top}((G,G_1),\cdot,X)$ acting on the space of potentials with bounded distortion is Lipschitz continuous with Lipschitz constant equal to one. This proves property (2). The later implies that $t\mapsto P_{\rm top}((G,G_1),t\varphi,X)$ is Lebesgue-almost everywhere differentiable, which concludes the proof of the theorem. □

C. Topological entropy and growth rate of periodic points

In the remaining of this section we prove that the topological entropy is a lower bound for the exponential growth rate of periodic points for semigroup of expanding maps. Clearly the theorems of the previous section apply to the topological entropy since it corresponds to the constant to zero potential.

Theorem 28. *Let G be the semigroup generated by a set $G_1=\{g_1,\dots,g_k\}$ of uniformly expanding maps on a Riemannian manifold X. Then:*

(a) G satisfies the periodic orbital specification property,

(b) periodic points $Per(G)$ are dense in X, and

(c) the mean growth of periodic points is bounded from below as

$$0<h_{top}((G,G_1),X)\le\limsup_{n\to\infty}\frac{1}{n}\log\Big(\frac{1}{m^n}\sum_{\underline{g}\in G_n^*}\sharp Fix(\underline{g})\Big).$$

Proof. Take $n\ge1$ arbitrary and fixed. It follows from Lemmas 17 and 18 that there exists $\delta_0>0$ satisfying: for any $0<\delta\le\delta_0$ there exists a uniform $N(\delta)\ge1$ so that for any $x\in X$, any $\underline{g}_n\in G_n$ and $\underline{g}_N\in G_N^*$ with $N\ge N(\delta)$ it holds

$$\underline{g}_N(\underline{g}_n(B(x,g,\delta)))=X.$$

Consider $\delta > 0$, $x_1, x_2, \ldots, x_k \in X$, natural numbers $n_1, n_2, \ldots, n_k$ and group elements $\underline{g}_{n_j,j} = g_{i_{n_j},j} \ldots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ $(j = 1 \ldots k)$ be given and let us prove that G satisfies the periodic orbital specification property, that is, there exists a periodic orbit shadowing the previously defined pieces of orbit. For that let us define $x_{k+1} = x_1$ and $\underline{g}_{n_{k+1}} = \underline{g}_1 \in G_{n_1}$.

By the proof of Theorem 16, there exists $p(\delta) \geq 1$ so that for any $p_1, \ldots, p_k \geq p(\delta)$, for $\underline{h}_{p_j} \in G^*_{p_j}$ we have that $\underline{h}_{p_i}(B(\underline{g}_{n_i}(x_i), \delta)) = X$. Hence, there is a well defined inverse branch (which we denote by $\underline{g}_{n_i}^{-1}\underline{h}_{p_i}^{-1}$ for simplicity) so that

$$\underline{g}_{n_i}^{-1}\underline{h}_{p_i}^{-1}(B(x_{i+1}, g_{n_{i+1}}, \delta)) \subset B(x_i, g_{n_i}, \delta)$$

and $\underline{g}_{n_i}^{-1}\underline{h}_{p_i}^{-1} \mid_{B(x_{i+1},g,\delta)}$ is a contraction. Since, $B(x_{k+1}, g_{n_{k+1}}, \delta) = B(x_1, g_{n_1}, \delta)$,

$$\underline{g}_{n_1}^{-1}\underline{h}_{p_1}^{-1} \ldots \underline{g}_{n_k}^{-1}\underline{h}_{p_k}^{-1}(B(x_{k+1}, g_{n_{k+1}}, \delta)) \subset B(x_1, g_{n_1}, \delta)$$

and the composition $\underline{g}_{n_1}^{-1}\underline{h}_{p_1}^{-1} \ldots \underline{g}_{n_k}^{-1}\underline{h}_{p_k}^{-1}$ is a uniform contraction, then there exists a unique repelling fixed point for $\underline{h}_{p_k}\underline{g}_{n_k} \ldots \underline{h}_{p_1}\underline{g}_{n_1}$ in the dynamical ball $B(x_1, g_{n_1}, \delta)$. By construction, the fixed point for $\underline{h}_{p_k}\underline{g}_{n_k} \ldots \underline{h}_{p_1}\underline{g}_{n_1}$ shadows the specified pieces of orbits. This proves that G satisfies the periodic orbital specification property in (a). Clearly (b) is a consequence of the first claim (a).

Now, take $\underline{g} \in G^*_n$ and observe that for any maximal $(\underline{g}, n, 2\delta)$-separated set E, the dynamical balls $\{B(x, \underline{g}, \delta) : x \in E\}$ form a pairwise disjoint collection. Let $p(\delta)$ be given by the previous periodic orbital specification property. For any arbitrary $\underline{k} \in G^*_{n+p(\delta)}$ one can write $\underline{k} = \underline{h}_g\, \underline{g}$ for $\underline{g} \in G^*_n$ and $\underline{h}_g \in G^*_{p(\delta)}$. Notice that, proceeding as before,

$$\underline{k}(B(x, \underline{g}, \delta)) = \underline{h}_g(B(\underline{g}(x), \delta)) = X$$

for every $x \in E$ and so there is a unique fixed point for $\underline{k}$ on the dynamical ball $B(x, \underline{g}, \delta)$. This yields $\text{Fix}(\underline{k}) \geq \sharp E$ and so

$$\sum_{|\underline{k}|=n+p(\delta)} \sharp\text{Fix}(\underline{k}) \geq \sum_{|\underline{g}|=n} \sharp\text{Fix}(\underline{h}_g\, \underline{g}) \geq \sum_{|\underline{g}|=n} s(\underline{g}, n, 2\delta).$$

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log\Big(\frac{1}{m^n} \sum_{|\underline{k}|=n} \sharp\text{Fix}(\underline{k})\Big) &= \limsup_{n\to\infty} \frac{1}{n} \log\Big(\frac{1}{m^{n+p(\delta)}} \sum_{|\underline{k}|=n+p(\delta)} \sharp\text{Fix}(\underline{k})\Big) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log\Big(\frac{1}{m^n} \sum_{|\underline{k}|=n+p(\delta)} \sharp\text{Fix}(\underline{k})\Big) \\
&\geq \limsup_{n\to\infty} \frac{1}{n} \log\Big(\frac{1}{m^n} \sum_{|\underline{g}|=n} s(\underline{g}, n, 2\delta)\Big).
\end{aligned}$$

Taking $\delta \to 0$ in the left hand side the previous inequality and recalling Theorem 14 this proves (c) and finishes the proof of the theorem. $\square$

Some comments are in order. Firstly it is not hard to check that an analogous result holds for the notion of entropy $h((G, G_1), X)$, leading to

$$h((G, G_1), X) \leq \limsup_{n\to\infty} \frac{1}{n} \log \sharp Per(G_n).$$

Secondly, since any expanding map satisfies the periodic specification property then periodic measures are dense in the space of invariant probability measures (see e.g.[17] (Proposition 21.8)). Hence, given a finitely generated semigroup of expanding maps G it is clear that whenever the set $\mathcal{M}(G)$ of probability measures invariant by every element $g \in G$ is non-empty then the set of periodic measures

$$\mathcal{P}_{per}(G) = \bigcup_{n\geq 1} \bigcup_{\underline{g}\in G_n} \Big\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\underline{g}_j(x)} : x \in Fix(\underline{g}) \Big\}$$

is dense in the set of probability measures $\mathcal{M}(G)$. Finally, weighted versions of the previous theorem for potentials with bounded distortion are also very likely to hold.

V. APPLICATIONS

In this section we provide some classes of examples of semigroup actions that combine hyperbolicity and specification properties. We also provide some examples for which while we compare the notions of topological entropy used here with some others previously introduced and available in the literature, and discuss the relation between entropy, periodic points and specification properties.

The following example illustrates that in the notion of specification some 'linear independence condition' on the set of generators must be assumed in order to obtain that the group has the specification property.

Example 29. *Consider the integer valued matrix*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \tag{15}$$

which induces a linear (topologically mixing) Anosov f_A on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ that satisfies the specification property. Hence, the $\mathbb{Z}$ action $\mathbb{Z} \times \mathbb{T}^2 \to \mathbb{T}^2$ given by $(n, x) \mapsto f_A^n(x)$ satisfies the specification property.

Now, take $B = A^{-2} \in SL(2, \mathbb{Z})$ which also induces a linear Anosov f_B on the torus and satisfies the specification property. Nevertheless, the $\mathbb{Z}^2$-action $\mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ given by $((m, n), x) \mapsto f_A^m(f_B^n(x)) = f_A^{m-2n}(x)$ clearly does not satisfy the specification property because every element in the (unbounded) subgroup $\{(2n, n) : n \in \mathbb{Z}\} \subset \mathbb{Z}^2$ induces the identity map. This indicates that generators should be taken in an irreducible way, that is, that there are $n_1, ..., n_k \in \mathbb{Z}$ not all simultaneously zero so that $g_1^{n_1} ... g_k^{n_k} = Id_G$.

The next modification of the previous example illustrates that the irreducibility of the generators in the sense that two generators A and B satisfy $A^m B^n \neq Id$ for all $m, n \in \mathbb{Z}$ not simultaneously zero is not the unique obstruction.

Example 30. *Let A, B be the two matrices in $SL(4, \mathbb{Z})$ given by*

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{I}_2 & \mathcal{A} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix}, \quad \text{where} \quad \mathcal{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}),$$

$\mathcal{I}_2 \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ denotes the identity matrix and $0 \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ is the null matrix. It is not difficult to see that A and B are hyperbolic matrices (hence the diffeomorphisms induced by A and B satisfy the specification property), these commute but $B \neq A^m$ for all $m \in \mathbb{Z}$. Consider the $\mathbb{Z}^2$-action $T : \mathbb{Z}^2 \times \mathbb{T}^4 \to \mathbb{T}^4$ of $\mathbb{Z}^2$ on the torus $\mathbb{T}^4$ defined by $((m, n), x) \mapsto A^m B^n(x)$. Since the element

$$A^{-1}B = \begin{pmatrix} \mathcal{I}_2 & 0 \\ \mathcal{I}_2 & \mathcal{I}_2 \end{pmatrix}$$

does not satisfy the specification property one can deduce from Lemma 2 that this group action does not satisfy the specification property. Similarly, it is not hard to check that this group action does not satisfy neither of the orbital specification properties.

It follows from the discussion on the previous section that C^1-robust specification property implies that the corresponding generators are uniformly hyperbolic and, in particular, the action is structurally stable. Our twofold purpose in the next example is: (i) to exhibit broad families of non-hyperbolic smooth maps that satisfy orbital specification properties although generators do not necessarily have the specification property; (ii) present examples where the weak orbital specification property holds while the strong orbital property does not.

Example 31. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-expanding map of the circle and $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ be the rotation of angle α. Let G be the semigroup generated by $G_1 = \{id, f, R_\alpha\}$. This example can be modified for the semigroup G to be free (e.g. by taking a irrational rotation and an expanding map with trivial centralizer c.f. discussion in the Example 32).*

Claim 1: The action induced by the semigroup G on the unit circle $\mathbb{S}^1$ does not satisfy the strong orbital specification property.

Proof of Claim 1. Take $\delta > 0$ and $x_1, \neq x_2$ in the circle, $n_1 = n_2 = n \geq 1$ and the maps $\underline{g}_{n_1} = f^{n_1}$ and $\underline{g}_{n_2} = f^{n_2}$. For any $p \geq 1$ take $\underline{h}_p = R_\alpha^p = R_{\alpha p}$ the rotation of angle αp. If n is large then the dynamical balls $B_f(x_1, n_1, \delta)$ and $B_f(x_2, n_2, \delta)$ are disjoint and small. In particular, there exists $p \geq 1$ so that $\underline{h}_p(B_f(x_1, n_1, \delta)) \cap B_f(x_2, n_2, \delta) = \emptyset$. In particular the semigoup action G on $\mathbb{S}^1$ does not satisfy the strong specification orbital property. $\square$

Claim 2: The action induced by the semigroup G on the unit circle $\mathbb{S}^1$ satisfies the weak orbital specification property.

Proof of Claim 2. Since f is C^1-expanding, by the proof of Lemma 17, there exists $\delta_0 > 0$ so that for any $0 < \delta \leq \delta_0$, any $x \in X$ and any $n \in \mathbb{N}$ it follows that $f^n(B_f(x, n, \delta)) = B(f^n(x), \delta)$. Moreover, there exists $N = N(\delta) \geq 1$ so that any ball of radius δ is mapped onto $\mathbb{S}^1$ by f^N. We can now prove the claim. Given $\delta > 0$ take $p(\delta) = N(\delta) \geq 1$. For any $p \geq p(\delta)$ let $\tilde{G}_p \subset G_p^*$ denote the set of elements $\underline{h}_p \in G_p^*$ for which the following holds: given arbitrary points $x_1, \dots, x_k \in X$, any positive integers $n_1, \dots, n_k \geq 1$, any elements $\underline{g}_{n_j,j} = g_{i_{n_j},j} \dots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ and any elements $h_{p_j} \in \tilde{G}_{p_j}$ with $p_j \geq p(\delta)$ there exists $x \in X$ so that $d(\underline{g}_{\ell,1}(x)\ ,\ \underline{g}_{\ell,1}(x_1)) < \delta$ for every $\ell = 1 \dots n_1$ and

$$d(\ \underline{g}_{\ell,j}\, \underline{h}_{p_{j-1}}\ \cdots\ \underline{g}_{n_2,2}\, \underline{h}_{p_1}\, \underline{g}_{n_1,1}(x)\ ,\ \underline{g}_{\ell,j}(x_j)\ ) < \delta$$

for every $j = 2 \dots k$ and $\ell = 1 \dots n_j$. We claim that $\lim_{p\to+\infty} |\tilde{G}_p|/|G_p^*| = 1$. We notice that $\underline{g}_{n_j,j}(B(x, \underline{g}_{n_j,j}, \delta)) = B(\underline{g}_{n_j,j}(x), \delta)$ is a ball of radius δ for any $1 \leq j \leq k$. So, if the expanding map is f is combined at least $p(\delta)$ times in any way in the words $\underline{h}_p$ we get $\underline{h}_p(B(y, \delta)) = \mathbb{S}^1$ for any y which clearly implies that $\underline{h}_p \in \tilde{G}_p$. Thus for any $p \geq p(\delta)$

$$G_p^* \setminus \tilde{G}_p \subset \Big\{ \underline{h}_p = h_{i_p} \dots h_{i_2} h_{i_1} \in G_p : \sharp\{1 \leq j \leq p : h_{i_j} = f\} < p(\delta) \Big\}.$$

Clearly, for any $0 < \gamma < 1$

$$\frac{|G_p^* \setminus \tilde{G}_p|}{2^{\gamma p}} \leq 2^{-\gamma p} \sum_{k=0}^{p(\delta)-1} \left(\begin{array}{c} p \\ k \end{array} \right) \leq p(\delta)\, 2^{-\gamma p}\, p^{p(\delta)} \to 0 \tag{16}$$

as p tends to infinity, which proves our claim. $\square$

Since the assumption (H) in Theorem 15 is a direct consequence of the previous equation (16) *then we deduce that this semigroup action has positive topological entropy.*

Clearly we can modify the previous strategy to deal with other different kind of semigroups with more generators. Our next purpose is to provide an example of a semigroup with exponential growth that is not a free semigroup but still satisfy the assumptions of Theorem 27.

Example 32. *Let $X = \mathbb{S}^1$ be the circle and consider the expanding maps on S^1 given by $g_1(x) = 2x \pmod 1$, that $g_2(x) = 3x \pmod 1$. It is clear that these maps comute (that is, $g_1 \circ g_2 = g_2 \circ g_1$) and that $g_1^k \neq g_2^\ell$ for every $k, \ell \in \mathbb{Z}_+$ (since 2 and 3 are relatively prime). Now, consider another C^1-expanding map g_3 such that its centralizer $Z(g_3)$ is trivial, meaning*

$$Z(g_3) := \{h : \mathbb{S}^1 \to \mathbb{S}^1 \text{ expanding } : h \circ g_3 = g_3 \circ h\} = \{g_3^\ell \colon \ell \in \mathbb{Z}_+\}.$$

In particular the subgroup generated by g_2 and g_1 is disjoint from $Z(g_3)$. In other words, $g_3 \circ g_2^\ell \circ g_1^k \neq g_2^\ell \circ g_1^k \circ g_3$ for every $\ell, k \in \mathbb{Z}_+$. The existence of such g_3 is garanteed by[1]. Let G be the semigroup of expanding maps with generators $G_1 = \{g_1, g_2, g_3\}$. By construction, the subgroup $\tilde{G}$ of G generated by $\tilde{G}_1 = \{g_1, g_3\}$ is a free semigroup then

$$\lim_{n\to\infty} \frac{1}{n} \log |G_n| \geq \log 2 > 1$$

and the semigroup has exponential growth. Since the generators do not have finite order then any elements $\underline{g} \in G_n$ is a concatenation $\underline{g} = g_{i_n} \dots g_{i_1}$ with $g_{i_j} \in G_1$. By commutativity, all concatenations of j elements g_1 and k elements g_2 coincide with the expanding map $g_1^j\, g_2^k$ and consequently there are exactly $n+1$ elements in G_n obtained as concatenations of the elements g_1 and g_2. This semigroup has exponential growth and is not abelian but still satisfies the conditions of Theorem 27 for every Hölder continuous potential $\varphi : X \to \mathbb{R}$ and, in particular, the pressure function $t \mapsto P_{top}((G, G_1), t\varphi, X)$ is differentiable Lebesgue-almost everywhere.

In what follows we shall provide a simple example of a $\mathbb{Z}^d$-semigroup action where we can already discuss the relation between the notion of topological entropy that we introduced in comparison with some of the previous ones. We focus on the case of semigroups of expanding maps for simplicity of computations while we notice that an example of actions of total automorphisms as considered in Example 29 could be constructed analogously.

Example 33. *Let $X = \mathbb{S}^1$ be the circle and the $\mathbb{Z}^3$-group action $T : \mathbb{Z}^3 \times \mathbb{S}^1 \to \mathbb{S}^1$ defined by $((m, n, k), x) \mapsto g_1^m g_2^n g_3^k(x)$, where $g_1(x) = 2x \pmod 1$, $g_2(x) = 3x \pmod 1$ and $g_3(x) = 5x \pmod 1$ are commuting expanding maps of the circle. By commutativity and the fact that the numbers $2, 3, 5$ are relatively prime it is easy to check that $|G_n| = (n+1)(n+2)/2$. First we shall compute the topological pressure as considered by Bis in[3]. If $s(n, \delta)$ denotes the number of (n, δ)-separated sets by G the topological entropy in[3] is defined by*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{|G_{n-1}|} \log s(n, \delta). \tag{17}$$

In our context, for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{1}{|G_{n-1}|} \log s(n, \delta) \leq \limsup_{n\to\infty} \frac{2}{n^2} \log(5^n) = 0$$

proving that the entropy in (17) is zero. For the sake of completeness let us mention that it is remarked in[3] that having positive topological entropy with this definition does not depend on the generators. Ruelle[33] considered a slightly different but similar notion of topological entropy but that does coincide with (17) in this context.

Let us now proceed to compute the notion of topological entropy considered by Ghys, Langevin, Walczak [21] and Bis[4]. According to their definition entropy is computed as

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \delta) = \log 5$$

and it measures the maximal *entropy rate in the semigroup. Finally we observe that it follows from[12] that the topological entropy of the semigroup action, according to Definition 12, in the case the generators are expanding is given by*

$$h_{top}((G, G_1), X) = \log\Big(\frac{\deg g_1 + \deg g_2 + \deg g_3}{3}\Big) = \log\Big(\frac{10}{3}\Big) > 0.$$

Finally let us mention that this semigroup action satisfies the stron orbital specification properties and, consequently, it follows from Theorems 11 and 13 that every point in the circle is an entropy point with respect to both entropy notions.

Acknowledgements

F.G. is supported by BREUDS and P.V. is supported by a postdoctoral fellowship by CNPq-Brazil and are grateful to Faculdade de Ciências da Universidade do Porto for the excellent research conditions.

[1] Carlos Arteaga. Centralizers of expanding maps on the circle. *Proc. Amer. Math. Soc.*, 114(1):263–267, 1992.
[2] Walter Bauer and Karl Sigmund. Topological dynamics of transformations induced on the space of probability measures. *Monatsh. Math.*, 79:81–92, 1975.
[3] Andrzej Biś. Partial variational principle for finitely generated groups of polynomial growth and some foliated spaces. *Colloq. Math.*, 110(2):431–449, 2008.
[4] Andrzej Biś. An analogue of the variational principle for group and pseudogroup actions. *Ann. Inst. Fourier (Grenoble)*, 63(3):839–863, 2013.
[5] Andrzej Biś and Mariusz Urbański. Some remarks on topological entropy of a semigroup of continuous maps. *Cubo*, 8(2):63–71, 2006.
[6] Andrzej Biś and Paweł Walczak. Entropy of distal groups, pseudogroups, foliations and laminations. *Ann. Polon. Math.*, 100(1):45–54, 2011.
[7] A. M. Blokh. Decomposition of dynamical systems on an interval. *Uspekhi Mat. Nauk*, 38(5(233)):179–180, 1983.
[8] Rufus Bowen. Periodic points and measures for Axiom *A* diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.
[9] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.
[10] Rufus Bowen and David Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29(3):181–202, 1975.
[11] A. Bufetov. Topological entropy of free semigroup actions and skew-product transformations. *J. Dynam. Control Systems*, 5(1):137–143, 1999.
[12] A. I. Bufetov. Ergodic theorems for actions of several mappings. *Uspekhi Mat. Nauk*, 54(4(328)):159–160, 1999.
[13] Grant Cairns, Alla Kolganova, and Anthony Nielsen. Topological transitivity and mixing notions for group actions. *Rocky Mountain J. Math.*, 37(2):371–397, 2007.
[14] Nhan-Phu Chung and Hanfeng Li. Homoclinic groups, IE groups, and expansive algebraic actions. *Invent. Math.*, 199(3):805–858, 2015.
[15] Maria Pires de Carvalho, Fagner B. Rodrigues, and Paulo Varandas. Semigroup actions of expanding maps. *preprint*.
[16] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
[17] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.
[18] A. Eizenberg, Y. Kifer, and B. Weiss. Large deviations for $\mathbf{Z}^d$-actions. *Comm. Math. Phys.*, 164(3):433–454, 1994.
[19] A. Yu. Fishkin. An analogue of the Misiurewicz-Przytycki theorem for some mappings. *Uspekhi Mat. Nauk*, 56(1(337)):183–184, 2001.
[20] Shmuel Friedland. Entropy of graphs, semigroups and groups. In *Ergodic theory of* $\mathbf{Z}^d$ *actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 319–343. Cambridge Univ. Press, Cambridge, 1996.
[21] É. Ghys, R. Langevin, and P. Walczak. Entropie géométrique des feuilletages. *Acta Math.*, 160(1-2):105–142, 1988.
[22] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.
[23] R. I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.
[24] P. R. Grossi Sad. A $\mathbf{Z} \times \mathbf{Z}$ structurally stable action. *Trans. Amer. Math. Soc.*, 260(2):515–525, 1980.
[25] Anatole Katok. Fifty years of entropy in dynamics: 1958–2007. *J. Mod. Dyn.*, 1(4):545–596, 2007.
[26] Piotr Kościelniak and Marcin Mazur. Chaos and the shadowing property. *Topology Appl.*, 154(13):2553–2557, 2007.

[27]Douglas Lind and Klaus Schmidt. Symbolic and algebraic dynamical systems. In *Handbook of dynamical systems, Vol. 1A*, pages 765–812. North-Holland, Amsterdam, 2002.
[28]Dongkui Ma and Min Wu. Topological pressure and topological entropy of a semigroup of maps. *Discrete Contin. Dyn. Syst.*, 31(2):545–557, 2011.
[29]R. Miles and T Ward. Directional uniformities, periodic points and entropy. *preprint, http://arxiv.org/abs/1411.5295*.
[30]Krerley Oliveira and Xueting Tian. Non-uniform hyperbolicity and non-uniform specification. *Trans. Amer. Math. Soc.*, 365(8):4371–4392, 2013.
[31]C.-E. Pfister and W. G. Sullivan. Large deviations estimates for dynamical systems without the specification property. Applications to the β-shifts. *Nonlinearity*, 18(1):237–261, 2005.
[32]D. Ruelle and Ya. G. Sinaĭ. From dynamical systems to statistical mechanics and back. *Phys. A*, 140(1-2):1–8, 1986. Statphys 16 (Boston, Mass., 1986).
[33]David Ruelle. Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.*, 187:237–251, 1973.
[34]Kazuhiro Sakai, Naoya Sumi, and Kenichiro Yamamoto. Diffeomorphisms satisfying the specification property. *Proc. Amer. Math. Soc.*, 138(1):315–321, 2010.
[35]F. M. Schneider. Topological entropy of continuous actions of compactly generated groups. *preprint, http://arxiv.org/abs/1502.03980*.
[36]Karl Sigmund. On dynamical systems with the specification property. *Trans. Amer. Math. Soc.*, 190:285–299, 1974.
[37]Ja. G. Sinaĭ. Gibbs measures in ergodic theory. *Uspehi Mat. Nauk*, 27(4(166)):21–64, 1972.
[38]Daniel J. Thompson. Irregular sets, the β-transformation and the almost specification property. *Trans. Amer. Math. Soc.*, 364(10):5395–5414, 2012.
[39]Paulo Varandas. Non-uniform specification and large deviations for weak Gibbs measures. *J. Stat. Phys.*, 146(2):330–358, 2012.
[40]Paulo Varandas, N. Sumi, and Kenichiro Yamamoto. Partial hyperbolicity and specification. *Proc. Amer. Math. Soc.*, 2015,to appear.
[41]Paulo Varandas, N. Sumi, and Kenichiro Yamamoto. Specification and partial hyperbolicity for flows. *Dyn. Syst.*, 2015,to appear.
[42]Kenichiro Yamamoto. On the weaker forms of the specification property and their applications. *Proc. Amer. Math. Soc.*, 137(11):3807–3814, 2009.
[43]Dongmei Zheng and Ercai Chen. Bowen entropy for action of amenable groups. *preprint, http://arxiv.org/abs/1410.4645*.