

SRB MEASURES FOR PARTIALLY HYPERBOLIC ATTRACTORS OF ENDOMORPHISMS

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ABSTRACT. In the present paper we contribute to the thermodynamic formalism of partially hyperbolic attractors for non-singular endomorphisms admitting an invariant stable bundle and a positively invariant cone field with non-uniform cone expansion at a positive Lebesgue measure set of points. These include the case of attractors for Axiom A endomorphisms and partially hyperbolic endomorphisms derived from Anosov. We prove these attractors have finitely many SRB measures, that these are hyperbolic, and that the SRB measure is unique provided the dynamics is transitive. Moreover, we show that the SRB measures are statistically stable (in the weak* topology) and that their entropy varies continuously with respect to the endomorphism.

1. INTRODUCTION

The main goal of the use of ergodic theory in dynamical systems is to describe the statistical properties of the dynamics using invariant measures. In particular, the thermodynamic formalism aims the construction and description of the statistical properties of invariant measures that are physically relevant, meaning equilibrium states with respect to some potential. Sinai, Ruelle and Bowen ([44, 41, 9]) constructed a special class of invariant measures for hyperbolic attractors of diffeomorphisms f acting on a compact Riemannian manifold M : these have absolutely continuous disintegration with respect to the Lebesgue measure along unstable manifolds and are Gibbs equilibrium states for the geometric potential $\phi^u(x) = -\log |\det Df(x)|_{E_x^u}|$. Such measures are known as SRB (Sinai-Ruelle-Bowen) measures. Furthermore, the latter measures are physical, meaning their ergodic basin of attraction $B(\mu) = \{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \rightarrow \mu \text{ as } n \rightarrow \infty\}$ has positive Lebesgue measure in the ambient space. Modern methods for the construction of SRB measures for hyperbolic attractors and the study of their finer properties can be found in [7, 20] and references therein.

Since SRB measures were introduced in the realm of dynamical systems, their construction and the study of their statistical properties rapidly became a topic of interest of the mathematics and physics communities. Unfortunately, apart from the uniformly hyperbolic setting, where the existence of finite Markov partitions allows to semiconjugate the dynamics to subshifts of finite type (cf. [44, 41, 9]) there is no systematic approach for the construction of SRB measures. Note that SRB measures often coincide with equilibrium states for a geometric potential (cf. [22]). As for the construction of SRB measures beyond the setting of uniform hyperbolicity one should mention the construction of u-Gibbs measures for partially hyperbolic attractors that admit an unstable bundle by Pesin and Sinai [34], the existence and uniqueness of the SRB measure for the class of robustly transitive diffeomorphisms derived from Anosov constructed by Mañé by Carvalho [12], and the construction of SRB measures for $C^{1+\alpha}$ partially hyperbolic diffeomorphisms displaying a non-uniform hyperbolicity condition along the central direction by Alves, Bonatti and Viana [1, 11]. After Young's [51] axiomatic construction for studying decay of correlations, the decay of correlations for SRB measures can be studied through the existence of Markov towers and was shown to depend on the Lebesgue measure of the tails associated to non-uniform hyperbolicity. Some contributions on the study of the decay of correlations for the SRB measures of related classes of uniformly hyperbolic and partially hyperbolic diffeomorphisms include [28, 19, 14]. For other methods for the construction of SRB measures we refer the reader to [17, 16] and references therein. Moreover, the statistical stability of SRB measures in dynamical systems and the continuous dependence of the entropy of the SRB measures has been considered in [17, 18, 2, 50, 15, 13], among others.

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In parallel to the developments of partially hyperbolic diffeomorphisms, there have been important contributions to the study of (non-singular) endomorphisms. In the case that the endomorphisms admit no stable bundle, the geometrical constructions of Markov structures for endomorphisms with non-uniform expansion due to [3] provide weak conditions for the existence of SRB measures and estimates on their decay of correlations. The situation is substantially more complicated in the case of endomorphisms displaying contracting behavior, as e.g. strongly dissipative endomorphisms that arise in the context of bifurcation of homoclinic tangencies associated to periodic points of diffeomorphisms (see [33] and references therein). In the mid seventies, Przytycki [35] extended the notion of uniform hyperbolicity to the context of endomorphisms and studied Anosov endomorphisms. Here, due to the non-invertibility of the dynamics, the existence of an invariant unstable subbundle for uniformly hyperbolic basic pieces needs to be replaced by the existence of positively invariant cone-fields on which vectors are uniformly expanded by all positive iterates (we refer the reader to Subsection 4.3 for more details). In [38, 39] the authors constructed SRB measures for Axiom A attractors of endomorphisms, obtaining these as equilibrium states for the geometric potential (defined by means of the natural extension). A characterization of SRB measures for uniformly hyperbolic endomorphisms can also be given in terms of dimensional characteristics of the stable manifold ([47]). The thermodynamic formalism of hyperbolic basic pieces for endomorphisms had the contribution of Mihailescu and Urbanski [31, 32] that introduced and constructed inverse SRB measures for hyperbolic attractors of endomorphisms. Among the difficulties that arise when dealing with non-invertible hyperbolic dynamics one should refer the absence of contraction along inverse branches and the fact that unstable manifolds depend on entire pre-orbits, thus unstable manifolds may have a complicated geometrical structure (cf. [37, 47]).

The ergodic theory of partially hyperbolic endomorphisms is more incomplete. In the context of surface endomorphisms, a major contribution is due to Tsujii [46], which proved that for $r \geq 19$, C^r -generic partially hyperbolic endomorphisms with an unstable cone field admit finitely many SRB measures whose ergodic basins of attraction cover Lebesgue almost every point in the manifold (the regularity can be dropped to $r \geq 2$ on the space of non-singular endomorphisms). While it remains unknown if these SRB measures are (generically) hyperbolic, an extension to higher dimension should present significant difficulties as discussed by the author (cf. [46, page 43]). The strategy developed by Tsujii for surface endomorphisms was more recently adapted to deal with open sets of three-dimensional partially hyperbolic and dynamically coherent diffeomorphisms (see [8] for the precise statement).

Our purpose here is to contribute to the ergodic theory of partially hyperbolic attractors for endomorphisms and to present open conditions under which SRB measures exist (actually are hyperbolic and physical measures) and are statistically stable. The open class of partially hyperbolic attractors considered here is complementary to the one studied in [46] (see Section 8 for a discussion on the different assumptions of partial hyperbolicity). We consider attractors for $C^{1+\alpha}$ ($\alpha > 0$) non-singular endomorphisms on compact Riemannian manifolds that admit uniform contraction along a well defined invariant stable subbundle and preserve a positively invariant cone-field. Under a non-uniform cone-expansion property on a positive Lebesgue measure set of points, we prove the endomorphism admits an hyperbolic SRB measure. Moreover, if Lebesgue almost every point satisfies the latter condition then there are finitely many SRB measures (it is unique if the attractor is transitive) whose basins cover Lebesgue almost every point in the topological basin of the attractor. In the trivial case that the subbundle E^s is trivial we recover results obtained in [1] for non-uniform expanding maps. In the case the subbundle E^s is non-trivial and there exists uniform expansion along the cone direction we recover the setting of Axiom A attractors for endomorphisms considered in [37]. One should remark that, while SRB measures are measures that admit absolutely continuous disintegrations along Pesin's unstable foliation, the simple existence of unstable manifolds is not a direct consequence of our assumptions. For that reason, some of the main difficulties arise from the fact that the construction of unstable manifolds (obtained by Pesin's unstable manifold theorem, provided a hyperbolic invariant measure) is parallel to the construction of the SRB measure. For that reason the construction of the SRB measures, as accumulation points of Lebesgue measure over disks tangent to the cone field, must be followed closely by a precise track of the accumulation disks in terms of the space of pre-orbits. This is crucial to conclude later the SRB property.

Here we use the geometric ingredients developed in the construction SRB measures to prove that SRB measures depend continuously (with respect to the weak* topology) with perturbations of the endomorphism (see Section 2 for the precise statement). This proof is inspired by [49]. Moreover, the continuous dependence of SRB measures on the dynamics together with Pesin's entropy formula and Oseledets theorem allow us to show that the entropy of the SRB measure is given by an expression involving the stable Jacobian and varies continuously with the endomorphism. To the best of our knowledge it seems these results are new even in

the case of attractors for Axiom A endomorphisms. Finally, we exhibit open conditions that define partially hyperbolic non-singular endomorphisms for which our results apply. These include endomorphisms Derived from Anosov (inspired by the examples of Mañé studied in [12, 1]) obtained from Anosov endomorphisms by means of local bifurcations (e.g. pitchfork or Hopf bifurcations).

This paper is organized as follows. In Section 2 we describe the classes of partially hyperbolic endomorphisms considered here, we state our main results and give a glance on the proofs. In Section 3 we give some applications. In Section 4 we discuss some concepts to the full understanding of our arguments, as the notion of hyperbolicity for endomorphism, the concept of SRB measures, natural extensions and the characterization of hyperbolicity via cone fields. In Section 5 we give the proof of the finitude (uniqueness when the dynamics is transitive) of SRB measures. In Section 6 we prove the statistical stability of the SRB measures, while the proof of the continuity of the entropy of SRB measures appears in Section 7. Finally, in Section 8 we relate our results with others for partially hyperbolic endomorphisms and discuss some possible future directions.

2. MAIN RESULTS

2.1. Setting. Throughout, let M be a compact connected Riemannian d -dimensional manifold. Assume that $f : M \rightarrow M$ is a $C^{1+\alpha}$, $\alpha > 0$, non-singular endomorphism and that $\Lambda \subset M$ a compact positively f -invariant subset. Let $U \supset \Lambda$ be an open set such that $f(\bar{U}) \subset U$ and assume

$$\Lambda = \Lambda_f := \bigcap_{n \geq 0} f^n(\bar{U}),$$

where \bar{U} denotes the closure of the set U . Assume that there are a continuous splitting of the tangent bundle $T_U M = E^s \oplus F$ (where F is not necessarily Df -invariant) and constants $a, c > 0$ and $0 < \lambda < 1$ satisfying:

- (H1) $Df(x) \cdot E_x^s = E_{f(x)}^s$ for every $x \in U$;
- (H2) $\|Df^n(x)|_{E_x^s}\| \leq \lambda^n$ for every $x \in U$ and $n \in \mathbb{N}$;
- (H3) there is $a > 0$ so that the cone field of width $a > 0$ centered at F , $U \ni x \mapsto C_a(x) := \{u + v \in E_x^s \oplus F_x : \|u\| \leq a\|v\|\}$ satisfies $Df(x)(C_a(x)) \subseteq C_a(f(x))$ for every $x \in U$, and there is a positive Lebesgue measure set $H \subset U$ so that:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x))|_{C_a(f^j(x))})^{-1}\| \leq -2c < 0 \quad (2.1)$$

for every $x \in H$ (the expression $\|(Df(f^j(x))|_{C_a(f^j(x))})^{-1}\|$ is made precise at Subsection 4.1);

- (H4) $\|Df(x)v\| \|Df(x)^{-1}w\| \leq \lambda \|v\| \|w\|$ for every $v \in E_x^s$, all $w \in Df(x)(C_a(x))$ and $x \in U$.

Given a non-singular endomorphism as above, we will say that Λ is an *attractor*, E^s is a *uniformly contracting bundle* and that C_a is a *non-uniformly expanding cone field* (or that C_a exhibits non-uniform expansion). We will denote $d = \dim(M)$, $d_s = \dim(E^s)$ and $d_u := \dim(F) = \dim(C_a)$.

2.2. Statements. Our first result concerns the existence and finitude of SRB measures for partially hyperbolic endomorphisms (we refer the reader to Section 4.4.2 for the definition of SRB measures).

Theorem A. *Let f be a $C^{1+\alpha}$ non-singular endomorphism satisfying (H1) - (H4). Then there are finitely many SRB measures for f whose basins of attraction cover H , Lebesgue mod zero. Moreover, these measures are hyperbolic and, if $\text{Leb}(U \setminus H) = 0$ then there are finitely many SRB measures for f in U . Furthermore, if $f|_\Lambda$ is transitive then there is a unique SRB measure for f in U .*

Some comments are in order. SRB measures are ergodic and have absolute continuous disintegration with respect to Lebesgue measure along unstable manifolds (cf. Subsection 4.4.2). It is not hard to check that the existence of the subbundle E^s is a C^1 -open condition, as it is equivalent to the existence of a stable cone field, and that whenever it is trivial then (2.1) corresponds to the usual notion of non-uniform expansion in [1]. Condition (H3) implies on the existence of a disk $D \subset U$ which is tangent to the cone field so that $\text{Leb}_D(H) > 0$ (here Leb_D stands for the Lebesgue measure on D), which is a sufficient condition for the existence of SRB measures. Nevertheless, the full strength of (H3) is used to guarantee the finiteness of the SRB measures whose basin have non-trivial intersection with the set H in Theorem A. In the case of surfaces, an upper bound on the number of SRB measures can be given in terms of the number homoclinic classes (cf. [30]). Let $\mathcal{M}(M)$ denote the space of Borelian probability measures on M endowed with the weak* topology.

Our second result concerns the continuous dependence of the SRB measures in terms of the dynamics, the so called statistical stability of the SRB measures, in the weak* topology. For that, we assume that a family of non-singular endomorphisms satisfy conditions (H1)-(H4) with uniform constants. Denote by $End^{1+\alpha}(M)$ the set of $C^{1+\alpha}$ non-singular endomorphisms on M endowed with the $C^{1+\alpha}$ topology. Assume that Λ is an attractor for f and that $U \supset \Lambda$ an open neighborhood of Λ such that $\bigcap_{n \geq 0} f^n(\overline{U}) = \Lambda$. For every $g \in End^{1+\alpha}(M)$ that is C^1 -close to f consider $\Lambda_g := \bigcap_{n \geq 0} g^n(\overline{U})$. Assume that there are an open neighborhood $\mathcal{V} \subset End^{1+\alpha}(M)$ of f , constants $\lambda \in (0, 1)$, $c > 0$ and a family of cone fields $U \ni x \rightarrow C(x)$ of constant dimension $0 < d_u \leq dim(M)$, such that (H1)-(H4) holds with uniform constants on U : for every $g \in \mathcal{V}$ there exists a continuous splitting $T_U M = E^s(g) \oplus F(g)$ so that

- (H1) $Dg(x) \cdot E_x^s(g) = E_{g(x)}^s(g)$ for every $x \in U$;
- (H2) $\|Dg^n(x)|_{E_x^s(g)}\| \leq \lambda^n$ for every $x \in U$ and $n \in \mathbb{N}$
- (H3) the cone field $U \ni x \rightarrow C(x)$ contains the subspace $F(g)$, and is such that $Dg(x)(C(x)) \subseteq C(g(x))$ for every $x \in U$, and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Dg(g^j(x))|_{C(g^j(x))})^{-1}\| \leq -2c < 0$$

for Lebesgue almost every $x \in U$; and

- (H4) $\|Dg(x) \cdot v\| \|Dg(x)^{-1} \cdot w\| \leq \lambda \|v\| \cdot \|w\|$ for every $v \in E_x^s(g)$, all $w \in Dg(x)(C(x))$ and $x \in U$.

We will say that \mathcal{V} is an *open set of partially hyperbolic $C^{1+\alpha}$ -endomorphisms with uniform constants*. For each $g \in \mathcal{V}$, we denote by H_g the subset of U that consists of points satisfying the non-uniform cone-hyperbolicity condition in (H3).

Theorem B. *Given $\alpha > 0$, let \mathcal{U} be an open set of $C^{1+\alpha}$ non-singular endomorphisms so that every $g \in \mathcal{U}$ satisfies (H1) - (H4) with uniform constants. If $(g_n)_{n \in \mathbb{N}}$ is a sequence in \mathcal{U} converging to $g \in \mathcal{U}$ and, for every $n \in \mathbb{N}$, μ_n is an SRB measure for g_n on Λ_{g_n} then every accumulation point μ of $(\mu_n)_n$ is a convex combination of the SRB measures for g on Λ_g . In particular, if every $g \in \mathcal{U}$ is transitive and μ_g denotes the unique SRB measure for g then*

$$\begin{aligned} \mathcal{U} &\rightarrow \mathcal{M}(\Lambda_g) \subset \mathcal{M}(M) \\ g &\mapsto \mu_g \end{aligned}$$

is continuous with respect to the weak topology.*

Using the statistical stability of SRB measures we also prove the continuous dependence of the entropy function associated to the SRB measure.

Corollary A. *Let \mathcal{U} be a open set of transitive $C^{1+\alpha}$ non-singular endomorphisms that satisfy (H1)-(H4) with uniform constants. If μ_f denotes the unique SRB measure for $f \in \mathcal{U}$ then the entropy function*

$$\begin{aligned} H : \mathcal{U} &\rightarrow \mathbb{R}^+ \\ f &\mapsto h_{\mu_f}(f) \end{aligned}$$

is continuous, where $h_{\mu_f}(f)$ denotes the entropy of μ_f with respect to f .

It is clear that an attractor of an Axiom A endomorphism satisfies the assumptions of Theorems A and B. In particular we obtain the following direct consequences that we did not find in the literature:

Corollary B. *Let f be an $C^{1+\alpha}$ non-singular endomorphism and $U \subset M$ be such that $\Lambda_f = \bigcap_{n \geq 0} f^n(U)$ is a transitive hyperbolic attractor for f . Assume that \mathcal{U} is a $C^{1+\alpha}$ -open neighborhood of f such that for any $U \ni g \mapsto \Lambda_g$ denotes the analytic continuation of the hyperbolic attractor. Then $f|_{\Lambda_f}$ is statistically stable: if $(f_k)_k$ is a sequence of endomorphisms in \mathcal{U} so that $f_k \rightarrow f$ as $k \rightarrow \infty$ (in the $C^{1+\alpha}$ -topology) then $\mu_{f_k} \rightarrow \mu_f$ in the weak* topology.*

In the remaining of this section we exhibit C^1 -open sets of $C^{1+\alpha}$ partially hyperbolic non-singular endomorphisms for which it is checkable that assumptions (H1)-(H4) hold with uniform constants. Let $\mathcal{U} \subset End^{1+\alpha}(M)$ be an open set of non-singular endomorphisms on M for which there are constants $\lambda \in (0, 1)$, $c > 0$ and a family of cone fields $U \ni x \rightarrow C(x)$ of constant dimension $0 < d_u \leq dim(M)$, and there is an open region $\mathcal{O} \subset M$ and a partition $\{V_1, \dots, V_p, V_{p+1}, \dots, V_{p+q}\}$ of M , and $\sigma > q$, such that for every $g \in \mathcal{U}$ there exists a continuous splitting $T_U M = E(g) \oplus F(g)$ and $a > 0$ so that

- (A1) the family of cone fields $x \mapsto C_a^-(x)$ of width $a > 0$ centered at $E(g)$ satisfies

- (a) $Dg(x)^{-1}(C_a^-(g(x))) \subsetneq C_a^-(x)$ for every $x \in M$;
- (b) there exists $\lambda_s > 1$ such that $\|Dg(x)^{-1} \cdot v\| > \lambda_s \|v\|$ for every $v \in C_a^-(g(x))$ and $x \in M$;
- (A2) there exists $L : M \rightarrow (0, \infty)$, determined by (A4)(b), in such a way that
 - (a) $Dg(x)(C(x)) \subsetneq C(g(x))$ for every $x \in M$;
 - (b) $\|Dg(x) \cdot v\| > L(x)\|v\|$ for every $v \in C(x)$ and for every $x \in M$;
- (A3) there exists $\sigma > 1$ so that $\det |Df(x)|_E| > \sigma$ for every subspace E of dimension $\dim(F)$ contained in $C(x)$, for every $x \in M$;
- (A4) the set \mathcal{O} is contained in $\cup_{j=1}^q V_{p+j}$ and
 - (a) for every disk D tangent to the cone field $C(\cdot)$, the image $g(D \cap V_i) \cap V_j$ has at most one connected component for all $i, j \in \{1, 2, \dots, p+q\}$;
 - (b) $L(x) \geq \lambda_u > 1$ for every $x \in M \setminus \mathcal{O}$ and $L(x) \geq L$ for every $x \in \mathcal{O}$;
where the constants L and γ are determined by (7.3), (7.4) and Lemma 7.1.

It is clear from the definition that the previous conditions are C^1 -open and that whenever $L > 1$ (or equivalently $\mathcal{O} = \emptyset$) corresponds to the hyperbolic setting. Endomorphisms satisfying (A1)-(A4) with uniform constants arise naturally in the context of local bifurcations of hyperbolic endomorphisms (cf Section 3). In Section 7 we prove that the previous class of endomorphisms satisfies the assumptions of Theorems A and B and, consequently:

Corollary C. *Every $C^{1+\alpha}$ non-singular endomorphism $f \in \mathcal{U}$ has a finite number of hyperbolic SRB measures, whose basins of attraction cover Lebesgue almost every points in U . Moreover, every $f \in \mathcal{U}$ is statistically stable.*

2.3. Overview of the construction and stability of SRB measures. We consider endomorphisms that admit an invariant uniformly contracting direction and a non-uniformly expanding cone field centered on a complementary direction. Let us describe the main differences between the strategy for the construction of SRB measures, inspired from [1], in the current non-invertible setting. First, the partial hyperbolicity assumptions (H1), (H2) and (H4) guarantee that small disks tangent to the cone field C_a are preserved under iteration of the dynamics. By (H3), there exists a disk D that is tangent to the cone field C_a and so that (2.1) holds for a positive Lebesgue measure set in D . This is possible because $H \subset U$ has Lebesgue positive measure. In particular, one can expect SRB measures to arise from the ergodic components of the accumulation points of the Cèsaro averages

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j Leb_D \quad (2.2)$$

In opposition to [1], our assumptions do not imply the subbundle F to be Df -invariant even if f happens to be invertible. For that reason, we use the non-uniform expansion along the cone field C_a (given by (H3)) to prove the existence of positive frequency of cone-hyperbolic times for every points in $H \cap D$. This condition assures that for every $x \in H \cap D$ there are infinitely many values of n (with positive frequency) so that

$$\|Df^n(x) \cdot v\| \geq e^{cn} \|v\| \quad \text{for all } v \in C_a(f^j(x))$$

(cf. Lemma 4.1). The previous uniformity is crucial because unstable directions (to be defined *a posteriori* at almost every point by means of Oseledets theorem) will be contained in the cone field C_a but cannot be determined a priori. This will be also important to prove that these SRB measures are hyperbolic measures. Moreover, it guarantees that if \mathcal{D}_n denotes a suitable family of disks in D that are expanded by f^n and ν is an accumulation point of the measures $\frac{1}{n} \sum_{j=0}^{n-1} f_*^j Leb_{\mathcal{D}_m}$ then its support of these measure is contained in a union of disks obtained as accumulations of uniform size $f^n(D_n)$, for $D_n \in \mathcal{D}_n$, as $n \rightarrow \infty$.

A second key difference between our setting and the context of partially hyperbolic diffeomorphisms is that, due to the possible non-invertibility of the dynamics, the iterates of a disk tangent to the cone C_a may have self intersections and stop being a submanifold. In general the accumulation disks Δ of the family of disks $f^n(\mathcal{D}_n)$ tangent to C_a may have self-intersections. We overcome this difficulty by a selection procedure on the space of pre-orbits (which also involve the lifting of a reference measure to the natural extension) in order to prove that almost every orbit has a well defined disk, tangent to the cone field C_a , with backward contraction by concatenation of all corresponding inverse branches. This construction involves a careful selection of pre-orbits so that accumulation disks are contained in unstable manifolds parametrized by elements of the natural extension. In consequence, any invariant measure supported in these disks will be hyperbolic since it will have

only positive Lyapunov exponents in the direction complementary to the stable bundle. This fact together with the absolute continuity of the stable foliation ([21, 36]) assures, by a Hopf-like argument, that there are accumulation points of the sequence of probability measures μ_n given by (2.2) (hence f -invariant) for which some ergodic component is a physical measure for the endomorphism f . The proof of the SRB property involves a careful and technical choice of partitions adapted to unstable disks on the natural extension that resemble an inverse Markov tower. Their finitude follows from the fact that the basin of every SRB measure contains an open set with a definite proportion (with respect to Lebesgue) of the phase space and all SRB measures can be obtained by the previous procedure. This completes the guideline for proof of the existence and finitude of SRB measures.

The proof of the stability of the SRB measures relies on the fact that, in the case that assumptions (H1)-(H4) can be taken with uniform constants, one can prove that the size of unstable disks tangent to the cone field can be proved uniform. This allows us to prove that unstable disks and the (not necessarily invariant) absolutely continuous invariant measures obtained by the previously limit procedure vary continuously with the endomorphism. Finally, the continuity of the entropy of the SRB measures is a byproduct of the statistical stability together with Pesin's entropy formula for endomorphisms (see Section 7).

3. SOME ROBUST CLASSES OF PARTIALLY HYPERBOLIC NON-SINGULAR ENDOMORPHISMS

In this section we discuss some applications of our results in relation with some natural bifurcations of expanding and hyperbolic endomorphisms.

3.1. Non-uniformly expansion and mostly expanding partially hyperbolic diffeomorphisms. In [1], the authors constructed SRB measures for two robust classes of non-uniformly hyperbolic maps:

(I) $C^{1+\alpha}$ -non-singular endomorphisms f ($\alpha > 0$) on a compact Riemannian M for which

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -2c < 0 \quad (3.1)$$

for a positive Lebesgue measure set of points $x \in M$; and

(II) $C^{1+\alpha}$ -diffeomorphisms ($\alpha > 0$) that exhibit a Df -invariant dominated splitting $TM = E^s \oplus E^c$ so that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x)) |_{E^c_{f^j(x)}})^{-1}\| \leq -2c < 0 \quad (3.2)$$

for a positive Lebesgue measure set of points $x \in M$.

The class of non-singular endomorphisms considered in (I) fit in our setting (corresponding to the case that the subbundle E^s is trivial and in which case conditions (2.1) and (3.1) coincide). In the case that f is a partially hyperbolic diffeomorphism as in (II), condition (3.2) seems weaker than our non-uniform hyperbolicity assumption in (2.1). This is not the case. Indeed, the existence of a dominated splitting implies that there exists $a > 0$ small so that the cone field C_a^+ of width a around E^c is positively Df -invariant. By compactness of Λ , continuity and Df -invariance of the central subbundle E^c , one can reduce $a > 0$ if necessary in order to guarantee that

$$\|(Df(f^j(x)) |_{C_a^+(f^j(x))})^{-1}\| \leq e^c \|(Df(f^j(x)) |_{E^c_{f^j(x)}})^{-1}\|$$

for every $x \in \Lambda$ and, consequently, $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x)) |_{C_a^+(f^j(x))})^{-1}\| \leq -c$. This proves that such a partially hyperbolic diffeomorphism f satisfies (2.1) if and only if (3.1) holds (possibly with different constants). In particular, our results apply to the class of non-uniformly hyperbolic maps considered in [1] and Corollary A provides an alternative proof of the continuity of the SRB measures.

3.2. Endomorphisms derived from Anosov. Dynamical systems in the isotopy class of uniformly hyperbolic ones have been intensively studied in the last decades. First examples of C^1 -robustly transitive non-Anosov diffeomorphisms were considered by Mañé and their SRB measures were studied in [12, 14]. In this subsection we illustrate how some partial hyperbolic and non-hyperbolic endomorphisms can be obtained, by local perturbations, in the isotopy class of hyperbolic endomorphisms. In particular, it is enough to show such perturbations can be done in such a way the endomorphism satisfies the requirements of Subsection 7. There are non-hyperbolic topologically mixing partially hyperbolic endomorphisms on \mathbb{T}^2 [45]. We make the

construction of the examples in dimension 3 for simplicity although similar statements hold in higher dimension. In what follows we give examples of dynamical systems with hyperbolic periodic points with different index but we could consider also the case of existence of periodic points with an indifferent direction.

3.2.1. *Partially hyperbolic endomorphisms obtained by pitchfork bifurcations.* Consider $M = \mathbb{T}^3$, let $g : M \rightarrow M$ be a linear Anosov endomorphism induced by a hyperbolic matrix $A \in \mathcal{M}_{3 \times 3}(\mathbb{Z})$ displaying three real eigenvalues and such that $TM = E^s \oplus E^u$ is the Dg -invariant splitting, where $\dim E^u = 2$. For instance, take a matrix of the form

$$A = \begin{pmatrix} n & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

for an integer $n \geq 2$. The map g is a special Anosov endomorphism, meaning that the unstable space independs of the pre-orbits and it admits a finest dominated splitting $E^s \oplus E^u \oplus E^{uu}$. If $p \in M$ is a fixed point for g and $\delta > 0$ is sufficiently small, one can write g on the ball $B(p, \delta)$ (in terms of local coordinates in $E_p^s \oplus E_p^u$) by $g(x, y, z) = (f(x), h(y, z))$, where $x \in E_p^s$, $(y, z) \in E_p^u$, f is a contraction on E_p^s and g is an expansion on E_p^u . Let $\lambda_2, \lambda_3 \in \mathbb{R}$ be the eigenvalues of $Dg(p) |_{E_p^u}$. Suppose that $|\lambda_2| \geq |\lambda_3| > 1$. Consider an isotopy $[0, 1] \ni t \mapsto h_t$ satisfying: (i) $h_0 = h$; (ii) for every $t \in [0, 1]$ the diffeomorphism h_t has a fixed point p_t (continuation of p) which, without loss of generality, we assume to coincide with p ; (iii) $h_t : E_{t,p}^c \rightarrow E_{t,p}^c$ is a $C^{1+\alpha}$ map, where $E_{t,p}^c := E_p^u$ for every $t \in [0, 1]$; (iv) the eigenvalues of Dh_t at p are λ_2 and $\rho \in \mathbb{R}$ with $|\rho| < 1$ (determined by (3.3) and (3.4)) (v) if $g_t(x, y, z) = (f(x), h_t(y, z))$ then $g_t |_{M \setminus B(p, \delta)} = g |_{M \setminus B(p, \delta)}$ for every $t \in [0, 1]$. Reducing $\delta > 0$ if necessary, we may assume that $g_t |_{B(p, \delta)}$ is injective for every $t \in [0, 1]$.

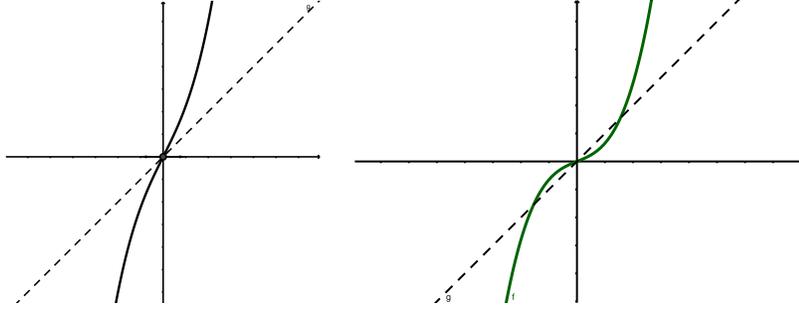


FIGURE 1. Pitchfork bifurcation in dimension one: repelling fixed point (on the left) an attracting fixed point whose topological boundary of attraction is formed by two repelling fixed points (on the right)

Roughly, the fixed point p goes through a pitchfork bifurcation along a one dimensional subspace contained in the unstable subspace associated to the original dynamics on the open set $\mathcal{O} = B(p, \delta)$. Given $a \in (0, 1)$ and the splitting $TM = E_t^s \oplus E_t^c$ consider the families of cone fields

$$\mathcal{C}_a^s(x) := \{v = v_s \oplus v_c : \|v_c\| \leq a \|v_s\|\} \quad \text{and} \quad \mathcal{C}_a^u(x) := \{v = v_s \oplus v_c : \|v_s\| \leq a \|v_c\|\}$$

and assume without loss of generality that $\|(u, v, w)\| = |u| + |v| + |w|$ for $(u, v, w) \in E^s \oplus E^u \oplus E^{uu}$ (local coordinates for g). If $a > 0$ is small then the families of cone fields are Dg_1 -invariant. Indeed, one can write $v = v_s \oplus v_c \in \mathcal{C}^s(g_1(p))$, with $v_* \in E_{1, g_1(p)}^*$, $*$ $\in \{s, c\}$. Assume ρ is such that

$$|\lambda_1 \cdot \rho^{-1}| < 1. \tag{3.3}$$

Then $\|Dg_1(g_1(p))^{-1} \cdot v_c\| \leq |\rho|^{-1} \|v_c\| \leq |\rho|^{-1} a \|v_s\| \leq |\lambda_1| |\rho|^{-1} \cdot a \|Dg_1(p)^{-1} \cdot v_s\|$, which proves that $Dg_1(p)^{-1} \cdot \mathcal{C}_a^s(g_1(p)) \subset \mathcal{C}_{|\lambda_1 \rho^{-1}| a}^s(p)$. Analogously, if $v \in \mathcal{C}_a^u(p)$ then $\|Dg_1(p) \cdot v_s\| \leq |\lambda_2^{-1} \lambda_1| a \|Dg_1(p) \cdot v_c\|$ and, consequently, $Dg_1(p) \cdot \mathcal{C}_a^u(p) \subset \mathcal{C}_{|\lambda_1 \lambda_2^{-1}| a}^u(g_1(p))$. By continuity, since δ is assumed small, we get the invariance of the cone fields for all points on the ball $B(p, \delta)$. Now observe that $g_1 |_{B(p, \delta)}$ expands volume along $E_{1,x}^c$ as it coincides with $g |_{B(p, \delta)}$. Moreover, if ρ is so

$$|\lambda_2 \cdot \rho| > 1 \tag{3.4}$$

then $|\det(Dg_1(p) |_{E_{1,p}^c})| = |\lambda_2| |\rho| > 1$ and, using that $a > 0$ is small and $x \mapsto \det(Dg_1(x) |_{E_{1,x}^c})$ is continuous, we conclude that $\inf_{x \in M} |\det(Dg_1(x) |_{E_{1,x}^c})| > 1$ and that the same property holds for the Jacobian along

disks tangent to the cone field. Simple computations show that $\|Dg_1(p) \cdot v\| \geq (1-a) \cdot \rho \|v\|$ for $v \in \mathcal{C}_a^u(p)$ and, by continuity of the derivative, we get that $\|Dg_1(x) \cdot v\| \geq L \|v\|$ for every $x \in B(p, \delta)$ and $v \in \mathcal{C}_a^u(x)$ with $L = L(\rho)$ close to $(1-a)\rho$, hence close to 1. If $x \notin B(p, \delta)$, then $g_1|_{M \setminus B(p, \delta)} = g|_{M \setminus B(p, \delta)}$ and so $\|Dg_1(x) \cdot v\| \geq (1-a) \cdot \lambda_3 \|v\|$ for every $v \in \mathcal{C}_a^u(x)$. Finally, since δ is assumed small, any partition $\{V_1, V_2, \dots, V_k, V_{k+1}\}$ of \mathbb{T}^3 such that $V_{k+1} = B(p, \delta)$ and each V_i contains a ball of radius δ and is contained in a ball of radius 2δ satisfies the requirements of (A4)(a) injectivity domains for f (here $q = 1$ and $\sigma > 1$). By Corollary C, the partially hyperbolic endomorphism g_1 has a finite number of SRB measures and it is statistically stable. Moreover, if g is robustly transitive (as e.g. in [45]) then Corollary A assures that the entropy map $f \mapsto h_{\mu_f}(f)$ is continuous in a $C^{1+\alpha}$ -small neighborhood of g .

3.2.2. Partially hyperbolic attractors obtained through Hopf bifurcations. Consider $M = \mathbb{T}^3$ and $g : M \rightarrow M$ a linear Anosov endomorphism of class $C^{1+\alpha}$. Let $p \in M$ be a fixed point and let $T_p M = E_p^s \oplus E_p^u$ be the hyperbolic splitting where $\dim E_p^u = 2$. We can mimic the previous construction, considering a Hopf's bifurcation through the fixed point p instead of a pitchfork bifurcation. Assume that Dg has two conjugated eigenvalues whose absolute value is $\rho > 1$. Consider an arc $[-1, 1] \ni t \rightarrow g_t$ of $C^{1+\alpha}$ endomorphisms so that: (i) for every $t \in [-1, 1]$, g_t is an endomorphism of class $C^{1+\alpha}$ with a fixed point p_t (we will assume that $p_t = p$ for every $t \in [-1, 1]$) (ii) $g_{-1} = g$, (iii) for every $t < 0$, g_t is an Anosov endomorphism, (iv) if $E_{t,p}^c$ is the continuation of the unstable space $E_{-1,p}^u$ to $t \in [-1, 1]$ then $Dg_0|_{E_{0,p}^c}$ has eigenvalues of absolute value equal to 1, (v) the Hopf's bifurcation occurs for $t = 0$, meaning that for $t > 0$ sufficiently small we have that the eigenvalues of $Dg_t(p)|_{E_t^c}$ direction are complex conjugate with absolute value is less than 1, there is an g_t -invariant repelling circle $C_t \subset B(p, \delta)$ around p and p is an attracting fixed point, and (vi) $g_t|_{M \setminus B(p, \delta)} = g|_{M \setminus B(p, \delta)}$ for every $t \in [-1, 1]$. An argument similar as the one used in the previous example guarantees that (A1)-(A4) are satisfied by $g_1|_\Lambda$, where $\Lambda = M \setminus B(p)$ and $B(p)$ denotes the topological basin of attraction of p . So, exist a finite number of SRB measures for $g_1|_\Lambda$ and that $g_1|_\Lambda$ is statistically stable.

4. PRELIMINARIES

The present section is devoted to some preliminary discussion on the notion of uniform and non-uniform hyperbolicity for non-singular endomorphisms. The reader acquainted with these notions may opt to omit the reading of this section and returning to it when necessary.

4.1. Cone hyperbolicity and cone-hyperbolic times. Here we define non-uniform expansion along cone fields and define cone-hyperbolic times, which extend the concept of hyperbolic times from [1, Definition 2.6]. Let V and W be vector spaces. Denote by $\mathcal{L}(V, W)$ the vector space of linear maps from V to W . If $E \subset V$ is a cone in V we endow the vector space $\mathcal{L}_E(V, W) := \{T|_E : T \in \mathcal{L}(V, W)\}$ with the norm $\|T|_E\| = \sup_{0 \neq v \in E} \frac{\|T \cdot v\|}{\|v\|}$. Given a non-singular endomorphism $f : M \rightarrow M$, a Df -invariant and convex cone field C and $x \in M$, denote by $(Df(x)|_{C(x)})^{-1}$ the map

$$Df(x)^{-1}|_{Df(x)(C(x))} : Df(x)(C(x)) \rightarrow C(x) \subset T_x M.$$

Given a normed vector space V such that $V = E \oplus F$ and $a > 0$, the *cone of width $a > 0$ centered at E* is defined by $C_a := \{v = v_1 \oplus v_2 \in E \oplus F : \|v_2\| \leq a \|v_1\|\}$. The *dimension of the cone field C_a* , denoted by $\dim(C_a)$, is the dimension of E . In particular, if $TM = E \oplus F$ be a continuous splitting of the tangent bundle, the *cone field of width $a > 0$ centered on E* is the continuous map $x \mapsto C_a(x)$ that assigns to each $x \in M$ the cone of width $a > 0$ centered at E_x . Finally, a submanifold $D \subset M$ is *tangent to the cone field* $x \mapsto C(x)$ if $\dim(D) = \dim(C)$ and $T_x D \subset C(x)$, for every $x \in D$.

As unstable subspaces will be almost everywhere defined *a posteriori* and will be contained in the cone field, some hyperbolicity will be required along cone fields. Let $\Lambda \subset M$ be a compact positively invariant subset of M and $\Lambda \ni x \mapsto C(x)$ be a Df -invariant cone field on Λ .

Definition 4.1. *Let M be a compact Riemannian manifold M . Let $f : M \rightarrow M$ be a non-singular endomorphism and $c > 0$. We say that $n \in \mathbb{N}$ is a c -cone-hyperbolic time for $x \in M$ (with respect to C) if $Df(f^j(x))(C(f^j(x))) \subset C(f^{j+1}(x))$ for every $0 \leq j \leq n-1$ and*

$$\prod_{j=n-k}^{n-1} \|(Df(f^j(x))|_{C(f^j(x))})^{-1}\| \leq e^{-ck} \quad \text{for every } 1 \leq k \leq n. \quad (4.1)$$

It is easy to see that if n and m are c -cone hyperbolic times, say $n < m$ then $n - m$ is a c -cone hyperbolic time for $f^n(x)$. One key property of cone-hyperbolic times is as follows:

Lemma 4.1. *Let M be a compact Riemannian manifold, f be a non-singular endomorphism on M and $c > 0$. If n is a c -cone-hyperbolic time for $x \in M$ then $\|Df^{n-j}(f^j(x)) \cdot v\| \geq e^{c(n-j)}\|v\|$ for every $v \in C(f^j(x))$ and $0 \leq j \leq n - 1$.*

Proof. Fix $0 \leq j \leq n - 1$ and $v \in C(f^j(x))$, and set $w = Df^{n-j}(f^j(x)) \cdot v$. By definition of cone-hyperbolic times one has $Df(f^\ell(x))(C(f^\ell(x))) \subset C(f^{\ell+1}(x))$ for every $0 \leq \ell \leq n - 1$. In particular $Df^{\ell-j}(f^j(x)) \cdot v \in Df(f^{\ell-1}(x))(C(f^{\ell-1}(x))) \subset C(f^\ell(x))$, for every $j + 1 \leq \ell \leq n - 1$. As n is a c -cone-hyperbolic time for x ,

$$\begin{aligned} \|v\| &= \|Df(f^j(x))^{-1} \circ Df(f^{j+1}(x))^{-1} \circ \dots \circ Df(f^{n-1}(x))^{-1} \cdot w\| \\ &\leq \prod_{k=j}^{n-1} \|(Df(f^k(x))|_{C(f^k(x))})^{-1}\| \cdot \|w\| \leq e^{-c(n-j)}\|w\|, \end{aligned}$$

which guarantees that $\|Df^{n-j}(f^j(x)) \cdot v\| \geq e^{c(n-j)}\|v\|$ and proves the lemma. \square

4.2. Natural extension. Let (M, d) be a compact metric space and $f : M \rightarrow M$ be a continuous map. The natural extension of M by f is the set $M^f := \{\hat{x} = (x_{-j})_{j \in \mathbb{N}} : x_{-j} \in M \text{ and } f(x_{-j}) = x_{-j+1}, \text{ for every } j \in \mathbb{N}\}$. We can induce a metric on M^f from the metric d by $\hat{d}(\hat{x}, \hat{y}) := \sum_{j \in \mathbb{N}} 2^{-j} d(x_{-j}, y_{-j})$. The space M^f endowed with the metric \hat{d} is a compact metric space and the topology induced by \hat{d} is equivalent with the topology induced by $M^{\mathbb{N}}$ endowed with Tychonov's product topology. The projection $\pi : M^f \rightarrow M$ given by $\pi(\hat{x}) := x_0$ is a continuous map. The lift of f is the map $\hat{f} : M^f \rightarrow M^f$ given by $\hat{f}(\hat{x}) := (\dots, x_{-n}, \dots, x_{-1}, x_0, f(x_0))$, and it is clear that $f \circ \pi = \pi \circ \hat{f}$. Finally, for each $\hat{x} \in M^f$ we take $T_{\hat{x}}M^f := T_{\pi(\hat{x})}M$ and set

$$\begin{aligned} D\hat{f}(\hat{x}) : T_{\hat{x}}M^f &\rightarrow T_{\hat{f}(\hat{x})}M^f \\ v &\mapsto Df(\pi(\hat{x})). \end{aligned}$$

If $\Lambda \subset M$ is a compact positively invariant, that is, $f(\Lambda) \subset \Lambda$, the natural extension of Λ by f is the set of pre-orbits that lie on Λ , that is, $\Lambda^f := \{\hat{x} = (x_{-j})_{j \in \mathbb{N}} : x_{-j} \in \Lambda \text{ and } f(x_{-j}) = x_{-j+1} \text{ for every } j \in \mathbb{N}\}$. The projection π induces a continuous bijection between \hat{f} -invariant probability measures and f -invariant probability measures: assigns every $\hat{\mu}$ in M^f to the push forward $\pi_*\hat{\mu} = \mu$ (see e.g. [37] for more details).

4.3. Uniform and non-uniform hyperbolicity for endomorphisms.

4.3.1. Uniform hyperbolicity for endomorphisms. The notion of uniform hyperbolicity for endomorphisms was introduced in [35]. If f is a non-singular endomorphism for each $x \in M$ there is a neighborhood V_x of x such that $f|_{V_x} : V_x \rightarrow f(V_x)$ admits an inverse $f^{-1} : f(V_x) \rightarrow V_x$ (whose derivative will be denoted simply by $Df(x)^{-1}$). We say a positively f -invariant set $\Lambda \subset M$ is *uniformly hyperbolic* if there are families of cone fields $\Lambda \ni x \mapsto C^s(x)$ and $\Lambda \ni x \mapsto C^u(x)$ of constant dimension satisfying: (i) $Df(x)^{-1}(C^s(f(x))) \subsetneq C^s(x)$ and $Df(x)(C^u(x)) \subsetneq C^u(f(x))$ for every $x \in \Lambda$; and (ii) there exists $\sigma > 1$ such that $\|Df(x)^{-1} \cdot v\| \geq \sigma\|v\|$ for every $v \in C^s(f(x))$ and $\|Df(x) \cdot w\| \geq \sigma\|w\|$ for every $w \in C^u(x)$. The existence of a contracting cone field C^s at item (i) assures that the stable subbundle E^s , defined by $E_x^s := \bigcap_{n \in \mathbb{N}} Df^n(x)^{-1}(C^s(f^n(x)))$ for every $x \in \Lambda$, is Df -invariant and uniformly contracting. Uniform hyperbolicity can be used to define invariant splittings by means of the natural extension of f , that is the (compact) set M^f of preimages of f . Indeed, given a pre-orbit $\hat{x} = (x_{-n})_{n \in \mathbb{N}} \in M^f$ of some point $x = x_0$, the expansion along the unstable cone field implies that the set $E_{\hat{x}}^u := \bigcap_{n \in \mathbb{N}} Df^n(x_{-n})(C^u(x_{-n}))$, is a subspace (see e.g Proposition 5.7). Moreover, $Df(x) \cdot E_{\hat{x}}^u = E_{\hat{y}}^u$, whenever $\hat{y} = (\dots, x_{-2}, x_{-1}, x_0, f(x_0))$. Clearly, $E_{\hat{x}}^u$ defines a subspace that is uniformly contracting by backward iterates along the pre-orbit $\hat{x} = (x_{-n})_{n \in \mathbb{N}}$ of x_0 . In particular, for each $\hat{x} \in M^f$, $T_{\hat{x}}M^f = E_x^s \oplus E_{\hat{x}}^u$ defines a $D\hat{f}$ -invariant splitting on $TM^f := \bigcup_{\hat{x} \in M^f} T_{\hat{x}}M^f$. We remark that, in general, the natural extension admits no differential structure.

4.3.2. Lyapunov exponents and non-uniform hyperbolicity. We need the following:

Proposition 4.1. [37, Proposition I.3.5] *Suppose that M is a compact Riemannian manifold. Let $f : M \rightarrow M$ be a C^1 map preserving a probability measure $\hat{\mu}$. There is a Borelian set $\hat{\Delta} \subset M^f$ with $\hat{f}(\hat{\Delta}) = \hat{\Delta}$ and $\hat{\mu}(\hat{\Delta}) = 1$ satisfying that for every $\hat{x} \in \hat{\Delta}$ there is a splitting $T_{\hat{x}}M^f = E_1(\hat{x}) \oplus \dots \oplus E_r(\hat{x})(\hat{x})$ and numbers $+\infty > \lambda_1(\hat{x}) > \lambda_2(\hat{x}) > \dots > \lambda_r(\hat{x})(\hat{x}) > -\infty$ (Lyapunov exponents) and multiplicities $m_i(\hat{x})$ for $1 \leq i \leq r(\hat{x})$ such that:*

- (1) $D\hat{f}(\hat{f}^n(\hat{x})) : T_{\hat{f}^n(\hat{x})}M^f \rightarrow T_{\hat{f}^{n+1}(\hat{x})}M^f$ is a linear isomorphism for every $n \in \mathbb{Z}$;
- (2) the functions $r : \hat{\Delta} \rightarrow \mathbb{N}$, $m_i : \hat{\Delta} \rightarrow \mathbb{N}$ and $\lambda_i : \hat{\Delta} \rightarrow \mathbb{R}$ are \hat{f} -invariant ;
- (3) $\dim(E_i(\hat{x})) = m_i(\hat{x})$ for every $1 \leq i \leq r(\hat{x}) \leq \dim M$;
- (4) the splitting is $D\hat{f}$ -invariant, that is, $D\hat{f}(\hat{x}) \cdot E_i(\hat{x}) = E_i(\hat{f}(\hat{x}))$, for every $1 \leq i \leq r(\hat{x})$;
- (5) $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D\hat{f}^n(\hat{x}) \cdot u\| = \lambda_i(\hat{x})$ for every $u \in E_i(\hat{x}) \setminus \{0\}$ and for every $1 \leq i \leq r(\hat{x})$;
- (6) if $\rho_1(\hat{x}) \geq \rho_2(\hat{x}) \geq \dots \geq \rho_d(\hat{x})$ represent the numbers $\lambda_i(\hat{x})$ repeated $m_i(\hat{x})$ times for each $1 \leq i \leq r(\hat{x})$ and $\{u_1, u_2, \dots, u_d\}$ is a basis for $T_{\hat{x}}M^f$ satisfying $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D\hat{f}^n(\hat{x}) \cdot u_i\| = \rho_i(\hat{x})$ for every $1 \leq i \leq d$, then for any subsets $P, Q \subset \{1, 2, \dots, d\}$ with $P \cap Q = \emptyset$ one has $\lim_{n \rightarrow \infty} \frac{1}{n} \log \angle(D\hat{f}^n(\hat{x}) \cdot E_P, D\hat{f}^n(\hat{x}) \cdot E_Q) = 0$ where E_P and E_Q are, respectively, the subspaces generated by $\{u_i\}_{i \in P}$ and $\{u_i\}_{i \in Q}$.

Moreover, if $\hat{\mu}$ is ergodic then the previous functions are constant $\hat{\mu}$ -almost everywhere.

Given an f -invariant probability measure μ , we say that μ is *hyperbolic* if it has no zero Lyapunov exponents. Associated to this formulation of the concept of Lyapunov exponents we have the existence of unstable manifolds for almost every point with respect to some invariant measure with some positive Lyapunov exponents. Consider the subspaces $E_{\hat{x}}^u := \bigoplus_{\lambda_i(\hat{x}) > 0} E_i(\hat{x})$ and $E_{\hat{x}}^{cs} := \bigoplus_{\lambda_i(\hat{x}) \leq 0} E_i(\hat{x})$.

Proposition 4.2. [37, Proposition V.4.4] *There is a countable number of compact subsets $(\hat{\Delta}_i)_{i \in \mathbb{N}}$, of M^f with $\bigcup_{i \in \mathbb{N}} \hat{\Delta}_i \subset \hat{\Delta}$ and $\hat{\mu}(\hat{\Delta} \setminus \bigcup_{i \in \mathbb{N}} \hat{\Delta}_i) = 0$ such that:*

- (1) for each $\hat{\Delta}_i$ there is $k_i \in \mathbb{N}$ satisfying $\dim(E^u(\hat{x})) = k_i$ for every $\hat{x} \in \hat{\Delta}_i$ and the subspaces $E_{\hat{x}}^u$ and $E_{\hat{x}}^{cs}$ depend continuously on $\hat{x} \in \hat{\Delta}_i$;
- (2) for any $\hat{\Delta}_i$ there is a family of C^1 embedded k_i -dimensional disks $\{W_{loc}^u(\hat{x})\}_{\hat{x} \in \hat{\Delta}_i}$ in M and real numbers $\lambda_i, \varepsilon \ll \lambda_i, r_i < 1, \gamma_i, \alpha_i$ and β_i such that the following properties hold for each $\hat{x} \in \hat{\Delta}_i$:
 - (a) there is a C^1 map $h_{\hat{x}} : O_{\hat{x}} \rightarrow E_{\hat{x}}^{cs}$, where $O_{\hat{x}}$ is an open set of $E_{\hat{x}}^u$ which contains $\{v \in E_{\hat{x}}^u : \|v\| < \alpha_i\}$ satisfying: (i) $h_{\hat{x}}(0) = 0$ and $Dh_{\hat{x}}(0) = 0$; (ii) $Lip(h_{\hat{x}}) \leq \beta_i$ and $Lip(Dh_{\hat{x}}(\cdot)) \leq \beta_i$; and (iii) $W_{loc}^u(\hat{x}) = \exp_{x_0}(\text{graph}(h_{\hat{x}}))$.
 - (b) for any $y_0 \in W_{loc}^u(\hat{x})$ there is a unique $\hat{y} \in M^f$ such that $\pi(\hat{y}) = y_0$ and
$$\text{dist}(x_{-n}, y_{-n}) \leq \min\{r_i e^{-\varepsilon_i n}, \gamma_i e^{-\lambda_i n} \text{dist}(x_0, y_0)\}, \text{ for every } n \in \mathbb{N},$$
 - (c) if $\hat{W}_{loc}^u(\hat{x}) := \{\hat{y} \in M^f : y_{-n} \in W_{loc}^u(\hat{f}^{-n}(\hat{x})), \text{ for every } n \in \mathbb{N}\}$ then $\pi : \hat{W}_{loc}^u(\hat{x}) \rightarrow W_{loc}^u(\hat{x})$ is bijective and $\hat{f}^{-n}(\hat{W}_{loc}^u(\hat{x})) \subset \hat{W}_{loc}^u(\hat{f}^{-n}(\hat{x}))$.
 - (d) for any $\hat{y}, \hat{z} \in \hat{W}_{loc}^u(\hat{x})$ it holds $\text{dist}_{\hat{f}^{-n}(\hat{x})}^u(y_{-n}, z_{-n}) \leq \gamma_i e^{-\lambda_i n} \text{dist}_{\hat{x}}^u(y_0, z_0)$ for every $n \in \mathbb{N}$, where $\text{dist}_{\hat{x}}^u$ is the distance along $W_{loc}^u(\hat{x})$.

4.4. SRB measures. The notion of SRB measure for endomorphisms depends intrinsically on the existence of unstable manifolds, whose geometry and construction are more involving than for diffeomorphisms as unstable manifolds may have self intersections (hence do not generate an invariant foliation).

4.4.1. Rokhlin's disintegration theorem. Let (X, \mathcal{A}, μ) be a probability space and \mathcal{P} be a partition of X . We say that \mathcal{P} is a measurable partition if there is a sequence of countable partitions of X , $(\mathcal{P}_j)_{j \in \mathbb{N}}$, such that $\mathcal{P} = \bigvee_{j=0}^{\infty} \mathcal{P}_j \pmod{0}$. Consider the projection $p : X \rightarrow \mathcal{P}$ that associate to each $x \in X$ the atom $\mathcal{P}(x)$ that contains x . We say that a subset $\mathcal{Q} \subset \mathcal{P}$ is measurable if, and only if $p^{-1}(\mathcal{Q})$ is measurable. It is easy to see that the family of all measurable sets of \mathcal{P} is a σ -algebra of \mathcal{P} . Set $\tilde{\mu} := p_*\mu$.

Definition 4.2. *Let (X, \mathcal{A}, μ) be a probability space and let \mathcal{P} be a partition of X . A disintegration of μ with respect to \mathcal{P} is a family $(\mu_P)_{P \in \mathcal{P}}$ of probability measures on X so that:*

- (1) $\mu_P(P) = 1$ for $\tilde{\mu}$ a.e. $P \in \mathcal{P}$;
- (2) for every measurable set $E \subset X$ the map $\mathcal{P} \ni P \mapsto \mu_P(E)$ is measurable;
- (3) $\mu(E) = \int \mu_P(E) d\tilde{\mu}(P)$ for every measurable set $E \subset X$.

Rokhlin's theorem [40] guarantees that probability measures can be disintegrated with respect to measurable partitions, and that if $(\mu_P)_{P \in \mathcal{P}}$ and $(\mu'_P)_{P \in \mathcal{P}}$ are disintegrations of the measure μ with respect to \mathcal{P} then $\mu_P = \mu'_P$ for $\tilde{\mu}$ almost every $P \in \mathcal{P}$.

4.4.2. *SRB property for endomorphisms.* In order to define SRB measures we need the notion of partitions (on M^f) adapted to unstable manifolds of (f, μ) .

Definition 4.3. Let μ be an f -invariant probability measure with at least one positive Lyapunov exponent at μ almost every point. A measurable partition \mathcal{P} of M^f is subordinated to unstable manifolds if for $\hat{\mu}$ a.e. $\hat{x} \in M^f$ (1) $\pi|_{\mathcal{P}(\hat{x})} : \mathcal{P}(\hat{x}) \rightarrow \pi(\mathcal{P}(\hat{x}))$ is bijective, and (2) there is a submanifold $W_{\hat{x}}$ with dimension $k(\hat{x})$ in M such that $W_{\hat{x}} \subset W^u(\hat{x})$, $\pi(\mathcal{P}(\hat{x})) \subset W_{\hat{x}}$ and $\pi(\mathcal{P}(\hat{x}))$ contains an open neighborhood of x_0 in $W_{\hat{x}}$, where $\mathcal{P}(\hat{x})$ denotes the element of the partition \mathcal{P} that contains \hat{x} and $k(\hat{x})$ denotes the number of positive Lyapunov exponents at \hat{x} .

We are now in a position to define the SRB property for invariant measures.

Definition 4.4. We say that an f -invariant and ergodic probability measure μ is an SRB measure if it has at least one positive Lyapunov exponent almost everywhere and for every partition \mathcal{P} subordinated to unstable manifolds one has $\pi_* \hat{\mu}_{\mathcal{P}(\hat{x})} \ll \text{Leb}_{W_{\hat{x}}}$ for $\hat{\mu}$ a.e. $\hat{x} \in M^f$, where $(\hat{\mu}_{\mathcal{P}(\hat{x})})_{\hat{x} \in M^f}$ is a disintegration of $\hat{\mu}$ with respect to \mathcal{P} .

5. EXISTENCE AND UNIQUENESS OF SRB MEASURES

The present section is devoted to the proof of Theorem A. In Subsections 5.1 and 5.2 we describe the geometry of the iterates of submanifolds that are tangent to the non-uniformly expanding cone field and construct families of disks with non-uniform hyperbolicity. Afterwards, the core of the proof is, after building a (non-invariant) hyperbolic measure ν on M , to construct a suitable lift $\hat{\nu}$ of ν to the natural extension and prove its absolute continuity along unstable disks (see Subsections 5.4 and 5.5). Finally, in Subsection 5.6 we use these lifted measures prove the existence and finiteness of SRB measures on the partially hyperbolic attractor.

5.1. Geometry of disks which are tangent to C_a . In this subsection we collect a result on the regularity disks tangent to cone fields from [1]. Consider D a disk in M which is tangent to the cone field C_a and assume that D satisfies $\text{Leb}_D(H) > 0$, where H is the subset given by the hypothesis (H3). For $\varepsilon > 0$ and $x \in M$ denote by $T_x M(\varepsilon) := \{v \in T_x M : \|v\| \leq \varepsilon\}$ the ball of radius ε -centered on $0 \in T_x M$. Fix $\delta_0 > 0$ such that the exponential map restricted to $T_x M(\delta_0)$ is a diffeomorphism onto its image for every $x \in M$, and set $B(x, \varepsilon) := \exp_x(T_x M(\varepsilon))$ for every $0 < \varepsilon \leq \delta_0$. As D is tangent to the cone field there exists $0 < \delta \leq \delta_0$ such that if $y \in B(x, \delta) \cap D$ then there exists a unique linear map $A_x(y) : T_x D \rightarrow E_x^s$ whose graph is parallel to $T_y D$. Moreover, by compactness of M , δ can be taken independent of x . Thus one can express the Hölder variation of the tangent bundle in these local coordinates as follows: given $C > 0$ and $0 < \alpha < 1$ we say that the tangent bundle TD of D is (C, α) -Hölder if $\|A_x(y)\| \leq C \cdot \text{dist}_N(x, y)^\alpha$, for every $y \in B(x, \delta_0) \cap D$ and $x \in D$. Moreover, the α -Hölder constant of the tangent bundle TD is given by the number: $\kappa(TD, \alpha) := \inf \{C > 0 : TD \text{ is } (C, \alpha)\text{-Hölder}\}$. Since our assumptions imply that disks tangent to the cone field are preserved under positive iteration by the dynamics and assumption (H4) guarantees domination, the following control of the distortion along disks tangent to the cone field is a consequence of the arguments of [1, Proposition 2.2 and Corollary 2.4]:

Proposition 5.1. *There are $\alpha \in (0, 1)$, $\beta \in (0, 1)$ and $C_0 > 0$ such that if D is a submanifold of M which is tangent to the cone field C_a and if $f(D)$ is also a submanifold of M then $\kappa(Tf(N), \alpha) \leq \beta \cdot \kappa(TN, \alpha) + C_0$. In consequence, there is $L_1 > 0$ such that if D is a submanifold of M which is tangent to C_a and if $k \in \mathbb{N}$ is such that $f^j(D)$ is a submanifold of M for every $1 \leq j \leq k$ then the map*

$$J_k : \begin{array}{ccc} f^k(D) & \rightarrow & \mathbb{R} \\ x & \mapsto & \log |\det(Df(x)|_{T_x f^k(D)})|, \end{array}$$

is (L_1, α) -Hölder continuous. Moreover, the constant L_1 depends only on f .

5.2. Non-uniform expansion along disks tangent to the cone-field. Hypothesis (H3) implies that there exists a disk $D \subset U$ tangent to C_a and a subset $H \subset D$ with $\text{Leb}_D(H) > 0$ satisfying that:

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x))|_{C_a(f^j(x))})^{-1}\| \leq -2c < 0, \quad (5.1)$$

for every $x \in H$. A direct computation (which we leave as a simple exercise to the reader) shows that Pliss' lemma, obtained independently by Liao, (see e.g. [29, Lema IV.11.8]) together with (5.1) is enough to deduce the following:

Lemma 5.1. *For all $x \in H$ and n sufficiently large there is $\theta > 0$ (that depends only on f and c) and a sequence $1 \leq n_1(x) < \dots < n_l(x) \leq n$ of c -cone-hyperbolic times for x with $l \geq \theta n$.*

Since M is compact and f is a non-singular endomorphism, there is $\delta_1 > 0$ such that all inverse branches are well defined on the ball $B(x, \delta_1)$ for every $x \in M$. More precisely, given $x \in M$ and $y \in f^{-1}(\{x\})$ there is a well defined diffeomorphism $f_y^{-1} : B(x, \delta_1) \rightarrow V_y$ onto an open neighborhood V_y of y satisfying $f \circ f_y^{-1} = id_{B(x, \delta_1)}$. Reducing δ_1 if necessary, we may assume that $V_z \cap V_y = \emptyset$ for every distinct $y, z \in f^{-1}(\{x\})$. Actually, we have the following uniform continuity of the inverse branches:

Lemma 5.2. *For any $\varepsilon > 0$ there exists $0 < \delta = \delta(\varepsilon) < \delta_1$ such that $f_y^{-1}(B(x, \delta)) \subset B(y, \varepsilon)$ for every $x \in M$ and $y \in f^{-1}(\{x\})$.*

Proof. Fix $\varepsilon > 0$. Given $x \in M$ and $y \in f^{-1}(\{x\})$, there is $0 < \delta_y < \delta_1$ such that $dist_M(f_y^{-1}(z), y) < \varepsilon/2$ whenever $dist_M(z, x) < \delta_y$. As M is a compact connected manifold then $\delta'(x) := \min \{\delta_y : y \in f^{-1}(\{x\})\} > 0$ for every $x \in M$. By compactness, the open covering $(B(x, \delta'(x)/8))_{x \in M}$ of M admits a finite subcover $(B(x_j, \delta'(x_j)/8))_{1 \leq j \leq m}$.

We claim that $\delta := \frac{1}{2} \min \{\delta'(x_j) : 1 \leq j \leq m\} > 0$ satisfies the requirements of the lemma. Indeed, take $x \in M$, $y \in f^{-1}(\{x\})$ and $z \in B(x, \delta)$, and let $1 \leq j \leq m$ be such that $z \in B(x_j, \delta'(x_j)/8)$. Hence $dist_M(x, x_j) < \delta'(x_j)$. Moreover, taking $u = f_y^{-1}(x_j) \in f^{-1}(\{x_j\})$ the maps f_y^{-1} and f_u^{-1} coincide at $B(x, \delta_1) \cap B(x_j, \delta_1)$. Since z and x belong to this intersection then $dist_M(f_y^{-1}(z), y) = dist_M(f_y^{-1}(z), f_y^{-1}(x)) = dist_M(f_u^{-1}(z), f_u^{-1}(x)) < \varepsilon$, which proves the lemma. \square

Lemma 5.3. *If $c > 0$ is given by (5.1) there exists $\delta_2 > 0$ (depending on c) so that for every $x \in \Lambda$, every $y \in B(x, \delta_2) \cap \Lambda$ and every $z \in f^{-1}(\{x\})$ it hold that:*

$$\|Df(f_z^{-1}(y))^{-1}v\| \leq e^{\frac{\varepsilon}{2}} \|(Df(f_z^{-1}(x))|_{C_a(f_z^{-1}(x))})^{-1}\| \|v\| \quad \text{for every } v \in C_a(y). \quad (5.2)$$

Proof. The map $\Lambda \ni x \mapsto \|(Df(x)|_{C_a(x)})^{-1}\|$ is continuous, hence uniformly continuous. So, given $\eta > 0$ there is $\xi > 0$ such that if $dist_M(x, y) < \xi$ then $\| \|(Df(x)|_{C_a(x)})^{-1}\| - \|(Df(y)|_{C_a(y)})^{-1}\| < \eta$. Moreover, as f is a non-singular endomorphism and \bar{U} is compact then $D := \min_{x \in \bar{U}} \|(Df(x)|_{C_a(x)})^{-1}\| > 0$. Take $\eta = (e^{\frac{\varepsilon}{2}} - 1)D$, let $\xi = \xi(\eta, c) > 0$ be given by the latter uniform continuity and let $\delta_2 = \delta(\xi) > 0$ be the radius given by Lemma 5.2. Thus, if $x, y \in \Lambda$ are such that $dist_M(x, y) < \delta_2$ and $z \in f^{-1}(\{x\})$ then $dist_M(x_1, y_1) < \xi$, where $x_1 := f_z^{-1}(x) = z$ and $y_1 := f_z^{-1}(y)$. Therefore $\| \|(Df(x_1)|_{C_a(x_1)})^{-1}\| - \|(Df(y_1)|_{C_a(y_1)})^{-1}\| < (e^{\frac{\varepsilon}{2}} - 1)D < (e^{\frac{\varepsilon}{2}} - 1)\|(Df(x_1)|_{C_a(x_1)})^{-1}\|$ and so $\|(Df(y_1)|_{C_a(y_1)})^{-1}\| < e^{\frac{\varepsilon}{2}} \|(Df(x_1)|_{C_a(x_1)})^{-1}\|$. Then, $\|Df(y_1)^{-1} \cdot v\| \leq e^{\frac{\varepsilon}{2}} \|(Df(x_1)|_{C_a(x_1)})^{-1}\| \cdot \|v\|$ or, equivalently, $\|Df(f_z^{-1}(y))^{-1}v\| \leq e^{\frac{\varepsilon}{2}} \|(Df(f_z^{-1}(x))|_{C_a(f_z^{-1}(x))})^{-1}\| \|v\|$, for every $v \in C_a(f(y_1)) = C_a(y)$. This proves the lemma. \square

For a submanifold $D \subset M$ we will denote by $B_D(\cdot, \varepsilon)$ the ball of radius $\varepsilon > 0$ in D with the distance induced by the Riemannian metric on D . Using the continuity of the parallel transport and the Hölder control of the variation of the tangent bundle of disks tangent to C_a (recall Proposition 5.1), if D is a C^1 disk embedded in U tangent to C_a then there exists $\xi > 0$ such that $B_D(x, \xi) = exp_x(gra_f(\psi_x))$ for every $x \in D$, where $\psi_x : B(0, \xi) \subset T_x D \rightarrow E_x^s$ is a Lipschitz map and $Lip(\psi_x) \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ depending only on $a > 0$. For that reason we will say that $\Delta(q, \delta) \subset D$ is a disk of radius $\delta > 0$ tangent to the cone field if $\Delta(q, \delta) = exp_q(gra_f(\psi))$ where the graph of $\psi : B(0, \delta) \subset T_q D \rightarrow E_x^s$ is contained in $C_a(q)$. Note that, there is a constant $R > 0$ (depending only on the cone field) such that $B_D(q, R\delta) \subset \Delta(q, \delta) \subset B_D(q, 2R\delta)$. In what follows we show that cone-hyperbolic times imply on a growth to a large scale along on some subdisk of D . If D is a submanifold of M then we denote by $dist_D$ the Riemannian distance on D then $dist_M \leq dist_D$.

Proposition 5.2. *Let D be a C^1 disk embedded in U that is tangent to C_a . There is $\delta > 0$ such that if n is a c -cone-hyperbolic time for $x \in D$ and $d(x, \partial D) > 8\delta$ then there is an open neighborhood $D(x, n, 8\delta)$ of x in D diffeomorphic to a C^1 -disk $\Delta(f^n(x), 8\delta)$ by f^n . Moreover*

$$dist_{f^{n-k}(D(x, n, 8\delta))}(f^{n-k}(x), f^{n-k}(y)) \leq e^{-\frac{\varepsilon}{2}k} dist_{\Delta(f^n(x), 8\delta)}(f^n(x), f^n(y))$$

for every $y \in D(x, n, 8\delta)$ and $1 \leq k \leq n$.

Proof. The first part in the proof is to assure that there exists a uniform radius δ so that the image $f(\tilde{D})$ of any disk \tilde{D} of radius 8δ tangent to the cone field is a disk tangent to the cone field with inner diameter at least 8δ , and the second part of the proof is to assure the backward contraction property. The first part of the argument adapts the classical ideas that the existence of the invariant cone field and domination assumptions (H3) and (H4) imply that the classical graph transform map is well defined (see e.g. [43, Appendix III] or [10, Section 7]). Although the subbundle F is not necessarily Df -invariant, we can use the tangent spaces to $f^j(D)$ (which are contained in the cone field) as reference space to define the graph transform. We will make this precise in what follows.

By compactness of M , there exist $\delta_0, \delta_1 > 0$ such that $\exp_y : B(0, \delta_0) \subset T_y M \rightarrow \exp_y(B(0, \delta_0))$ and $f|_{B(y, \delta_1)} : B(y, \delta_1) \rightarrow f(B(y, \delta_1))$ are diffeomorphisms for every $y \in M$. Moreover, by uniform continuity of f , there is $\delta_3 > 0$ such that if $\text{dist}_M(y, z) < \delta_3$ then $\text{dist}_M(f(y), f(z)) < \delta_1$. Fix $\xi > 0$ so that if \tilde{D} is a disk tangent to the cone field then $B_{\tilde{D}}(x, \xi) = \exp_x(\text{graf}(\psi_x))$, $\psi_x : B(0, \xi) \subset T_x \tilde{D} \rightarrow E_x^s$ and set $8\delta := \frac{1}{2} \min\{\xi, \delta_0, \delta_1, \delta_3\}$. By the choice of δ , the map $F_x := \exp_{f(x)}^{-1} \circ f \circ \exp_x : B(0, 8\delta) \subset T_x M \rightarrow T_{f(x)} M$ is a diffeomorphism onto its image for every $x \in M$.

Fix a C^1 disk D tangent to C_a and $x \in D$ such that $d(x, \partial D) > 8\delta$. Then we claim that if $B_D(x, 8\delta) := \exp_x(\text{graf}(\psi_x))$ with $\psi_x : T_x D(8\delta) \subset T_x D \rightarrow E_x^s$ Lipschitz then the image $F_x(\text{graf}(\psi_x))$ is the graph of a Lipschitz map from an open disk in $T_{f(x)} f(D)$ to $E_{f(x)}^s$ whose domain contains a ball of radius 8δ around $0 \in T_{f(x)} f(D)$. The choice of $8\delta < \delta_1$ guarantees that $f(B_D(x, 8\delta))$ is a submanifold of M tangent to C_a .

Claim: $F_x(\text{graf}(\psi_x)) = \text{graf}(\psi_{f(x)})$ for some Lipschitz map $\psi_{f(x)} : T_{f(x)} f(D)(8\delta) \subset T_{f(x)} f(D) \rightarrow E_{f(x)}^s$.

Proof of the claim. Write $F_x(u + v) = Df(x)(u + v) + r_x(u + v) + t_x(u + v)$ where the error terms $r_x : T_x D(8\delta) \times E_x^s(8\delta) \rightarrow T_{f(x)} f(D)$ and $t_x : T_x D(8\delta) \times E_x^s(8\delta) \rightarrow E_{f(x)}^s$ are $C^{1+\alpha}$ maps. Hypothesis (H3)-(H4) assure that the angle between $T_x D$ and E_x^s is bounded away from zero. Therefore, the norm of the projection $p_x : T_{f(x)} M \rightarrow T_{f(x)} f(D)$ parallel to $E_{f(x)}^s$ is uniformly bounded away from zero. Reduce $\delta > 0$ if necessary so that for every $x \in M$ the error terms r_x and t_x are uniformly small (see e.g. [10, Section 7.1]) so that $\text{Lip}(r_x) \leq \text{Lip}[(p_x \circ Df(x) \circ (id, \psi_x))^{-1}]^{-1}$. Now, using the invariance of E^s and that $Df(x) \cdot T_x D = T_{f(x)} f(D)$, we get $\text{Lip}[p_x \circ F_x \circ (id, \psi_x) - p_x \circ Df(x) \circ (id, \psi_x)] \leq \text{Lip}(r_x) \leq \text{Lip}[(p_x \circ Df(x) \circ (id, \psi_x))^{-1}]^{-1}$. Theorem I.2 in [43] assures that the Lipschitz map $p_x \circ F_x \circ (id, \psi_x)$ is a homeomorphism onto its image, which guarantees that $F_x(\text{graf}(\psi_x))$ is a graph of a Lipschitz map $\psi_{f(x)} : T_{f(x)} f(D)(8\delta) \subset T_{f(x)} f(D) \rightarrow E_{f(x)}^s$. \square

As the previous estimates are uniform (independ of the disk D) then $f^j(D(x, n, 8\delta)) = F_x^j(\text{graf}(\psi_x^j))$ for every $0 \leq j \leq n$, where $F_x^j = \exp_{f^j(x)}^{-1} \circ f^j \circ \exp_x$ and $D(x, n, 8\delta)$ denotes the set of points $y \in D$ so that $d(f^i(x), f^i(y)) < 8\delta$ for every $0 \leq i \leq n$. Finally, the proof of the backward contraction property and that $f^n(D(x, n, 8\delta))$ is a disk of radius 8δ tangent to C is similar to [1, Lemma 2.7], the difference being that hyperbolic times are considered along the cone-field (these guarantee the necessary control of the curvature of the disks) and that backward contraction is given along the suitable concatenation of inverse branches. \square

We observe the constant $\delta > 0$ in Proposition 5.2 is independent of $x \in M$. The disks $D(x, n, 8\delta) \subset D$ are called *hyperbolic pre-disks* and their images $\Delta(f^n(x), 8\delta)$ are called *n-hyperbolic disks*. We need the following bounded distortion property for the Jacobian of f along disks tangent to the invariant cone field.

Proposition 5.3. [1, Proposition 2.8] *Suppose D is a C^1 disk embedded in U , $x \in D$ and n a c -cone hyperbolic time for x . There is $C_1 > 0$ such that*

$$C_1^{-1} \leq \frac{|\det Df^n(y)|_{T_y D(x, n, 8\delta)}}{|\det Df^n(z)|_{T_z D(x, n, 8\delta)}} \leq C_1$$

for every $y, z \in D(x, n, 8\delta)$, where $D(x, n, 8\delta)$ is the hyperbolic pre-disk associated to x .

Fix $\delta > 0$ given by Proposition 5.2 and $H = H(c) \subset U$ be the set of points with infinitely many c -cone hyperbolic times and such that $\text{Leb}_D(H) > 0$, guaranteed by hypothesis (H3). Reducing δ , if necessary, we can assume that $B := B(D, c, 8\delta) := \{x \in D \cap H : \text{dist}_D(x, \partial D) \geq 8\delta\}$ satisfies $\text{Leb}_D(B) > 0$. For each $n \in \mathbb{N}$ define $H_n := \{x \in B : n \text{ is a } c\text{-cone hyperbolic time for } x\}$.

Proposition 5.4. [1, Proposition 3.3] *There exists $\tau > 0$ and for each $n \in \mathbb{N}$ there is a finite set $H_n^* \subset H_n$ such that the hyperbolic pre-disks $(D(x, n, \delta))_{x \in H_n^*}$ are contained in D , pairwise disjoint and their union $\mathcal{D}_n := \bigcup_{x \in H_n^*} D(x, n, \delta)$ satisfies $\text{Leb}_D(\mathcal{D}_n \cap H_n) \geq \tau \text{Leb}_D(H_n)$.*

5.3. Construction of (non-invariant) hyperbolic measures. Let $\mathcal{B}(\Lambda)$ be the Borel σ -algebra in Λ . We will say that a measure $\nu : (\Lambda, \mathcal{B}(\Lambda)) \rightarrow [0, +\infty]$ is a subprobability if $\nu(\Lambda) \leq 1$. We denote by $\mathcal{S}(\Lambda)$ the set of all Borelian subprobabilities which, by the Schauder's Theorem, is a compact space (in the weak* topology). Define, for each $n \in \mathbb{N}$,

$$\mu_n := \frac{1}{n} \sum_{j=0}^{n-1} f_*^j Leb_D,$$

where Leb_D is the Lebesgue measure in D induced by its Riemannian structure. It is easy to check that any accumulation point μ of $(\mu_n)_n$ is an f -invariant measure. Now define for each $n \in \mathbb{N}$

$$\nu_n := \frac{1}{n} \sum_{j=0}^{n-1} f_*^j Leb_{\mathcal{D}_j} \quad (5.3)$$

where $Leb_{\mathcal{D}_j} := \sum_{x \in H_j^*} Leb_{D(x,j,\delta)}$, and write $\mu_n = \nu_n + \eta_n$. If μ is an accumulation point of $(\mu_n)_{n \in \mathbb{N}}$ in the weak* topology then $\mu = \nu + \eta$, where ν and η are subprobabilities obtained as an accumulation points of $(\nu_n)_{n \in \mathbb{N}}$ and $(\eta_n)_{n \in \mathbb{N}}$, respectively. The following result guarantees such a measure ν has positive mass.

Lemma 5.4. *There is $\alpha > 0$ such that $\nu_n(H) \geq \alpha$ for every large n . Consequently, if ν is an accumulation point of $(\nu_n)_{n \in \mathbb{N}}$ then $\nu(H) \geq \alpha$.*

Proof. The proof is identical to [1, Proposition 3.5]. □

We now describe the support of the measure ν . By construction, the measure ν_n is supported on the union of disks $\cup_{j=0}^{n-1} \Delta_j$, with $\Delta_j := \cup_{x \in H_j^*} \Delta(f^j x, \delta)$, which *may have* self intersections. In particular any accumulation point ν of $(\nu_n)_{n \in \mathbb{N}}$ is such that $supp(\nu) \subseteq \overline{\cap_{n \in \mathbb{N}} \cup_{j=0}^{n-1} \Delta_j}$. The next result asserts that $supp(\nu)$ is contained in a union of unstable manifolds of uniform size.

Proposition 5.5. *Given $y \in supp(\nu)$ exist $\hat{x} = (x_{-n})_{n \in \mathbb{N}} \in M^f$ such that y belongs to a disk $\Delta(\hat{x})$ of radius $\delta > 0$ accumulated by n_j -hyperbolic disks $\Delta(f^{n_j}(x), \delta)$ with $j \rightarrow \infty$, and satisfying that*

- (1) *the inverse branch $f_{x_{-n}}^{-n}$ is well defined in $\Delta(\hat{x})$ for every $n \geq 0$;*
- (2) *if for each $y \in \Delta(\hat{x})$ one takes $y_{-n} := f_{x_{-n}}^{-n}(y)$ then $dist_M(y_{-n}, x_{-n}) \leq e^{-\frac{\epsilon}{2} \cdot n} \delta$ for every $n \geq 0$.*

Proof. The idea is that any point in $\overline{\cap_{n \in \mathbb{N}} \cup_{j=0}^{n-1} \Delta_j}$ is accumulated by j -hyperbolic disks with j going to ∞ and that, using the contraction of j -hyperbolic disk along past j iterates, one can choose a full past trajectory (pre-orbit) along which we observe backward contraction at all iterates.

Given $y \in supp(\nu)$, there is a sequence $(y_j)_{j \in \mathbb{N}}$ with $y_j \in \Delta(f^{n_j}(x_{n_j}), \delta)$ and $x_{n_j} \in H_{n_j}^*$ such that $\lim_{j \rightarrow \infty} y_j = y$. Choosing a subsequence if necessary, we can assume that $\lim_{j \rightarrow \infty} f^{n_j}(x_{n_j}) = x$. As one can write $\Delta(f^{n_j}(x_{n_j}), \delta)$ as the image of the exponential map of the graph of a map $\psi_{f^{n_j}(x_{n_j})} : T_{f^{n_j}(x_{n_j})} D(\delta) \rightarrow E_{f^{n_j}(x_{n_j})}^s$ with $Lip(\psi_{f^{n_j}(x_{n_j})}) < \epsilon_0$. Moreover, by parallel transport, we can identify the disk $\Delta(f^{n_j}(x_{n_j}), \delta)$ with the graph of a map $g_{n_j} : T_x(f^{n_j}(D))(\delta) \rightarrow E_x^s$ in $T_x M$. Since $Lip(\psi_{f^{n_j}(x_{n_j})}) < \epsilon_0$ for any j and the parallel transport is a linear isomorphism then $(g_{n_j})_{j \in \mathbb{N}}$ is a sequence of uniformly bounded and equicontinuous maps which, by Arzelà-Ascoli's theorem, admits a convergent subsequence $\lim_{k \rightarrow \infty} g_{n_{j_k}} = g$. In particular, $\Delta(x) = exp_x(gra f(g))$ is a C^1 disk of radius $\delta > 0$ tangent to the cone field and obtained as a limit of the family of disks $(\Delta(f^{n_j}(x_{n_j}), \delta))_{j \in \mathbb{N}}$. For notational simplicity assume, without loss of generality, that

$$\Delta(x) := \lim_{j \rightarrow \infty} \Delta(f^j(x_j), \delta) \quad \text{and} \quad \lim_{j \rightarrow \infty} f^j(x_j) = x.$$

Since $\Delta(f^j(x_j), \delta) \subset B(f^j(x_j), \delta)$ then there exists $j_0 \in \mathbb{N}$ such that $\Delta(f^j(x_j), \delta) \subset B(x, \delta_1)$ for every $j \geq j_0$. As every point has $d = \deg(f) \geq 1$ preimages, there exists $x_{-1} \in f^{-1}(\{x\})$ such that $f_{x_{-1}}^{-1}(B(x, \delta)) \supset f^{j_k-1}(D(x_{j_k}, j_k, \delta))$ for some subsequence $(j_k)_{k \in \mathbb{N}}$, with $j_k \rightarrow \infty$ when $k \rightarrow \infty$ (see Figure 2 below). Moreover,

$$\begin{aligned} dist_M(f_{x_{-1}}^{-1}(f^{j_k}(y_{j_k})), f_{x_{-1}}^{-1}(f^{j_k}(x_{j_k}))) &\leq dist_{f^{j_k-1}(D(x_{j_k}, j_k, \delta))}(f^{j_k-1}(y_{j_k}), f^{j_k-1}(x_{j_k})) \\ &\leq e^{-\frac{\epsilon}{2}} dist_{\Delta(f^{j_k}(x_{j_k}), \delta)}(f^{j_k}(y_{j_k}), f^{j_k}(x_{j_k})), \end{aligned}$$

for every $y_{j_k} \in D(x_{j_k}, j_k, \delta)$ and for all $k \in \mathbb{N}$ (cf. Proposition 5.2) So, if $y \in \Delta(x)$ and $y = \lim_{j \rightarrow \infty} y_{j_k}$ with $y_{j_k} \in \Delta(f^{j_k}(x_{j_k}), \delta)$ then $y_{-1} := f_{x_{-1}}^{-1}(y)$ satisfies $y_{-1} = \lim_{k \rightarrow \infty} f_{x_{-1}}^{-1}(y_{j_k})$ and $dist_M(y_{-1}, x_{-1}) \leq e^{-\frac{\epsilon}{2}} \delta$. By considering a subsequence of $(x_{j_k})_{k \in \mathbb{N}}$ we may assume $\lim_{k \rightarrow \infty} f^k(x_k) = x$ and $f^{k-1}(D(x_k, k, \delta)) \subset B(x_{-1}, \delta)$

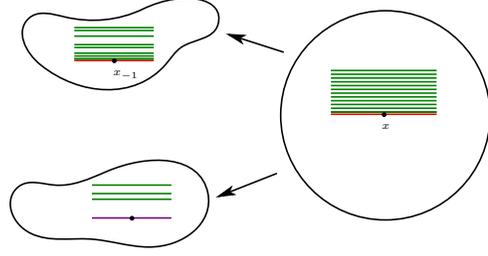


FIGURE 2. Construction of local unstable manifolds through selection of inverse branches

for every large $k \geq 1$. Then, there exists $x_{-2} \in f^{-1}(\{x_{-1}\}) \subset f^{-2}(\{x\})$ such that $f_{x_{-2}}^{-1}(B(x_{-1}, \delta)) \supset f^{k_s-2}(D(x_{k_s}, k_s, \delta))$ for some subsequence k_s of k , where $k_s \rightarrow \infty$ when $s \rightarrow \infty$. The previous argument shows that $\text{dist}_M(y_{-2}, x_{-2}) \leq e^{-\frac{\delta}{2}} \cdot 2\delta < \delta$. Proceeding recursively, we conclude that there exists $\hat{x} \in \pi^{-1}(\{x\})$ such that the inverse branches $f_{x_{-n}}^{-n}$ are well defined in $\Delta(x)$ and that, for every $y \in \Delta(x)$, $\text{dist}_M(y_{-n}, x_{-n}) \leq e^{-\frac{\delta}{2} \cdot n} \delta$ for every $n \in \mathbb{N}$, where $y_{-n} = f_{x_{-n}}^{-n}(y)$. The disk $\Delta(\hat{x}) := \Delta(x)$ is as desired. \square

The previous result says that each point of the support of ν is contained in a disk $\Delta(\hat{x})$, where $\hat{x} \in M^f$ parameterizes the inverse branches along which we observe backward contraction at the disk. Let us denote by \hat{H}_∞ the set of all elements of M^f obtained in this way. Although ν is not necessarily invariant it is supported on true unstable disks.

5.4. Lift of non necessarily invariant measures to M^f . In this section we will construct a lift for ν , that is, a measure $\hat{\nu}$ on M^f such that $\pi_* \hat{\nu} = \nu$. Although the uniqueness of the lift holds only for invariant measures (cf. [37, Proposition I.3.1]), our method of the construction of this measure will allow us to verify the absolute continuity of $\hat{\nu}$ along unstable sets. As observed in Section 4.2, in general the natural extension is not a manifold. Nevertheless, in the case of non-singular endomorphisms it can be locally identified by suitable charts. We will use the following:

Proposition 5.6. *Let M be a compact connected Riemannian manifold and let $f : M \rightarrow M$ be a non-singular endomorphism. There exists $\rho > 0$ such that for every $x \in M$ and $n \in \mathbb{N}$ there is a family of pairwise disjoint open sets $\mathcal{J}_n(x) := \{V_\lambda : \lambda \in I_n(x)\}$ satisfying:*

- (1) $f^{-n}(B(x, \rho)) := \bigcup_{\lambda \in I_n(x)} V_\lambda$
- (2) $f^n|_{V_\lambda} : V_\lambda \rightarrow B(x, \rho)$ is a diffeomorphism, for every $\lambda \in I_n(x)$.

Proof. [4, Theorem 6.5.1]. \square

Recall that $\#f^{-1}(x) = d$ for every $x \in M$ and so the set $I_n(x)$ given by the previous proposition satisfies that $\#I_n(x) = \#f^{-n}(\{x\}) = d^n$ for every $n \geq 1$ (hence it is independent of x). Moreover, given $V_\lambda \in \mathcal{J}_n(x)$ we have that $f^j(V_\lambda) \in \mathcal{J}_{n-k}(x)$, for every $1 \leq k \leq n-1$. This motivates the following coding. We fix $I_n = I_n(x) = \{1, 2, \dots, d\}^n$ for every $n \in \mathbb{N}$ and establish an enumeration for $\mathcal{J}_n(x)$ that will allow us to identify the images of the elements of $\mathcal{J}_n(x)$ in $\mathcal{J}_k(x)$, for $1 \leq k \leq n-1$. For $n=1$ write $\mathcal{J}_1(x) := \{V_1, \dots, V_d\}$. For $n=2$ one can write $\mathcal{J}_2(x) := \{V_{i_2 i_1} : (i_2, i_1) \in I_2\}$ where $f(V_{i_2 i_1}) = V_{i_1} \in \mathcal{J}_1(x)$. By induction, having defined

$$\mathcal{J}_{n-1}(x) := \{V_{i_{n-1} i_{n-2} \dots i_1} : (i_{n-1}, \dots, i_2, i_1) \in I_{n-1}\}.$$

one can write $\mathcal{J}_n(x) := \{V_{i_n i_{n-1} \dots i_1} : (i_n, \dots, i_1) \in I_n\}$ where each $V_{i_n i_{n-1} \dots i_1}$ is determined by $f(V_{i_n i_{n-1} \dots i_1}) = V_{i_{n-1} \dots i_1} \in \mathcal{J}_{n-1}(x)$. This establishes an enumeration for the pre-images of a ball of radius ρ . Furthermore, recalling that $\pi : M^f \rightarrow M$ is the projection in the first coordinate, given $x \in M$ one can identify $\pi^{-1}(B(x, \rho))$ with a product of $B(x, \rho)$ by a fixed Cantor set. Indeed, given $x \in M$ and $\Gamma := \{1, \dots, d\}^{\mathbb{N}}$, the map

$$\begin{aligned} \varphi_x : B(x, \rho) \times \Gamma &\rightarrow \pi^{-1}(B(x, \rho)) \subset M^f \\ (z, (i_n)_{n \in \mathbb{N}}) &\mapsto ((f^n|_{V_{i_n \dots i_1}})^{-1}(z))_{n \in \mathbb{N}} \end{aligned} \quad (5.4)$$

is a homeomorphism (see [4, Theorem 6.5.1]) and it satisfies $\pi \circ \varphi_x(z, c) = z$ for every $(z, c) \in B(x, \rho) \times \Gamma$.

We make use of the previous structure to lift the Lebesgue measure restricted to hyperbolic pre-disks $D(x, n, \delta)$ to M^f . Fix an arbitrary probability measure \mathbb{P} on Γ , reduce δ (if necessary) so that $D(x, n, \delta) \subset$

$B(x, \rho)$ for every $x \in H_n$ and $n \in \mathbb{N}$ and consider the measure

$$\hat{\nu}_n := \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_*^j \hat{m}_j, \quad (5.5)$$

in $\pi^{-1}(D(x, j, \delta)) \subset M^f$, where $\hat{m}_j := \sum_{x \in H_j^*} \hat{m}_{j,x}$ and for each $j \in \mathbb{N}$ and $x \in H_j^*$

$$\hat{m}_{j,x} := (\varphi_x)_* [Leb_{D(x,j,\delta)} \times \mathbb{P}]. \quad (5.6)$$

It is not hard to check that $\pi_* \hat{\nu}_n = \nu_n$ and that if $\lim_{k \rightarrow \infty} \hat{\nu}_{n_k} = \hat{\nu}$ then $\nu = \pi_* \hat{\nu}$ is an accumulation point of $(\nu_n)_{n \in \mathbb{N}}$. Moreover, any accumulation point of the sequence $(\nu_n)_n$ can be obtained as a limit of the push-forwards by π of elements of the sequence $(\hat{\nu}_n)_n$. Finally, using that $\text{supp}(\hat{\nu}_n) \subset \bigcup_{j=0}^{n-1} \bigcup_{x \in H_j^*} \hat{f}^j(\pi^{-1}(D(x, j, \delta)))$

we conclude that $\text{supp}(\hat{\nu}) \subset \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{j=0}^{n-1} \bigcup_{x \in H_j^*} \hat{f}^j(\pi^{-1}(D(x, j, \delta)))}$ for any accumulation point $\hat{\nu}$ of $(\hat{\nu}_n)_n$. By Proposition 5.5, given $y \in \text{supp}(\nu)$ there is a disk $\Delta(\hat{x})$ which is contractive along the pre-orbit $\hat{x} \in M^f$. We proceed to prove $\hat{\nu}$ is supported on a union of unstable manifolds. For that, if $\Delta(\hat{x})$ ($\hat{x} \in \hat{H}_\infty$) is a disk given by Proposition 5.5 then there exists a sequence of points $(z_{j_k})_{k \in \mathbb{N}}$ satisfying $z_{j_k} \in H_{j_k}^*$ for every $k \in \mathbb{N}$ such that $\lim_{k \rightarrow \infty} \Delta(f^{j_k}(z_{j_k}), \delta) = \Delta(\hat{x})$. Since $\lim_{k \rightarrow \infty} f^{j_k - n}(z_{j_k}) = x_{-n}$ for every $n \in \mathbb{N}$ (where the limit is taken over $j_k \geq n$) then it is natural to define:

$$\begin{aligned} \hat{\Delta}(\hat{x}) := & \left\{ \hat{y} \in M^f : \hat{y} = \lim_{s \rightarrow \infty} \hat{y}_{j_{k_s}}, \text{ for some subsequence } (\hat{y}_{j_{k_s}})_{s \in \mathbb{N}} \right. \\ & \left. \text{where } \hat{y}_{j_{k_s}} \in \hat{f}^{j_{k_s}}(\pi^{-1}(D(z_{j_{k_s}}, j_{k_s}, \delta))) \text{ and } (z_{j_{k_s}})_{s \in \mathbb{N}} \subset (z_{j_k})_{k \in \mathbb{N}} \right\}. \end{aligned} \quad (5.7)$$

Roughly, the set $\hat{\Delta}(\hat{x})$ is formed by all points obtained as accumulation points of a sequence $(\hat{y}_{j_k})_{k \in \mathbb{N}}$ where $\hat{y}_{j_k} \in \hat{f}^{j_k}(\pi^{-1}(D(z_{j_k}, j_k, \delta)))$ for every $k \geq 1$. As before the support of $\hat{\nu}$ is contained in the union of this family of sets. Moreover, we have the following:

Lemma 5.5. *Given $\hat{y} \in \text{supp}(\hat{\nu})$, there exists $\hat{x} \in M^f$ and $\hat{\Delta}(\hat{x}) \subset M^f$ (given by (5.7)) such that $\hat{y} \in \hat{\Delta}(\hat{x})$ and $\pi|_{\hat{\Delta}(\hat{x})} : \hat{\Delta}(\hat{x}) \rightarrow \Delta(\hat{x})$ is a continuous bijection.*

Proof. Let $\Delta(\hat{x}) = \lim_{k \rightarrow \infty} \Delta(f^{j_k}(z_{j_k}), \delta)$ be given by Proposition 5.5 and $\hat{\Delta}(\hat{x})$ be given by (5.7). The continuity of π implies that $\pi(\hat{\Delta}(\hat{x})) \subset \Delta(\hat{x})$. In fact, if $\hat{y} \in \hat{\Delta}(\hat{x})$ then $\hat{y} = \lim_{s \rightarrow \infty} \hat{y}_{j_{k_s}}$ where $\hat{y}_{j_{k_s}} \in \hat{f}^{j_{k_s}}(\pi^{-1}(D(z_{j_{k_s}}, j_{k_s}, \delta)))$. So, $\pi(\hat{y}) = \lim_{s \rightarrow \infty} \pi(\hat{y}_{j_{k_s}})$. Moreover, as $\pi(\hat{y}_{j_{k_s}}) \in \Delta(f^{j_{k_s}}(z_{j_{k_s}}), \delta)$ and $\Delta(\hat{x}) = \lim_{k \rightarrow \infty} \Delta(f^{j_k}(z_{j_k}), \delta)$, we conclude that $\pi(\hat{y}) \in \Delta(\hat{x})$. The inclusion $\Delta(\hat{x}) \subset \pi(\hat{\Delta}(\hat{x}))$ is analogous. Indeed, if $y \in \Delta(\hat{x})$ satisfies that $y = \lim_{k \rightarrow \infty} y_{j_k}$ then, choosing \hat{y}_{j_k} such that $\pi(\hat{y}_{j_k}) = y_{j_k}$ and $\hat{y}_{j_k} \in \hat{f}^{j_k}(\pi^{-1}(D(z_{j_k}, j_k, \delta)))$, the compactness of M assures that there exists some convergent subsequence $(\hat{y}_{j_{k_s}})_{s \in \mathbb{N}}$ to some point \hat{y} . Moreover $\hat{y} \in \hat{\Delta}(\hat{x})$ and, since π is continuous, $\pi(\hat{y}) = y$. We will show that $\pi|_{\hat{\Delta}(\hat{x})}$ is injective. We claim that if $\hat{y} \in \hat{\Delta}(\hat{x})$ then $\text{dist}(y_{-n}, x_{-n}) \leq e^{-\frac{\varepsilon}{2}n}\delta$, for all $n \geq 1$. In fact, given $n \geq 1$, Proposition 5.5 implies that $f_{x_{-n}}^{-n}(\Delta(f^{j_k}(z_{j_k}), \delta)) = f^{j_k - n}(D(z_{j_k}, j_k, \delta))$ for every $j_k \geq n$. Then $\hat{y} \in \Delta(\hat{x})$. Therefore $y_{-n} = \lim_{s \rightarrow \infty} f_{x_{-n}}^{-n}(y_{j_{k_s}})$, because $y_{-n} = \pi \circ \hat{f}^{-n}(\hat{y})$ and $f_{x_{-n}}^{-n}(f^{j_{k_s}}(z_{j_{k_s}})) = \pi \circ \hat{f}^{-n}(\hat{y}_{j_{k_s}})$. Moreover, by Proposition 5.5, $\text{dist}(y_{-n}, x_{-n}) \leq e^{-\frac{\varepsilon}{2}n}\delta$ for every $n \geq 1$. This guarantees that if $\pi(\hat{y}) = \pi(\hat{u})$ then $\hat{y} = \hat{u}$ (because points in $\hat{\Delta}(\hat{x})$ are determined by their images by the inverse branches $f_{x_{-n}}^{-n}$, for all $n \in \mathbb{N}$). Therefore, $\pi|_{\hat{\Delta}(\hat{x})}$ is injective and, consequently, $\pi|_{\hat{\Delta}(\hat{x})} : \hat{\Delta}(\hat{x}) \rightarrow \Delta(\hat{x})$ is a bijection. \square

Recall \hat{H}_∞ is the set of points $\hat{x} \in M^f$ given by Lemma 5.5 and that $\text{supp}(\hat{\nu}) \subset \bigcup_{\hat{x} \in \hat{H}_\infty} \hat{\Delta}(\hat{x})$. For all points in the latter set we will construct an invariant unstable subbundle, which will coincide with the unstable Oseledets subspace associated to the SRB measure supported at these points (cf. Proposition 5.8).

Proposition 5.7. *Given $\hat{x} \in \hat{H}_\infty$ and $\hat{y} \in \hat{\Delta}(\hat{x})$ there is a d_u -dimensional subspace $\mathcal{E}_{\hat{y}}^u$ of $T_{\hat{y}} = T_{y_0}M$ so that*

$$\|(Df^m(y_{-m}))^{-1} \cdot v\| \leq e^{-\frac{\varepsilon}{2}m} \|v\| \text{ for every } v \in \mathcal{E}_{\hat{y}}^u \text{ and } m \in \mathbb{N}. \quad (5.8)$$

Proof. Fix $\hat{y} \in \hat{\Delta}(\hat{x})$ and set $\mathcal{E}_{\hat{y}}^u := \bigcap_{n \in \mathbb{N}} Df^n(y_{-n})(C_a(y_{-n}))$. We will show that $\mathcal{E}_{\hat{y}}^u$ is a subspace of $T_{y_0}M$ whose dimension is d_u and satisfies (5.8). By Lemma 5.5, we know that $\hat{y} = \lim_{k \rightarrow \infty} \hat{y}_{n_k}$, for some subsequence $(\hat{y}_{n_k})_{k \in \mathbb{N}}$ where $\hat{y}_{n_k} \in \hat{f}^{n_k}(\pi^{-1}(D(z_{n_k}, n_k, \delta)))$ and $z_{n_k} \in H_{n_k}^*$ for every $k \in \mathbb{N}$. Writing $\hat{y}_{n_k} := (y_{-n_k}^k)_{n \in \mathbb{N}}$ and

$\hat{y} = (y_{-n})_{n \in \mathbb{N}}$ we have that $\hat{y} = \lim_{k \rightarrow \infty} \hat{y}_{n_k}$ if, and only if $\lim_{k \rightarrow \infty} y_{-j}^k = y_{-j}$ for every $j \in \mathbb{N}$. We need the following:

Claim: *If $m \in \mathbb{N}$ and $w \in Df^m(y_{-m})(C_a(y_{-m}))$ then $\|(Df^m(y_m))^{-1} \cdot w\| \leq e^{-\frac{\varepsilon}{2}m} \|w\|$.*

Proof of the claim. In fact, if k is large so that $n_k \geq m$, one can use that $\hat{y}_{n_k} \in \hat{f}^j(\pi^{-1}(D(z_{n_k}, n_k, \delta)))$ to conclude that $y_{-j}^k \in f^{n_k-j}(D(z_{n_k}, n_k, \delta))$ for every $0 \leq j \leq n_k$. In particular $y_{-j}^k \in f^{n_k-j}(D(z_{n_k}, n_k, \delta))$ for every $0 \leq j \leq m$. Proposition 5.2 assures $\text{dist}_M(y_{-j}^k, f^{n_k-j}(z_{n_k})) \leq e^{-\frac{\varepsilon}{2}j} \delta$ for every $0 \leq j \leq m$. Using the continuity of the derivative, as in Lemma 5.3, we obtain that

$$\|Df(y_{-j-1}^k)^{-1} \cdot u_j\| \leq e^{\frac{\varepsilon}{2}} \|(Df(f^{n_k-j-1}(z_{n_k}))|_{C_a(f^{n_k-j-1}(z_{n_k}))})^{-1}\| \cdot \|u_j\|, \quad (5.9)$$

for every $u_j \in C_a(y_{-j}^k)$ and for every $0 \leq j \leq m-1$. Thus, if $w_k \in Df^m(y_{-m}^k)(C_a(y_{-m}^k))$ then

$$\|(Df^m(y_{-m}^k))^{-1} \cdot w_k\| \leq e^{\frac{\varepsilon}{2}m} \prod_{j=n_k-m}^{n_k-1} \|(Df(f^j(z_{n_k}))|_{C_a(f^j(z_{n_k}))})^{-1}\| \cdot \|w_k\| \leq e^{-\frac{\varepsilon}{2}m} \|w_k\|.$$

Here we used that $(Df^j(y_{-j}^k))^{-1} \cdot w_k \in C_a(y_{-j})$ for every $0 \leq j \leq m-1$, because the cone field is invariant and $w_k \in Df^m(y_{-m}^k)(C_a(y_{-m}^k))$, and that z_{n_k} has n_k as c -cone-hyperbolic time. Suppose that $w \in Df^m(y_{-m})(C_a(y_{-m}))$. The continuity of the cone field allow us that we write w as the limit of a sequence $(w_k)_{k \in \mathbb{N}}$ where $w_k \in Df^m(y_{-m}^k)(C_a(y_{-m}^k))$. Since f is C^2 and the cone field is continuous then $\|(Df^m(y_{-m}^k))^{-1} \cdot w_k\| \leq e^{-\frac{\varepsilon}{2}m} \|w_k\|$ for every k . So, passing the limit when $k \rightarrow \infty$ we have that $\|(Df^m(y_{-m}))^{-1} \cdot w\| \leq e^{-\frac{\varepsilon}{2}m} \|w\|$, for every $w \in Df^m(y_{-m})(C_a(y_{-m}))$. This proves the claim. \square

Now the remaining proof of the proposition, that $\mathcal{E}_{\hat{y}}^u$ is a vector subspace, follows a standard route. For each $n \geq 1$ consider the set \mathcal{S}_n formed by subspaces of $T_{y_0}M$ given by

$$\mathcal{S}_n := \{E \leq T_{y_0}M : E \subset Df^n(y_{-n})(C_a(y_{-n})) \text{ and } \dim(E) = d_u\},$$

where $E \leq T_{y_0}M$ means that E is a vector subspace of $T_{y_0}M$. It is not hard to check that \mathcal{S}_n is a compact set (in the Grassmannian topology). Moreover, \mathcal{S}_n is non-empty subset because $F_n := Df^n(y_{-n}) \cdot F_{y_{-n}} \in \mathcal{S}_n$, for every $n \in \mathbb{N}$, where F is the subbundle (not necessarily Df -invariant) used to define the cone field C_a . Furthermore, $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ for every $n \in \mathbb{N}$: if $E := Df^{n+1}(y_{-n-1}) \cdot Z \in \mathcal{S}_{n+1}$ then $E := Df^n(y_{-n}) \cdot [Df(y_{-n-1}) \cdot Z] \in \mathcal{S}_n$. Therefore $\bigcap_{n \in \mathbb{N}} \mathcal{S}_n \neq \emptyset$ and $G \in \bigcap_{n \in \mathbb{N}} \mathcal{S}_n$ if, and only if $G \subset \mathcal{E}_{\hat{y}}^u$. Hence, in order to conclude the proof of the proposition we are left to show that $\# \bigcap_{n \in \mathbb{N}} \mathcal{S}_n = 1$. Suppose there exist $G \neq G' \in \bigcap_{n \in \mathbb{N}} \mathcal{S}_n$ and take $v \in G \setminus G'$. On the one hand $\|[Df^n(y_{-n})]^{-1} \cdot v\| \leq e^{-\frac{\varepsilon}{2}n} \|v\|$, for every $n \in \mathbb{N}$. On the other hand, writting $v = v_s + v'$ where $v_s \in E_{y_0}^s \setminus \{0\}$ and $v' \in G'$, we get that, for every $n \in \mathbb{N}$,

$$\|[Df^n(y_{-n})]^{-1} \cdot v\| \geq \|[Df^n(y_{-n})]^{-1} \cdot v_s\| - \|[Df^n(y_{-n})]^{-1} \cdot v'\| \geq \lambda^{-n} \cdot \|v_s\| - e^{-\frac{\varepsilon}{2}n} \|v'\|,$$

which can hold if and only if $\|v_s\| = 0$. This proves that $v \in G'$, leading to a contradiction. Thus, $\bigcap_{n \in \mathbb{N}} \mathcal{S}_n$ contains a unique vector subspace $\mathcal{E}_{\hat{y}}^u$. This completes the proof of the proposition. \square

Remark 5.1. *Proposition 5.7 implies that for every $\hat{x} \in \hat{H}_\infty$ and $\hat{y} \in \hat{\Delta}(\hat{x})$ there exists a unique splitting $T_{\hat{y}} = E_{y_0}^s \oplus \mathcal{E}_{\hat{y}}^u$ with the backward contraction property (5.8). Given $\hat{y} \in \bigcup_{\hat{x} \in \hat{H}_\infty} \hat{\Delta}(\hat{x})$, the vector subspace*

$$\mathcal{E}_{\hat{f}^{-1}(\hat{y})}^u := \bigcap_{n \in \mathbb{N}} Df^n(y_{-1-n})(C(y_{-1-n}))$$

is clearly an unstable space in $T_{\hat{f}^{-1}(\hat{y})} = T_{y_{-1}}M$. Thus, for every $\hat{x} \in \hat{S}_\infty := \bigcup_{n \geq 0} \hat{f}^{-n}(\hat{\Delta}_\infty)$ there is a Df -invariant splitting $T_{\hat{x}} = E_{x_0}^s \oplus \mathcal{E}_{\hat{x}}^u$, such that $\|D\hat{f}^n(\hat{x})|_{E_{x_0}^s}\| \leq \lambda^n < 1$ and $\|D\hat{f}^{-n}(\hat{x})|_{\mathcal{E}_{\hat{x}}^u}\| \leq e^{-\frac{\varepsilon}{2}n} < 1$ for every $n \in \mathbb{N}$. By [37] we have the existence of local stable manifolds $\{W_{loc}^s(\hat{x})\}_{\hat{x} \in \hat{S}_\infty}$.

The following lemma is useful to distinguish unstable disks and to prove that some ergodic component of the accumulation points of the measures defined by (2.2) have the SRB property.

Lemma 5.6. *There is $\varepsilon > 0$ such that if $\hat{x}, \hat{y} \in \hat{H}_\infty$ and $\text{dist}(x_0, y_0) < \varepsilon$ then either $\Delta(\hat{x}) = \Delta(\hat{y})$ or $\Delta(\hat{x}) \cap \Delta(\hat{y}) = \emptyset$.*

Proof. Fix $z \in M$ and $\varepsilon > 0$ small so that $B(z, \varepsilon) \times \Gamma$ is contained in the domain of the homeomorphism φ_z given by equation (5.4). Consider the set $H_z := \{\hat{x} \in \hat{H}_\infty : \Delta(\hat{x}) \cap B(z, \varepsilon) \neq \emptyset\}$. We claim that for any $\hat{x}, \hat{y} \in H_z$ the sets $\hat{\Delta}(\hat{x}) \cap \pi^{-1}(B(z, \varepsilon))$ and $\hat{\Delta}(\hat{y}) \cap \pi^{-1}(B(z, \varepsilon))$ either coincide or are disjoint. Indeed, since $B(z, \varepsilon) \times \Gamma$ is contained in the domain of the homeomorphism φ_z , $\hat{\Delta}(\hat{x}) \cap \pi^{-1}(B(z, \varepsilon)) = \varphi_z((\Delta(\hat{x}) \cap B(z, \varepsilon)) \times \{\xi\})$ and $\hat{\Delta}(\hat{y}) \cap \pi^{-1}(B(z, \varepsilon)) = \varphi_z((\Delta(\hat{y}) \cap B(z, \varepsilon)) \times \{\zeta\})$ for some $\xi, \zeta \in \Gamma$. In particular, if $\xi \neq \zeta$ then $(\hat{\Delta}(\hat{x}) \cap \pi^{-1}(B(z, \varepsilon))) \cap (\hat{\Delta}(\hat{y}) \cap \pi^{-1}(B(z, \varepsilon))) = \emptyset$. Otherwise, \hat{x} and \hat{y} are pre-orbits that follow the same pre-orbit of z and, by [37, Proposition VII.2.1], either $\hat{\Delta}(\hat{x}) \cap \pi^{-1}(B(z, \varepsilon)) = \hat{\Delta}(\hat{y}) \cap \pi^{-1}(B(z, \varepsilon))$ or $(\hat{\Delta}(\hat{x}) \cap \pi^{-1}(B(z, \varepsilon))) \cap (\hat{\Delta}(\hat{y}) \cap \pi^{-1}(B(z, \varepsilon))) = \emptyset$. \square

5.5. Absolute continuity. A main difference between the case our setting to the case of diffeomorphisms is caused by the fact that the family of unstable disks $\{\Delta(\hat{x})\}_{\hat{x} \in \hat{H}_\infty}$ is not pairwise disjoint. Indeed, there may have infinitely many unstable disks passing through a single point. By Lemma 5.6, there is $\varepsilon \in (0, 1)$ such that if $\text{dist}(x_0, y_0) < \varepsilon$ then either $\Delta(\hat{x}) \cap \Delta(\hat{y}) = \emptyset$ or $\Delta(\hat{x}) = \Delta(\hat{y})$. Intuitively, this means that at balls of radius $\varepsilon > 0$ we only observe disjoint disks (cf. Figure 3).

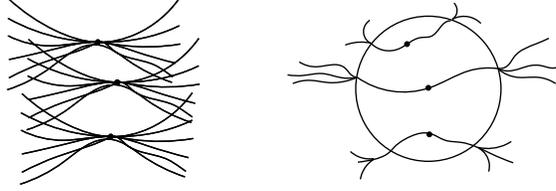


FIGURE 3. Intersection of arbitrary unstable disks (on the left) and intersection of unstable disks only for close pre-orbits (on the right).

Recall that the set $\hat{H}_\infty \subset M^f$ was defined by Proposition 5.5: every $\hat{x} \in \hat{H}_\infty$ is an accumulation point of a sequence $(f^{n_j}(z_{n_j}))_{j \in \mathbb{N}}$, where $z_{n_j} \in H_{n_j}$ and $n_j \rightarrow \infty$ when $j \rightarrow \infty$. Moreover, $\pi|_{\hat{\Delta}(\hat{x})}: \hat{\Delta}(\hat{x}) \rightarrow \Delta(\hat{x})$ is a bijection. Consider the foliated cylinder

$$C_r(\hat{x}) := \bigcup_{y \in \Delta(\hat{x})} W_r^s(y), \quad (5.10)$$

where $W_r^s(y)$ is the connected component of $W_{loc}^s(y) \cap B(y, r)$ that contains y . We will construct a covering of the support of $\hat{\nu}$ by sets of the form $C_r(\hat{x})$. We will show that in each set of this covering that has $\hat{\nu}$ -positive measure we can find a partition in unstable disks and prove that $\pi_* \hat{\nu}_\gamma \ll \text{Leb}_{W_{loc}^u(\hat{z})}$, for almost every γ with respect to such partition. More details are as follows.

Let $\delta > 0$ be the size of the unstable disks obtained in Proposition 5.5 and $0 < r < \delta$ be a lower bound for the size of local stable manifolds. Assume without loss that $C_r(\hat{x}) \subset B(x_0, \varepsilon)$. Using the homeomorphism given by expression (5.4) we can identify $\pi^{-1}(C_r(\hat{x}))$ with $C_r(\hat{x}) \times \Gamma$. Fix $\varepsilon_1 > 0$ and consider $\{\Delta_{\hat{x}, l}\}_{l=1}^n$ a finite open covering of $\overline{\Delta(\hat{x})}$ by balls in such way that the intersection of the sub-cylinders

$$C_r(\hat{x}, l) := \bigcup_{y \in \Delta_{\hat{x}, l}} W_r^s(y) \quad (5.11)$$

with any disk Δ that is tangent to the cone field C_a has diameter smaller than ε_1 in Δ . Clearly, $\pi^{-1}(C_r(\hat{x}, l))$ is identified with $C_r(\hat{x}, l) \times \Gamma$. Let $H^s : C_r(\hat{x}) \rightarrow \Delta(\hat{x})$ be the projection obtained via holonomy along stable manifolds given by $H^s(y) := W_r^s(y) \cap \Delta(\hat{x})$ for every $y \in C_r(\hat{x})$.

We say that $\Delta(f^j(z), \delta)$ crosses $C_r(\hat{x}, l)$ if $H^s|_{\Delta(f^j(z), \delta) \cap C_r(\hat{x}, l)}$ is a diffeomorphism onto $\Delta_{\hat{x}, l}$. Given (\hat{x}, l) , fixed, by some abuse of notation, we will denote by $\Delta(f^j(z), \delta)$ the intersection of the disk $\Delta(f^j(z), \delta)$ with $C_r(\hat{x}, l)$. The choice of ε_1 assures that any disk $\Delta(f^j(z), \delta)$ that intersects $C_r(\hat{x}, l)$ crosses $C_r(\hat{x}, l)$. Define for each $j \in \mathbb{N}$

$$\hat{\mathcal{K}}_j(\hat{x}, l) := \left\{ \hat{f}^j(\pi^{-1}(D(z, j, \delta))) : \Delta(f^j(z), \delta) \text{ crosses } C_r(\hat{x}, l) \right\} \quad (5.12)$$

and

$$\hat{\mathcal{K}}_\infty(\hat{x}, l) := \left\{ \hat{\Delta}(\hat{z}) : \Delta(\hat{z}) \text{ crosses } C_r(\hat{x}, l) \right\}. \quad (5.13)$$

Observe that, since $\{D(z, j, \delta)\}_{z \in H_j^*}$ is a pairwise disjoint family of disks (cf. Proposition 5.4), then the

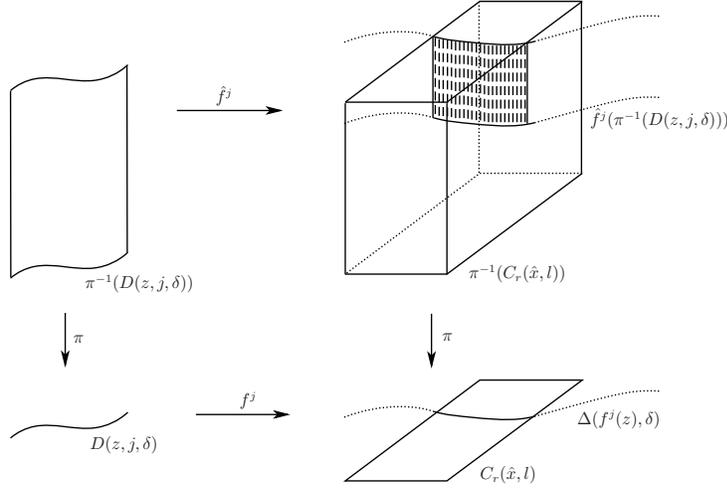


FIGURE 4. Elements in $\hat{\mathcal{K}}_j(\hat{x}, l)$

family $\{\pi^{-1}(D(z, j, \delta))\}_{z \in H_j^*}$ is also pairwise disjoint and, consequently, the same holds for the elements of $\hat{\mathcal{K}}_j(\hat{x}, l)$. Denote by $\hat{K}_j(\hat{x}, l)$ the union of the elements of $\hat{\mathcal{K}}_j(\hat{x}, l)$, for each $0 \leq j \leq \infty$.

Remark 5.2. *By construction, the support of $\hat{\nu}_n|_{\pi^{-1}(C_r(\hat{x}, l))}$ (recall $\hat{\nu}_n$ is the lift of ν_n (5.5)) is contained in $\bigcup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)$ and the support of any accumulation point $\hat{\nu}$ of the sequence $(\hat{\nu}_n)_n$ is contained in $\hat{\mathcal{K}}_\infty(\hat{x}, l)$.*

Since we are interested in describing the disks in the support of the measures $\hat{\nu}_n$ and their accumulation disks it will be useful to show that these measures are not concentrated near the boundary of the disks. For that reason, given $\varepsilon > 0$, let $V_\varepsilon(\partial\Delta(f^j(z), \delta))$ be the ε -neighborhood of the boundary of $\Delta(f^j(z), \delta)$, for $z \in H_j^*$ and $j \in \mathbb{N}$. Define $\Delta_\varepsilon(f^j(z), \delta) := \Delta(f^j(z), \delta) \setminus V_\varepsilon(\partial\Delta(f^j(z), \delta))$ and $D_\varepsilon(z, j, \delta) := f_z^{-j}(\Delta_\varepsilon(f^j(z), \delta))$. Consider the measures

$$\hat{\nu}_{n,\varepsilon} := \frac{1}{n} \sum_{j=0}^{n-1} \hat{f}_*^j \hat{m}_{j,\varepsilon}. \quad (5.14)$$

where $\hat{m}_{j,\varepsilon} := \sum_{z \in H_j^*} \hat{m}_{j,z,\varepsilon}$ and $\hat{m}_{j,z,\varepsilon} := (\varphi_z)_*(Leb_{D_\varepsilon(z,j,\delta)} \times \mathbb{P})$. Then we have the following:

Lemma 5.7. *If $\varepsilon > 0$ is sufficiently small then $\hat{\nu}_{n,\varepsilon}(M^f) \geq \frac{\alpha}{2}$ for every n large enough.*

Proof. Note that $\hat{\nu}_n(M^f) = \nu_n(M) \geq \nu_n(H) \geq \alpha$, for every n sufficiently large (Lemma 5.4). Thus:

$$\begin{aligned} \hat{\nu}_n(M^f) - \hat{\nu}_{n,\varepsilon}(M^f) &\leq \frac{1}{n} \sum_{j=0}^{n-1} \sum_{z \in H_j^*} \hat{f}_*^j \hat{m}_{j,z}(\pi^{-1}(\Delta(f^j(z), \delta) \setminus \Delta_\varepsilon(f^j(z), \delta))) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{z \in H_j^*} \pi_* \hat{f}_*^j \hat{m}_{j,z}(\Delta(f^j(z), \delta) \setminus \Delta_\varepsilon(f^j(z), \delta)) \\ &\leq \frac{1}{n} \sum_{j=0}^{n-1} \sum_{z \in H_j^*} f_*^j Leb_{D(z,j,\delta)}(\Delta(f^j(z), \delta) \setminus \Delta_\varepsilon(f^j(z), \delta)). \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small, the Lebesgue measure of the union of the ε -neighborhoods of the boundary of the disks $\Delta(f^j(z), \delta)$ is a small fraction of the measure of the union of the disks $\Delta(f^j(z), \delta)$. By bounded distortion (cf. Proposition 5.3) this is also true for $f_*^j Leb_D$. Therefore we may reduce ε , if necessary, so that $\sum_{z \in H_j^*} f_*^j Leb_{D(z,j,\delta)}(\Delta(f^j(z), \delta) \setminus \Delta_\varepsilon(f^j(z), \delta)) \leq \frac{\alpha}{2}$, which guarantees that $\hat{\nu}_{n,\varepsilon}(M^f) \geq \hat{\nu}_n(M^f) - \frac{\alpha}{2} \geq \frac{\alpha}{2}$ and proves the lemma. \square

As consequence of the previous lemma we also get;₁₉

Lemma 5.8. *Assume that $(\hat{\nu}_{n_k})_k$ converges to $\hat{\nu}$. There exists (\hat{x}, l) and $\kappa(\hat{x}, l) > 0$ such that $\hat{\nu}(\hat{K}_\infty(\hat{x}, l)) > \kappa(\hat{x}, l)$ and $\hat{\nu}_n(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)) > \kappa(\hat{x}, l)$ for every n in the subsequence of (n_k) .*

Proof. Up to consider a subsequence of $(n_k)_{k \in \mathbb{N}}$, we may assume that $\hat{\nu}_{n_k, \varepsilon}$ converges to some measure $\hat{\nu}_\varepsilon$ (in the weak* topology) so that $\hat{\nu}_\varepsilon(M^f) \geq \frac{\alpha}{2}$ (cf. Lemma 5.7). Note that $\text{supp}(\hat{\nu}_\varepsilon) \subset \cap_{n \in \mathbb{N}} \overline{\cup_{j=0}^{n-1} \hat{K}_{j, \varepsilon}}$ and this set is covered by the union of the interiors of the boxes $\pi^{-1}(C_r(\hat{x}))$. By compactness, there is a finite number of \hat{x} such that $\text{supp}(\hat{\nu}_\varepsilon) \subset \cup_{\hat{x} \in \hat{H}_\infty} (\pi^{-1}(C_r(\hat{x})))$. In consequence, there exists \hat{x} such that $\hat{\nu}_\varepsilon(\pi^{-1}(C_r(\hat{x}))) > 0$ and consequently there is (\hat{x}, l) and $\kappa(\hat{x}, l) > 0$ such that $\hat{\nu}_\varepsilon(\pi^{-1}(C_r(\hat{x}, l))) \geq \kappa(\hat{x}, l) > 0$. By definition of $C_r(\hat{x}, l)$, any disk $\Delta(f^j(z), \delta)$ that intersects $C_r(\hat{x}, l)$ crosses $C_r(\hat{x}, l)$. This implies that $\hat{\nu}_{n, \varepsilon}(\pi^{-1}(C_r(\hat{x}, l))) = \hat{\nu}_{n, \varepsilon}(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l))$. Reducing $r > 0$ and $\Delta_{\hat{x}, l}$ if necessary, we can assume that the $\hat{\nu}$ -measure of the boundary of $\pi^{-1}(C_r(\hat{x}, l))$ is zero. Since $\hat{\nu}_\varepsilon \leq \hat{\nu}$ then $\lim_{k \rightarrow \infty} \hat{\nu}_{n_k, \varepsilon}(\pi^{-1}(C_r(\hat{x}, l))) = \hat{\nu}_\varepsilon(\pi^{-1}(C_r(\hat{x}, l))) \geq \kappa(\hat{x}, l)$. Thus, $\hat{\nu}_n(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)) \geq \hat{\nu}_{n, \varepsilon}(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)) \geq \kappa(\hat{x}, l)$ for every n in the subsequence (n_k) . Finally, since we have that $\limsup_{k \rightarrow \infty} \hat{\nu}_{n_k}(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)) \leq \hat{\nu}(\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l))$ and $\cap_{n \in \mathbb{N}} \overline{\cup_{j=0}^{n-1} \hat{K}_j(\hat{x}, l)} \subset \hat{K}_\infty(\hat{x}, l)$, it follows that $\hat{\nu}(\hat{K}_\infty(\hat{x}, l)) \geq \kappa(\hat{x}, l) > 0$. \square

We now use the local product structure from expression (5.4) to describe the sets $\hat{f}^j(\pi^{-1}(D(z, j, \delta))) \in \hat{K}_j(\hat{x}, l)$. We begin with the following remark.

Remark 5.3. *Fix $\hat{f}^j(\pi^{-1}(D(z, j, \delta))) \in \hat{K}_j(\hat{x}, l)$. The set $\pi^{-1}(D(z, j, \delta))$ is identified with $D(z, j, \delta) \times \Gamma$ via the homeomorphism φ_z . Since the set $\pi^{-1}(D(z, j, \delta)) \subset M^f$ consists of pre-orbits of $D(z, j, \delta)$ then $\hat{f}^j(\hat{\pi}^{-1}(D(z, j, \delta)))$ consists of the pre-orbits of $\Delta(f^j(z), \delta) = f^j(D(z, j, \delta))$ that visit the set $D(z, j, \delta)$. In other words, $\hat{f}^j(\hat{\pi}^{-1}(D(z, j, \delta)))$ is the set of pre-orbits of $\Delta(f^j(z), \delta)$ where the first j pre-images are fixed. Hence the set $\hat{f}^j(\hat{\pi}^{-1}(D(z, j, \delta)))$ is homeomorphic to $\Delta(f^j(z), \delta) \times [x_{-j}, \dots, x_{-1}]_{x_0}$ by φ_{x_0} , where*

$$[x_{-j}, \dots, x_{-1}]_{x_0} := \{(i_n)_{n \in \mathbb{N}} \in \Gamma : x_{-k} \in V_{i_k \dots i_1}(x_0), \text{ para cada } 1 \leq k \leq j\}$$

is a cylinder of length j in Γ . Moreover, given $j \in \mathbb{N}$ and $z \in H_j^*$, if $\Delta(f^j(z), \delta) \subset B(x, \rho)$ consider

$$[z, f(z), \dots, f^j(z)]_x := \{(i_n)_{n \in \mathbb{N}} \in \Gamma : f^{j-k}(z) \in V_{i_n i_{n-1} \dots i_1}(x), 0 \leq k \leq j\},$$

as the set of pre-orbits of x such that $f^{j-k}(z) = f^{-k}(f^j(z))$. Then $[z, f(z), \dots, f^j(z)]_x$ is a cylinder of length j in Γ and the set $\hat{f}^j(\pi^{-1}(D(z, j, \delta))) \in \hat{K}_j(\hat{x}, l)$ is identified with $\Delta(f^j(z), \delta) \times [z, f(z), \dots, f^j(z)]_{x_0}$.

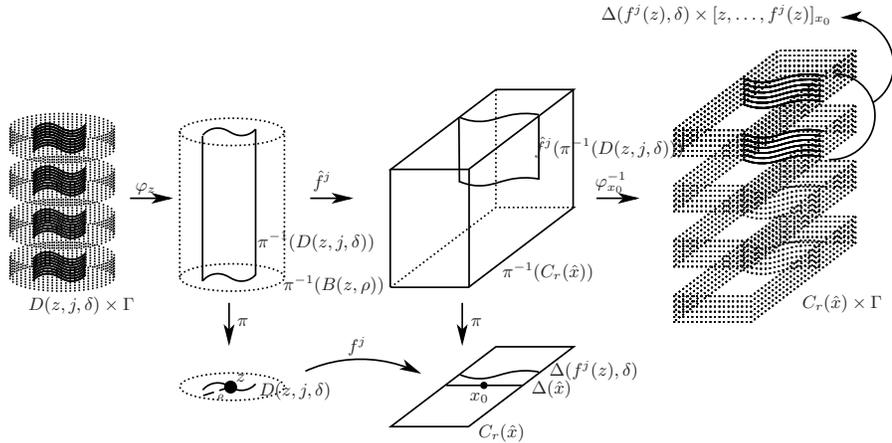


FIGURE 5. Representation of $\hat{f}^j(\pi^{-1}(D(z, j, \delta)))$ in $C_r(\hat{x}) \times \Gamma$

In order to prove absolute continuity we construct an auxiliary sequence of partitions that allow us to study an element $\hat{\nu}_\Delta$ of the disintegration of $\hat{\nu}$ with respect to $\hat{K}_\infty(\hat{x}, l)$ in terms of the sequence of measures $(\hat{\nu}_n)_{n \in \mathbb{N}}$. This construction is one of the most technical steps in the proof.

Fix a pair (\hat{x}, l) such that $\hat{\nu}(\hat{K}_\infty(\hat{x}, l)) > 0$ (cf. Lemma 5.8). The elements of $\hat{K}_\infty(\hat{x}, l)$ are obtained as a limit of $\hat{f}^j(\pi^{-1}(D(z, j, \delta)))$, where $\Delta(f^j(z), \delta) =_{20} f^j(D(z, j, \delta))$ are disks that cross $C_r(\hat{x}, l)$. When no

confusion is possible, by notational simplicity, we shall omit (\hat{x}, l) from the sets $\hat{\mathcal{K}}_j(\hat{x}, l)$, $\hat{K}_j(\hat{x}, l)$ and $C_r(\hat{x}, l)$ previously defined. We denote $\hat{x} = (x_{-n})_{n \in \mathbb{N}}$, $\hat{\Delta} := \hat{\Delta}(\hat{x})$ and $\Delta := \Delta(\hat{x})$. Define

$$K^\dagger := \bigcup_{j \in \mathbb{N} \cup \{\infty\}} \hat{K}_j \times \{j\}.$$

Then, an element of K^\dagger is a pair (\hat{a}, m) where $\hat{a} \in \hat{f}^m(\pi^{-1}(D(z, m, \delta)))$ for some $z \in H_m^*$. Recall that $\hat{f}^m(\pi^{-1}(D(z, m, \delta))) = \varphi_{x_0}(\Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0})$ where $[z, \dots, f^m(z)]_{x_0}$ is a cylinder of length m in $\Gamma = \{1, \dots, d\}^{\mathbb{N}}$. Since φ_{x_0} is a homeomorphism, to each $\hat{a} \in \hat{f}^m(\pi^{-1}(D(z, m, \delta)))$ one can associate a unique pair $(a, \lambda) \in \Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0}$ such that $\varphi_{x_0}(a, \lambda) = \hat{a}$ and $a = \pi(\hat{a})$. Having in mind this identification by φ_{x_0} , we will write an element $(\hat{a}, m) \in K^\dagger$ as a triple $(a, \lambda, m) \in \Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0} \times \mathbb{N}$ where $\varphi_{x_0}(a, \lambda) = \hat{a}$.

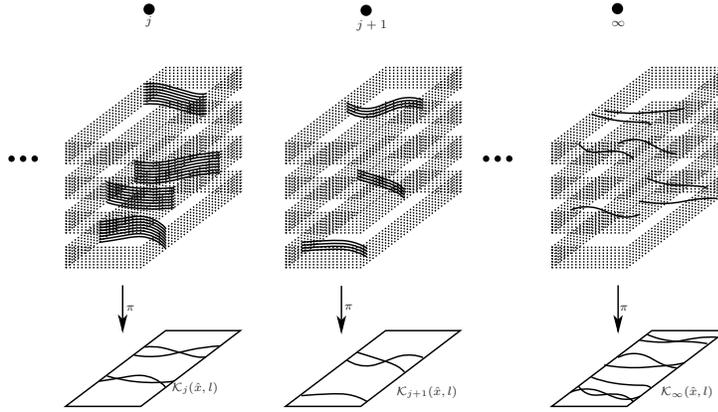


FIGURE 6. The space K^\dagger

In parallel to the construction of $\hat{\nu}_m$ consider the measure ξ_m^\dagger in K^\dagger given by

$$\xi_m^\dagger\left(\bigcup_{j \in \mathbb{N} \cup \{\infty\}} \hat{A}_j \times \{j\}\right) := \frac{1}{m} \sum_{j=0}^{m-1} \hat{f}_*^j \hat{m}_j(\hat{A}_j), \quad (5.15)$$

whenever $\bigcup_{j \in \mathbb{N} \cup \{\infty\}} \hat{A}_j \times \{j\}$ is a subset of K^\dagger (in particular $\hat{A}_j \subset \hat{K}_j$ for every j). The measure $\hat{f}_*^j \hat{m}_j$ is supported in \hat{K}_j and $\hat{\nu}_m(\hat{A}) = \hat{\nu}_m(\bigcup_{j=0}^{m-1} \hat{A} \cap \hat{K}_j) = \xi_m^\dagger(\bigcup_{j=0}^{m-1} (\hat{A} \cap \hat{K}_j) \times \{j\})$ for every $m \in \mathbb{N}$ and $\hat{A} \subset \hat{C}_r$. This implies that $\xi_m^\dagger(K^\dagger) > \kappa > 0$ for some subsequence of integers (given by Lemma 5.8) and that $(\xi_m^\dagger)_{m \in \mathbb{N}}$ admits an accumulation that is a positive measure ξ^\dagger supported in $\hat{K}_\infty \times \{\infty\} \subset K^\dagger$. In particular, $\xi^\dagger(\hat{B} \times \{\infty\}) = \hat{\nu}(\hat{B})$, for every $\hat{B} \subset \hat{K}_\infty$.

We will show that ξ^\dagger has absolutely continuous disintegration with respect to Lebesgue relatively with the partition $\{\hat{\Delta} \times \{\infty\}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$ of $\hat{K}_\infty \times \{\infty\}$ (meaning that $p_*^\dagger \xi^\dagger_{\hat{\Delta} \times \{\infty\}} \ll \text{Leb}_{\pi(\hat{\Delta})}$ where $p^\dagger : K^\dagger \rightarrow \Delta$ is given by $p^\dagger(a, \lambda, n) = H^s(a)$ and H^s is the stable holonomy in C_r) and use this fact to conclude that the same holds for $\hat{\nu}$. For that, we define a sequence of increasing and generating partitions $\{\mathcal{P}_k\}_{k \in \mathbb{N}}$ for K^\dagger that generate $\{\hat{\Delta} \times \{\infty\}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$.

Recall that we can identify $\pi^{-1}(C_r)$ with $C_r \times \Gamma$ by the homeomorphism φ_{x_0} . Consider $z' \in \Delta$ and let $W_r^s(z')$ be the local stable manifold of z' . Take $(\mathcal{W}_k)_{k \in \mathbb{N}}$ a sequence of countable partitions of $W_r^s(z')$ whose diameter goes to zero when k goes to infinity. Fix the sequence $(\Gamma_k)_{k \in \mathbb{N}}$ of partitions of Γ , where each Γ_k is the partition by k -cylinders in Γ (hence their diameter goes to zero when k goes to infinity). Thus, $(\mathcal{W}_k \times \Gamma_k)_{k \in \mathbb{N}}$ is an increasing sequence of countable partitions whose diameter goes to zero when k goes to zero (see Figure 7). So, $\bigvee_{j=0}^\infty \mathcal{W}_k \times \Gamma_k$ is the partition of $W_r^s(z') \times \Gamma$ by singletons.

For each $k \in \mathbb{N}$, we will identify the elements of $\hat{\mathcal{K}}_j$ that intersects the same atom of $\mathcal{W}_k \times \Gamma_k$ to construct the partition \mathcal{P}_k . The atoms of \mathcal{P}_k will consist of the union of elements in $\hat{\mathcal{K}}_j$ that intersect the same atom of $\mathcal{W}_k \times \Gamma_k$ for $j < k$ fixed or consist of the union of elements in $\hat{\mathcal{K}}_j$, for every $j \geq k$, that intersect the same

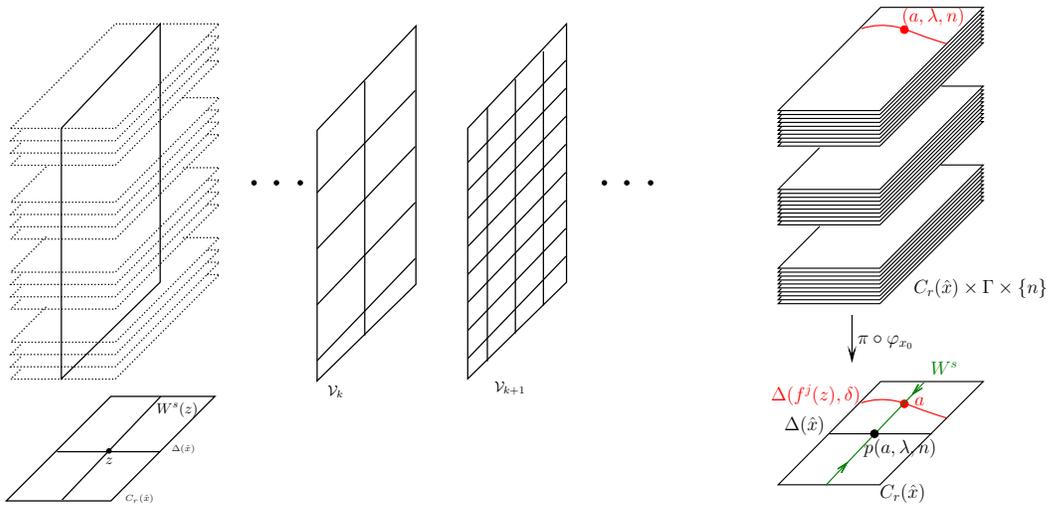


FIGURE 7. Sequence of partitions $(\mathcal{W}_k \times \Gamma_k)_{k \in \mathbb{N}}$ (on the left) and projection p^\dagger (on the right)

atom of $\mathcal{W}_k \times \Gamma_k$. More precisely, we say that $(a, \lambda, n) \in K^\dagger$ and $(b, \sigma, m) \in K^\dagger$ belong to the same atom of \mathcal{P}_k if the following conditions are satisfied:

- (1) there are y and z such that $a \in \Delta(f^n(y), \delta)$ and $b \in \Delta(f^m(z), \delta)$ and the disks $\Delta(f^n(y), \delta)$ and $\Delta(f^m(z), \delta)$ intersect the same atom of \mathcal{W}_k ; or
- (2) λ and σ belong to the same atom of Γ_k ;
- (3) either $[m \geq k$ and $n \geq k]$ or $m = n < k$.

Observe that, since $\Delta(f^n(y), \delta)$ and $\Delta(f^m(z), \delta)$ cross C_r , their intersection with $W_r^s(z')$ is never empty. Conditions (1), (2) and (3) define an equivalence relation in K^\dagger . This implies that \mathcal{P}_k is a partition of K^\dagger .

We claim that the atoms of \mathcal{P}_k consist of unions of sets that are obtained as products of hyperbolic disks by cylinders in Γ by the corresponding cone-hyperbolic times (see Figure 8). In fact, fix $(a, \lambda, m) \in K^\dagger$.

Suppose first that $m < k$. Then $(a, \lambda) \in \Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0}$ for some $z \in H_m^*$. Denote by $\mathcal{P}_k(a, \lambda, m)$ the atom of \mathcal{P}_k that contains (a, λ, m) . It is immediate from the definition that

$$\Delta(f^m(z), \delta) \times ([z, \dots, f^m(z)]_{x_0} \cap \Gamma_k(\lambda)) \times \{m\} \subset \mathcal{P}_k(a, \lambda, m),$$

where $\Gamma_k(\lambda)$ is the atom of the partition Γ_k that contains λ . But $\Gamma_k(\lambda)$ is a cylinder of length $k > m$ that has non empty intersection with the cylinder $[z, \dots, f^m(z)]_{x_0}$ whose length is m . So $[z, \dots, f^m(z)]_{x_0} \supset \Gamma_k(\lambda)$ and $[z, \dots, f^m(z)]_{x_0} \cap \Gamma_k(\lambda) = \Gamma_k(\lambda)$ and, consequently, $\Delta(f^m(z), \delta) \times \Gamma_k(\lambda) \times \{m\} \subset \mathcal{P}_k(a, \lambda, m)$. Now, observe that $(y, \sigma, n) \in \mathcal{P}_k(a, \lambda, m)$ if, and only if:

- $y \in \Delta(f^n(w), \delta)$ for some $w \in H_n^*$ and $\Delta(f^n(w), \delta)$ and $\Delta(f^m(z), \delta)$ intersect the same atom of \mathcal{W}_k ;
- $\sigma \in \Gamma_k(\lambda)$;
- $m = n$.

Then $\Delta(f^n(w), \delta) \times \Gamma_k(\lambda) \times \{m\} \subset \mathcal{P}_k(a, \lambda, m)$. Therefore, we conclude that, if $m < k$ then

$$\mathcal{P}_k(a, \lambda, m) = \bigcup_w \Delta(f^m(w), \delta) \times \Gamma_k(\lambda) \times \{m\},$$

where the union runs over the set of all $w \in H_m^*$ such that $\Delta(f^m(w), \delta)$ and $\Delta(f^m(z), \delta)$ intersect the same atom of \mathcal{W}_k and $[w, \dots, f^m(w)]_{x_0}$ has non empty intersection with $\Gamma_k(\lambda)$.

Now, suppose that $(a, \lambda, m) \in K^\dagger$ with $m \geq k$. As before, $(a, \lambda) \in \Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0}$ for some $z \in H_m^*$ and it follows from the definition that $\Delta(f^m(z), \delta) \times ([z, \dots, f^m(z)]_{x_0} \cap \Gamma_k(\lambda)) \times \{m\} \subset \mathcal{P}_k(a, \lambda, m)$. As $m \geq k$ we have that $[z, \dots, f^m(z)]_{x_0} \cap \Gamma_k(\lambda) = [z, \dots, f^m(z)]_{x_0}$, and $\Delta(f^m(z), \delta) \times [z, \dots, f^m(z)]_{x_0} \times \{m\} \subset \mathcal{P}_k(a, \lambda, m)$. The previous argument show that

$$\mathcal{P}_k(a, \lambda, m) = \bigcup_{j \geq k} \bigcup_{w_j} \Delta(f^j(w_j), \delta) \times [w_j, \dots, f^j(w_j)]_{x_0} \times \{j\} \cup \left(\bigcup_{\hat{x}} \Delta(\hat{x}) \times \{\sigma\} \times \{\infty\} \right)$$

where the union runs over all $w_j \in H_j^*$ such that $\Delta(f^j(w_j), \delta)$ and $\Delta(f^m(z), \delta)$ intersect the same atom of \mathcal{W}_k and $[w_j, \dots, f^j(w_j)]_{x_0}$ has non empty intersection with $\Gamma_k(\lambda)$, and over all $\hat{x} \in \hat{H}_\infty$ such that $\Delta(\hat{x})$ and $\Delta(f^m(z), \delta)$ intersect the same atom of \mathcal{W}_k and $\sigma \in \Gamma_k(\lambda)$. It is clear from the construction that the diameter

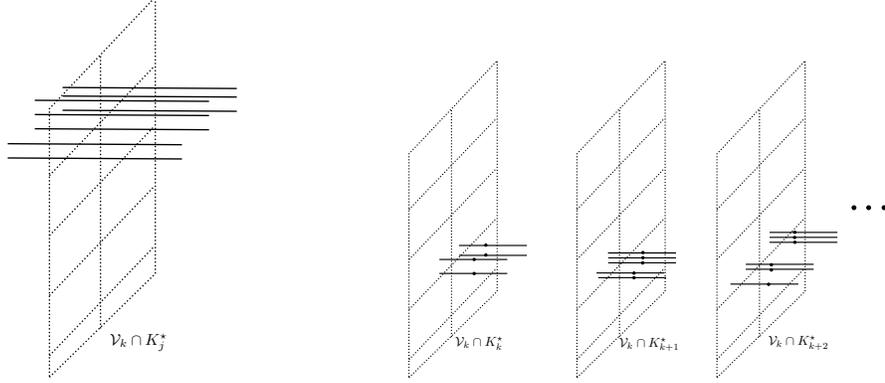


FIGURE 8. Atom of partition \mathcal{P}_k that contains (\hat{u}, j) , where $j < k$ (on the left) and atom of partition \mathcal{P}_k that contains (\hat{u}, j) , where $j \geq k$ (on the right)

of $\mathcal{W}_k \times \Gamma_k$ tends to zero as $k \rightarrow \infty$. Moreover $\mathcal{P}_1(a, \lambda, n) \supset \mathcal{P}_2(a, \lambda, n) \supset \dots \supset \mathcal{P}_k(a, \lambda, n) \supset \dots$ and

$$\bigcap_{k \in \mathbb{N}} \mathcal{P}_k(a, \lambda, n) = \Delta(f^n(z), \delta) \times \{\lambda\} \times \{n\}$$

where $a \in \Delta(f^n(z), \delta)$. In particular, given (a, λ, ∞) we have that $\bigcap_{k \in \mathbb{N}} \mathcal{P}_k(a, \lambda, \infty) = \Delta(\hat{y}) \times \{\lambda\} \times \{\infty\}$ where $a \in \Delta(\hat{y})$. We are now in a position to state the main result of this section.

Theorem 5.1. *There exists $C_3 > 1$ so that the following holds: given (\hat{x}, l) given by Lemma 5.8 there exists a family of conditional measures $(\hat{\nu}_\Delta)_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty(\hat{x}, l)}$ of $\hat{\nu}|_{\hat{K}_\infty(\hat{x}, l)}$ such that $\pi_* \hat{\nu}_\Delta \ll \text{Leb}_\Delta$, where $\Delta = \pi(\hat{\Delta})$, and $\frac{1}{C_3} \text{Leb}_\Delta(B) \leq \pi_* \hat{\nu}_\Delta(B) \leq C_3 \text{Leb}_\Delta(B)$ for every measurable subset $B \subset \Delta$ and for almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty(\hat{x}, l)$.*

Proof. Fix (\hat{x}, l) given by Lemma 5.8, that is, such that $\hat{K}_\infty(\hat{x}, l)$ has positive $\hat{\nu}$ -measure. Consider the space K^\dagger and the sequence of measures $(\xi_m^\dagger)_{m \in \mathbb{N}}$ given by (5.15). Consider the projection $p^\dagger : K^\dagger \rightarrow \Delta$ given by $p^\dagger(a, \lambda, n) = H^s(a)$. Now, recall that the measures ξ_m^\dagger involve the push-forward of the lift of Lebesgue measure at hyperbolic pre-disks as in (5.6) at page 16, where \mathbb{P} is a fixed probability measure in the Cantor set Γ . Hence, in order to prove the theorem it is enough to show that there exists $C > 1$ such that, for any measurable subset $B \subset \Delta$, any integer $k \geq 1$, any $\zeta^\dagger \in K^\dagger$ and any $n \geq 1$ we have:

$$C^{-1} \xi_n^\dagger(\mathcal{P}_k(\zeta^\dagger)) \text{Leb}_\Delta(B) \leq \xi_n^\dagger((p^\dagger)^{-1}(B) \cap \mathcal{P}_k(\zeta^\dagger)) \leq C \xi_n^\dagger(\mathcal{P}_k(\zeta^\dagger)) \text{Leb}_\Delta(B). \quad (5.16)$$

Note that the previous estimates are independent of the choice of the probability measure \mathbb{P} . Fix a measurable subset $B \subset \Delta$ and write $\mathcal{P}_k(\zeta^\dagger)$ as the union of sets $\Delta(f^j(z), \delta) \times \Omega(z, k) \times \{j\}$, where, either $\Omega(z, k) = [z, \dots, f^j(z)]_{x_0}$ or $\Omega(z, k) = \Gamma_k(\lambda)$ with $\zeta^\dagger = (a, \lambda, n)$. In both cases, $\Omega(z, k)$ is a cylinder in Γ . Then

$$\xi_n^\dagger(\mathcal{P}_k(\zeta^\dagger)) = \frac{1}{n} \sum_{j=0}^{n-1} \sum_z \hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}(\Delta(f^j(z), \delta) \times \Omega(z, k))). \quad (5.17)$$

where the sum is over all z such $\Delta(f^j(z), \delta) \times \Omega(z, k) \times \{j\} \subset \mathcal{P}_k(\zeta^\dagger)$. The arguments used in Remark 5.3 guarantee that $\varphi_z^{-1} \circ \hat{f}^{-j} \circ \varphi_{x_0}(\Delta(f^j(z), \delta) \times \Omega_{x_0}(z, k)) = D(z, j, \delta) \times \Omega_z$, where Ω_z still is a cylinder set in Γ . Thus, the summand in (5.17) is

$$\begin{aligned} (\text{Leb}_{D(z,j,\delta)} \times \mathbb{P})(D(z, j, \delta) \times \Omega_z) &= \text{Leb}_{D(z,j,\delta)}(D(z, j, \delta)) \cdot \mathbb{P}(\Omega_z) \\ &= \hat{f}_*^j \text{Leb}_{D(z,j,\delta)}(\Delta(f^j(z), \delta)) \cdot \mathbb{P}(\Omega_z). \end{aligned}$$

Then $\hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}(\Delta(f^j(z), \delta) \times \Omega_{x_0}(z, k))) = \hat{f}_*^j \text{Leb}_{D(z,j,\delta)}(\Delta(f^j(z), \delta)) \cdot \mathbb{P}(\Omega_z)$. Denote by $H_{f^j(z)}^s$ the holonomy map from the hyperbolic disk $\Delta(f^j(z), \delta)$ to the disk Δ . Since $(H_{f^j(z)}^s)^{-1}(B) \subset \Delta(f^j(z), \delta)$, it is not hard to see that the same properties above hold for $(H_{f^j(z)}^s)^{-1}(B)$ instead $\Delta(f^j(z), \delta)$ with the same subsets

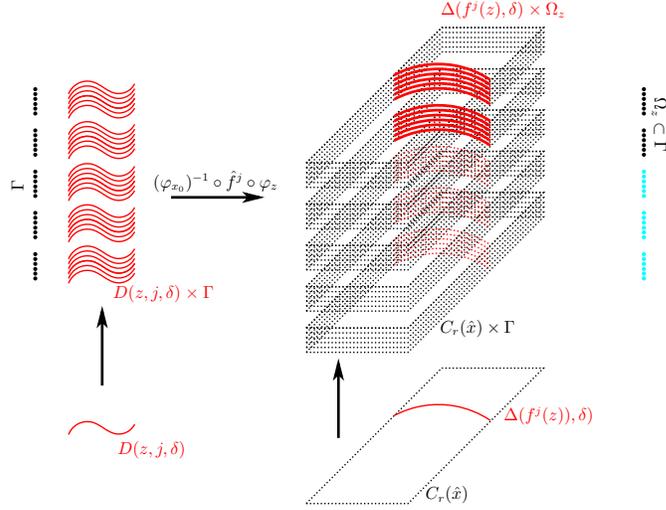


FIGURE 9. Change of cylinders.

of Γ . Then $\hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}((H_{f^j(z)}^s)^{-1}(B) \times \Omega_{x_0}(z, k))) = f_*^j \text{Leb}_{D(z,j,\delta)}((H_{f^j(z)}^s)^{-1}(B)) \cdot \mathbb{P}(\Omega_z)$. By bounded distortion (recall Proposition 5.3) there exists $C_1 > 1$ such that

$$C_1^{-1} \text{Leb}_{\Delta(f^j(z), \delta)}((H^s)^{-1}(B)) \leq f_*^j \text{Leb}_{D(z,j,\delta)}((H^s)^{-1}(B)) \leq C_1 \text{Leb}_{\Delta(f^j(z), \delta)}((H^s)^{-1}(B)).$$

Since the stable holonomy $H_{f^j(z)}^s$ is absolutely continuous with a Jacobian bounded away from zero and infinity (cf. [37, Theorem V.8.1]), there exists $C_2 > 1$ so that $C_2^{-1} \text{Leb}_{\Delta}(B) \leq \text{Leb}_{\Delta(f^j(z), \delta)}((H^s)^{-1}(B)) \leq C_2 \text{Leb}_{\Delta}(B)$ and, consequently, $T_1^{-1} T_2^{-1} \text{Leb}_{\Delta}(B) \leq f_*^j \text{Leb}_{D(z,j,\delta)}((H^s)^{-1}(B)) \leq T_1 T_2 \text{Leb}_{\Delta}(B)$. This proves that

$$\begin{aligned} C_*^{-1} \frac{\text{Leb}_{\Delta}(B)}{\text{Leb}(\Delta)} &\leq \frac{\hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}((H_{f^j(z)}^s)^{-1}(B) \times \Omega_{x_0}(z, k)))}{\hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}(\Delta(f^j(z), \delta) \times \Omega_{x_0}(z, k)))} \\ &= \frac{f_*^j \text{Leb}_{D(z,j,\delta)}((H_{f^j(z)}^s)^{-1}(B))}{f_*^j \text{Leb}_{D(z,j,\delta)}(\Delta(f^j(z), \delta))} \leq C_* \frac{\text{Leb}_{\Delta}(B)}{\text{Leb}(\Delta)} \end{aligned}$$

where $C_* = (T_1 T_2)^2$. Hence we conclude that

$$\begin{aligned} \xi_n^\dagger((p^\dagger)^{-1}(B) \cap \mathcal{P}_k(\zeta^\dagger)) &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_z \hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}((H^s)^{-1}(B) \times \Omega_{x_0}(z, k))) \\ &\leq C_* \frac{\text{Leb}_{\Delta}(B)}{\text{Leb}(\Delta)} \frac{1}{n} \sum_{j=0}^{n-1} \sum_z \hat{f}_*^j \hat{m}_{j,z}(\varphi_{x_0}(\Delta(f^j(z), \delta) \times \Omega_{x_0}(z, k))) \\ &= C_* \frac{\text{Leb}_{\Delta}(B)}{\text{Leb}(\Delta)} \xi_n^\dagger(\mathcal{P}_k^\dagger(\zeta^\dagger)) \end{aligned}$$

for every $B \subset \Delta$ (and the lower bound is obtained analogously). Thus, (5.16) holds with $C = \frac{C_*}{\text{Leb}(\Delta)}$.

To conclude the proof of the theorem, note that if $\lim_{k \rightarrow \infty} \hat{\nu}_{n_k} = \hat{\nu}$ in the weak* topology then $\lim_{k \rightarrow \infty} \xi_{n_k}^\dagger = \xi^\dagger$ and $\text{supp}(\xi^\dagger) \subset \cup_{\hat{\Delta} \in \hat{K}_\infty} \hat{\Delta} \times \{\infty\}$. Moreover, $\xi^\dagger(\Upsilon \times \{\infty\}) = \hat{\nu}(\Upsilon)$. Now, using that $\cap_k \mathcal{P}_k(\hat{\zeta}, \infty) = \hat{\Delta} \times \{\infty\}$ where $\hat{\Delta} \times \{\infty\}$ is the element containing ζ , and taking $n \rightarrow \infty$ at (5.16) we conclude that

$$C^{-1} \text{Leb}_{\Delta}(B) \leq \frac{\xi^\dagger((p^\dagger)^{-1}(B) \cap \mathcal{P}_k(\hat{\zeta}, \infty))}{\xi^\dagger(\mathcal{P}_k(\hat{\zeta}, \infty))} \leq C \text{Leb}_{\Delta}(B).$$

Since $B \subset \Delta$ is arbitrary we conclude the conditional measure of ξ^\dagger in $\hat{\Delta} \times \{\infty\}$ is absolutely continuous with respect to Leb_{Δ} . Equivalently, $\pi_* \hat{\nu}_{\hat{\Delta}}$ is absolutely continuous with respect to Leb_{Δ} . This finishes the proof of the theorem. \square

5.6. Finiteness and uniqueness of SRB measures. In this subsection we first prove that some ergodic component of the measure $\mu = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^j Leb_D$, is an SRB measure. Up to normalization we assume, without loss of generality, that the limiting measure μ described above is a probability measure. Let $\hat{\mu}$ denote the unique \hat{f} -invariant lift of μ . Write $\mu = \nu + \eta$, where ν is an accumulation point of the sequence of measures $(\nu_n)_n$ defined by (5.3) at page 14, and $\eta = \mu - \nu$. By some abuse of notation we assume $\nu = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} f_*^j Leb_{\mathcal{D}_j}$ and assume $\hat{\nu}$ is the lift of ν built along Subsection 5.4. In particular $\pi_* \hat{\nu}_{\hat{\Delta}} \ll Leb_{\pi(\hat{\Delta})}$ for every $\hat{\Delta} \in \hat{K}_\infty$, where $\{\hat{\nu}_{\hat{\Delta}}\}_{\hat{\Delta} \in \hat{K}_\infty}$ is a disintegration of $\hat{\nu}$ with respect to \hat{K}_∞ . Consider the set $R(\hat{f})$ of regular points, that is, the set of points $\hat{z} \in M^f$ such that the limits $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{f}^{-j}(\hat{z}))$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{f}^j(\hat{z}))$ exist and coincide for every continuous $\hat{\phi} : M^f \rightarrow \mathbb{R}$. The set $R(\hat{f})$ is a $\hat{\mu}$ -full measure as a consequence of the ergodic theorem (see e.g. [29, Corollary II.1.4]). Given a measurable partition \mathcal{P} on K_∞ and the Borelian measure ν , let $\tilde{\nu}$ denote the quotient measure induced by \mathcal{P} .

Lemma 5.9. *For $\tilde{\nu}$ -almost every $\Delta \in \mathcal{K}_\infty$ and every $\phi \in C(M)$ there exists $L_\Delta(\phi) \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = L_\Delta(\phi)$ for Leb_Δ -almost every $x \in \Delta$.*

Proof. Since $\hat{\mu}(R(\hat{f})) = 1$ we get $\hat{\nu}(M^f \setminus R(\hat{f})) = 0$. In consequence, $\hat{\nu}_{\hat{\Delta}}(M^f \setminus R(\hat{f})) = 0$ for $\tilde{\nu}$ -almost every $\hat{\Delta} \in \hat{K}_\infty$. Fix $\hat{\Delta} \in \hat{K}_\infty$ and consider the sets $R_{\hat{\Delta}}(\hat{f}) := R(\hat{f}) \cap \hat{\Delta}$ and

$$R_\Delta := \pi(R_{\hat{\Delta}}(\hat{f})). \quad (5.18)$$

Then it is clear that $\hat{\nu}_{\hat{\Delta}}(M^f \setminus R_{\hat{\Delta}}(\hat{f})) = 0$ for almost every $\hat{\Delta} \in \hat{K}_\infty$ and, consequently, R_Δ is a $\pi_* \hat{\nu}_{\hat{\Delta}}$ -full measure subset. In particular R_Δ is a full Leb_Δ -measure subset, because $\pi_* \hat{\nu}_{\hat{\Delta}}$ is absolutely continuous with respect to Leb_Δ with a density bounded away from zero and infinity (cf. Theorem 5.1).

Given a continuous observable $\phi : M \rightarrow \mathbb{R}$ we get $\phi \circ \pi \in C(M^f)$. Moreover, if $x, y \in R_\Delta$ there are $\hat{x}, \hat{y} \in R_{\hat{\Delta}}(\hat{f})$ so that $\pi(\hat{x}) = x$ and $\pi(\hat{y}) = y$ and the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(x_{-j}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^{-j}(\hat{x})) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^j(\hat{x})) \quad (5.19)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(y_{-j}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^{-j}(\hat{y})) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^j(\hat{y})) \quad (5.20)$$

do exist. Now, as the points \hat{x}, \hat{y} belong to $\hat{\Delta}$ we get that $\lim_{j \rightarrow \infty} d(x_{-j}, y_{-j}) = 0$ and the limits in (5.19) and (5.20) coincide. On the other hand, since $\hat{x}, \hat{y} \in R(\hat{f})$, we get that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^j(\hat{x}))$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ \pi(\hat{f}^j(\hat{y}))$ coincide. Thus, using that $f^j \circ \pi = \pi \circ \hat{f}^j$, we also conclude that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(y))$ are equal. As x and y were chosen arbitrarily, we have that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ is constant to some value $L_\Delta(\phi)$ for every $x \in R_\Delta$. Finally, since $Leb_\Delta(\Delta \setminus R_\Delta) = 0$ this completes the proof of the lemma. \square

We are now in a position to guarantee the existence of SRB measures.

Proposition 5.8. *The f -invariant probability measure $\mu = \nu + \eta$ has some ergodic component μ_* that is supported on Λ , is a hyperbolic measure and an SRB measure.*

Proof. Consider $R = \bigcup_{\Delta \in \mathcal{K}_\infty} R_\Delta$ (recall (5.18)) and set $\mathcal{R} = \bigcup_{n \in \mathbb{N}} f^{-n}(R)$. We have that \mathcal{R} is an f -invariant subset: $\mu(\mathcal{R} \cap f^{-1}(\mathcal{R})) = 0$ by Poincaré's recurrence theorem. We claim that for every $\phi \in C(M)$ there is $L(\phi) \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = L(\phi)$ for every $x \in R$, which means that the constant given by Lemma 5.9 depends on the disk Δ . This part of the proof follows the ideas from the Hopf's argument for ergodicity. By Lemma 5.9, there exists $L_\Delta(\phi) \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = L_\Delta(\phi)$ for every $x \in R_\Delta$. Given $\Delta, \Delta' \in \mathcal{K}_\infty$, consider the holonomy $H^s : \Delta \rightarrow \Delta'$ along the stable foliation. Since R_Δ and $R_{\Delta'}$ are full measure sets for Leb_Δ and $Leb_{\Delta'}$, respectively and the stable foliation is absolutely continuous ([37, Theorem V.8.1]) we have that $H^s(R_\Delta) = R_{\Delta'}$ modulo a zero Leb_Δ measure subset. As $\tilde{\phi}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$ (whenever well defined) is constant along stable manifolds, then we get

that $L_\Delta(\phi) = L_{\Delta'}(\phi) = L(\phi)$, for any $\Delta, \Delta' \in \mathcal{K}_\infty$, which proves the claim. Now, given $x \in \mathcal{R}$, there is $n_0(x) \in \mathbb{N}$ such that $f^{n_0}(x) \in R$. In particular, if $\phi \in C(M)$ is arbitrary, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(f^{n_0}(x))) = L(\phi) \quad \text{for every } x \in \mathcal{R}. \quad (5.21)$$

Moreover, $\mu(\mathcal{R}) \geq \mu(R) \geq \nu(R) = \pi_* \hat{\nu}(R) = \hat{\nu}(\pi^{-1}(R)) \geq \hat{\nu}(R(\hat{f})) = \hat{\nu}(\hat{K}_\infty) \geq \alpha > 0$. Thus one can define $\mu_* := \mu|_{\mathcal{R}}$, where $\mu|_{\mathcal{R}}(A) = \frac{\mu(A \cap \mathcal{R})}{\mu(\mathcal{R})}$, for every measurable set $A \subset M$. It is clear that μ_* is a positive f -invariant probability measure and, by (5.21), it is ergodic.

We now prove that μ_* is a hyperbolic measure. One can write $\mu_* = \mu|_{\mathcal{R}} = \nu|_{\mathcal{R}} + \eta|_{\mathcal{R}}$. As consequence of Proposition 5.7, for every $\Delta \in \mathcal{K}_\infty$ and $y \in \Delta$ there is a pre-orbit \hat{y} of y that admits a $D\hat{f}^{-1}$ -invariant splitting $T_{\hat{y}} = E_{\hat{y}}^s \oplus \mathcal{E}_{\hat{y}}^u$ where $E_{\hat{y}}^s$ is uniformly contracting and $\mathcal{E}_{\hat{y}}^u$ is uniformly contracting for the past along the pre-orbit \hat{y} . In particular, given $y \in R$ let $\hat{y} \in M^f$ be such pre-orbit and $T_{\hat{y}} = E_{\hat{y}}^s \oplus \mathcal{E}_{\hat{y}}^u$ be the corresponding splitting. Since μ_* is ergodic and $\mu_*(R) > \nu(R) > 0$ then, by Proposition 4.1, the Lyapunov exponents of μ_* coincide with the Lyapunov exponents at typical points \hat{y} which are all non-zero. Thus μ_* is a hyperbolic measure.

We are left to show that μ_* is an SRB measure. It is easy to check that $\hat{\mu}_* := \hat{\mu}|_{\pi^{-1}(\mathcal{R})}$ is the (uniquely defined) lift of μ_* . Moreover, we have the following:

Claim 5.1. *If $\{\hat{\mu}_{*,\hat{\Delta}}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$ is a disintegration of $\hat{\mu}_*$ with respect to the partition $\hat{\mathcal{K}}_\infty$ then $\pi_* \hat{\mu}_{*,\hat{\Delta}} \ll \text{Leb}_\Delta$, for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$.*

Proof of the claim. For each $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$ we have that $\pi|_{\hat{\Delta}} : \hat{\Delta} \rightarrow \Delta$ is a homeomorphism. Taking $\hat{m}_{\hat{\Delta}} := (\pi|_{\hat{\Delta}})_*^{-1} \text{Leb}_\Delta$ we have that $\pi_* \hat{\mu}_{*,\hat{\Delta}} \ll \text{Leb}_\Delta$ if, and only if $\hat{\mu}_{*,\hat{\Delta}} \ll \hat{m}_{\hat{\Delta}}$. Hence we will prove the second. Assume, by contradiction, that there exists $\hat{A} \subset \hat{\mathcal{K}}_\infty$ so that $\hat{\mu}_*(\hat{A}) > 0$ but $\hat{m}_{\hat{\Delta}}(\hat{A} \cap \hat{\Delta}) = 0$ for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$. As $\hat{\mu}_*(\hat{A}) > 0$ we get that $\hat{\mu}_{*,\hat{\Delta}}(\hat{A} \cap \hat{\Delta}) > 0$ for every $\hat{\Delta}$ in some subset $\hat{\mathcal{K}}'_\infty \subset \hat{\mathcal{K}}_\infty$ with positive $\hat{\mu}_*$ -measure. Consider the saturated set $\hat{B} = \bigcup_{j=-\infty}^{\infty} \hat{f}^j(\hat{A})$, which is \hat{f} -invariant subset and, by ergodicity, has full $\hat{\mu}_*$ -measure. Then $\hat{\mu}_{*,\hat{\Delta}}(\hat{B} \cap \hat{\Delta}) = 1$, for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$. On the other hand, using that $\hat{m}_{\hat{\Delta}}(\hat{A} \cap \hat{\Delta}) = 0$ for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$ and that $\hat{f}_*^j \hat{m}_{\hat{\Delta}} \ll \hat{m}_{\hat{f}^j(\hat{\Delta})}$ for every $j \in \mathbb{Z}$, we conclude that $\hat{m}_{\hat{\Delta}}(\hat{B} \cap \hat{\Delta}) = 0$ for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$. Hence $\hat{\mu}_{*,\hat{\Delta}} \perp \hat{m}_{\hat{\Delta}}$ for $\hat{\mu}_*$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$. However, this is a contradiction with the fact that $\hat{\mu}_{*,\hat{\Delta}} = \hat{\nu}_{\hat{\Delta}} + \hat{\eta}_{\hat{\Delta}}$ on $\hat{\mathcal{K}}_\infty$, with $\hat{\nu}_{\hat{\Delta}} \ll \hat{m}_{\hat{\Delta}}$ (cf. Theorem 5.1). This proves the claim. \square

We have that the partition $\mathcal{P} := \bigvee_{j \in \mathbb{N}} \hat{f}^{-j} \hat{\mathcal{K}}_\infty$ of \mathcal{R} is subordinated to unstable manifolds (see e.g. [37, Proposition VII.2.4]) and, the claim assures that $\pi_* \hat{\mu}_{*,\hat{P}} \ll \text{Leb}_{\pi(\hat{P})}$ for almost all $\hat{P} \in \mathcal{P}$. Now, if \mathcal{Q} is any other partition subordinated to unstable manifolds then the same property holds for $\mathcal{Q} \vee \mathcal{P}$. Then, since every element of \mathcal{Q} is obtained as sum of elements of $\mathcal{Q} \vee \mathcal{P}$ we conclude that μ_* is an SRB measure. This completes the proof of the proposition. \square

In order to conclude the finiteness of SRB measures that cover H we prove that each of their basins of attraction occupies a definite proportion of the phase space. This is done in two steps.

Lemma 5.10. *There exists $b > 0$ such that the following holds: if μ_* is an SRB measure given by Proposition 5.8 then there exists an open set $V \subset M$ contained in $B(\mu_*)$ (mod zero) so that $\text{Leb}(V) > b$.*

Proof. Note that $\hat{\mu}_*(B(\hat{\mu}_*)) = 1$. In particular, there is $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$ such that $\hat{\mu}_{*,\hat{\Delta}}$ almost every \hat{x} in $\hat{\Delta}$ belongs to $B(\hat{\mu}_*)$. Consequently, $\hat{m}_{\hat{\Delta}}$ -almost every $\hat{x} \in \hat{\Delta}$ belongs to $B(\hat{\mu}_*)$. Moreover, $\pi(B(\hat{\mu}_*)) \subset B(\mu_*)$. Thus Leb_Δ -almost every point $x \in \Delta$ belongs to $B(\mu_*)$. The union of stable manifolds of points in $B(\mu_*) \cap \Delta$ is also contained in $B(\mu_*)$, because the Birkhoff's average converges to the same limit along stable manifolds. Using the absolute continuity of the stable foliation and that stable manifolds have size uniformly bounded away from zero, we conclude that $V = \bigcup_{y \in \Delta} W_r^s(y)$ is an open set contained in $B(\mu_*)$, except for a zero measure subset. Thus $\text{Leb}(V)$ is uniformly bounded away from zero, which proves the lemma. \square

Actually every SRB measure can be obtained by the procedure described along Section 5. Indeed:

Lemma 5.11. *If μ is an SRB measure for $f|_\Lambda$ then there exists a disk $D \subset U$ such that $\text{Leb}_D(D \cap B(\mu) \cap H) = \text{Leb}_D(D)$.*

Proof. Let $\hat{\mathcal{P}}$ be a partition of M^f subordinated to unstable manifolds of μ . Then each atom of $\hat{\mathcal{P}}$ projects over a disk in some unstable manifold. The lift $\hat{\mu}$ of μ is an ergodic measure and is such $\hat{\mu}(B(\hat{\mu})) = 1$. In particular, if $\{\hat{\mu}_{g, \hat{P}}\}_{\hat{P} \in \hat{\mathcal{P}}}$ is a disintegration of $\hat{\mu}_g$ with respect to $\hat{\mathcal{P}}$ then $\hat{\mu}_{\hat{P}}(B(\hat{\mu})) = \hat{\mu}_{\hat{P}}(\hat{P})$ for $\hat{\mu}$ -almost every $\hat{P} \in \hat{\mathcal{P}}$. By the latter and the fact that μ is an SRB measure, there exists $\hat{P} \in \hat{\mathcal{P}}$ such that $\hat{\mu}_{\hat{P}}(B(\hat{\mu})) = \hat{\mu}_{\hat{P}}(\hat{P})$ and that $\mu_D := \pi_* \hat{\mu}_{\hat{P}} \ll \text{Leb}_D$, where $D := \pi(\hat{P})$. Notice that $\mu_D(B(\mu) \cap D) = \mu_D(D)$, because $\mu_D(B(\mu)) = \pi_* \hat{\mu}_{\hat{P}}(B(\mu)) \geq \hat{\mu}_{\hat{P}}(B(\hat{\mu})) = \hat{\mu}_{\hat{P}}(\hat{P}) = \mu_D(D)$. This proves that μ_D -almost every $x \in D$ belongs to $B(\mu)$ and, since μ_D and Leb_D are equivalent, Leb_D -almost every $x \in D$ belongs to $B(\mu)$. Consider $\mathcal{C} = \bigcup_{y \in D} W_{loc}^s(y)$. Since $\text{Leb}(H) > 0$ and the holonomy along the stable foliation is absolutely continuous we conclude that there exists a disk \tilde{D} crossing \mathcal{C} such that $\text{Leb}_{\tilde{D}}(H) = \text{Leb}_{\tilde{D}}(\tilde{D})$, which proves the lemma. \square

Corollary 5.1. *If μ is an SRB measure for $f|_\Lambda$ then there is a disk D tangent to the cone field \mathcal{C} so that $\omega_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \text{Leb}_D$ is convergent to μ (in the weak* topology), where Leb_D is the normalized Lebesgue measure in a disk D for which there are constants $C > 0$, $\xi \in (0, 1)$ and $c > 0$ such that:*

- (1) *given $k \geq 1$ and an injectivity domain $\mathcal{D}_k \subset D$ for f^k , the map*

$$f^j(\mathcal{D}_k) \ni x \mapsto \log |\det(Df(x)|_{T_x f^j \mathcal{D}_k})|$$

is (L, ξ) -Hölder continuous for every $0 \leq j \leq k - 1$.

- (2) *$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df(f^j(x))|_{T_{f^j(x)} f^j D})^{-1}\| \leq -2c < 0$ for Leb_D -almost every $x \in D$.*

Moreover the constants c and ξ depend only of the constants in (H1)-(H4).

Proof. Let $D \subset M$ be a disk given by Lemma 5.11. Proposition 5.1 implies that (1) holds. Property (2) is immediate from the fact that $\text{Leb}_D(H) = \text{Leb}_D(D)$ (cf. Lemma 5.11). We are left to prove that μ is the limit of the Césaro averages $\omega_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \text{Leb}_D$, $n \in \mathbb{N}$. Assume that $\omega = \lim_{k \rightarrow \infty} \omega_{n_k}$ is an accumulation point of $(\omega_n)_n$. Given an arbitrary $\phi \in C(M)$, using the dominated convergence theorem and the fact that $\text{Leb}_D(B(\mu)) = \text{Leb}_D(D)$, we conclude that

$$\int \phi d\omega = \lim_{k \rightarrow \infty} \int \frac{1}{n_k} \sum_{j=0}^{n_k-1} \phi \circ f^j d\text{Leb}_D = \int \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \phi \circ f^j d\text{Leb}_D = \int \phi d\mu.$$

As ϕ was taken arbitrarily, it follows that $\omega = \mu$. Hence $(\omega_n)_n$ is convergent to the SRB measure μ_g . Finally recall that the constants c and ξ , given by Proposition 5.1, depend only on the cone field \mathcal{C}_a and (H3). \square

Finally, to complete the proof of Theorem A we need only show uniqueness of the SRB measure in the case f is transitive. By Lemma 5.10, as basins of attraction of different SRB measures have empty intersection, there is a finite number of SRB measures for f . As each one of them contain a set which has Lebesgue measure bounded away from zero, we must have a finite number of them. Moreover, if f is transitive the SRB measure is unique. Indeed, if μ_1 and μ_2 are SRB measures given by Lemma 5.10 there are open sets $V_1 \subset B(\mu_1)$ and $V_2 \subset B(\mu_2) \pmod{0}$. The transitivity of f implies that exist $N_0 \in \mathbb{N}$ such that $f^{N_0}(V_1) \cap V_2 \neq \emptyset$. Taking a smaller open set, if necessary, we may assume that $f^{N_0}|_{U_1}$ is a diffeomorphism onto its image, and so $f^{N_0}(V_1) \cap V_2 \subset B(\mu_2)$ is an open set $\pmod{0}$. Similarly, $f^{-N_0}(f^{N_0}(U_1) \cap U_2) \cap U_1 \subset B(\mu_1) \pmod{0}$. Thus there exists $x \in M$ so that $x \in B(\mu_1)$ and $f^{N_0}(x) \in B(\mu_2)$, which implies that $\mu_1 = \mu_2$. Thus, if f is transitive the SRB measure is unique.

6. STATISTICAL STABILITY

The main goal of this section is the proof of Theorem B. More precisely, in this section we prove that if (H1), (H2), (H3) and (H4) hold robustly with uniform constants on the attractors of an open set \mathcal{V} of $C^{1+\alpha}$ endomorphisms then the corresponding SRB measures vary continuously (in the weak* topology). By continuous variation we mean that if $(f_n)_{n \in \mathbb{N}}$ is a sequence in \mathcal{V} converging to $f \in \mathcal{V}$ and if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of probability measures where μ_n is a SRB measure for f_n , for every $n \in \mathbb{N}$, then any accumulation point of $(\mu_n)_{n \in \mathbb{N}}$ in the weak* topology is a convex sum of the SRB measures for f .

First we establish the context. Denote by $\text{End}^{1+\alpha}(M)$ the set of non-singular endomorphisms of class $C^{1+\alpha}$ in M . Assume that Λ is an attractor for f and that $U \supset \Lambda$ an open neighborhood of Λ such that $\bigcap_{n \geq 0} f^n(\bar{U}) = \Lambda$. For every $g \in \text{End}^{1+\alpha}(M)$ that is $C^{1+\alpha}$ -close to f define $\Lambda_g := \bigcap_{n \geq 0} g^n(\bar{U})$ is an attractor

for g . Assume that there are an open neighborhood $\mathcal{V} \subset \text{End}^{1+\alpha}(M)$ of f , constants $\lambda \in (0, 1)$, $c > 0$ and a family of cone fields $U \ni x \rightarrow C(x)$ of constant dimension $0 < d_u \leq \dim(M)$, such that (H1)-(H4) holds with uniform constants on U : for every $g \in \mathcal{V}$ there exists a continuous splitting $T_U M = E^s(g) \oplus F(g)$ so that

- (H1) $Dg(x) \cdot E_x^s(g) = E_{g(x)}^s(g)$ for every $x \in U$;
- (H2) $\|Dg^n(x)|_{E_x^s(g)}\| \leq \lambda^n$ for every $x \in U$ and $n \in \mathbb{N}$
- (H3) the cone field $U \ni x \rightarrow C(x)$ contains the subspace $F(g)$, and is such that $Dg(x)(C(x)) \subseteq C(g(x))$ for every $x \in U$, and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Dg(g^j(x))|_{C(g^j(x))}^{-1})\| \leq -2c < 0$$

for Lebesgue almost every $x \in U$; and

- (H4) $\|Dg(x) \cdot v\| \|Dg(x)^{-1} \cdot w\| \leq \lambda \|v\| \cdot \|w\|$ for every $v \in E_x^s(g)$, all $w \in Dg(x)(C(x))$ and $x \in U$.

We will say that \mathcal{V} is an open set of partially hyperbolic $C^{1+\alpha}$ -endomorphisms with uniform constants. For each $g \in \mathcal{V}$, we denote by H_g the subset of U that consists of points satisfying the non-uniform hyperbolicity condition in (H3). As a direct consequence of Theorem A and the construction of SRB measures along Section 5 we have the following:

Corollary 6.1. *Every $g \in \mathcal{V}$ has finitely many SRB measures for g supported on Λ_g and, if $g|_{\Lambda_g}$ is transitive then the SRB measure is unique. Moreover, every SRB measure μ_g for g is the limit of the sequence of measures $(\frac{1}{n} \sum_{j=0}^{n-1} g_*^j \text{Leb}_D)_n$ for some C^1 -disk D tangent to the cone field.*

Given $g \in \mathcal{V}$, let μ_g be an SRB measure for g . In particular, for any partition $\hat{\mathcal{P}}$ of M^g subordinated to unstable manifolds we have that $\pi_* \hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})} \ll \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}$, for $\hat{\mu}_g$ almost every $\hat{x} \in M^g$, where $\hat{\mu}_g$ is the lift of μ_g in M^g , $\hat{\mathcal{P}}(\hat{x})$ is the atom of $\hat{\mathcal{P}}$ that contains \hat{x} , $\hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})}$ is the element of the disintegration of $\hat{\mu}_g$ in $\hat{\mathcal{P}}(\hat{x})$ and the map $\pi = \pi^g : M^g \rightarrow M$ is the natural projection. By the Radon-Nikodym theorem there is a measurable function $\rho_{\hat{x}} : \pi(\hat{\mathcal{P}}(\hat{x})) \rightarrow \mathbb{R}$ that is positive $\pi_* \hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})}$ -almost everywhere and so that

$$\pi_* \hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})}(B) = \int_B \rho_{\hat{x}} d\text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))} \quad (6.1)$$

for every measurable subset $B \subset \pi(\hat{\mathcal{P}}(\hat{x}))$. Recall $\pi|_{\hat{\mathcal{P}}(\hat{x})} : \hat{\mathcal{P}}(\hat{x}) \rightarrow \pi(\hat{\mathcal{P}}(\hat{x}))$ is a bijection and that $\pi(\hat{\mathcal{P}}(\hat{x}))$ is contained in $W_{loc}^u(\hat{x})$. Define $\hat{m}_{\hat{\mathcal{P}}(\hat{x})} := (\pi|_{\hat{\mathcal{P}}(\hat{x})})^* \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}$, where $(\pi|_{\hat{\mathcal{P}}(\hat{x})})^* \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}(\hat{A}) = \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}(\pi|_{\hat{\mathcal{P}}(\hat{x})}(\hat{A}))$ for every measurable subset $\hat{A} \subset \hat{\mathcal{P}}(\hat{x})$. Observe that $\pi_* \hat{m}_{\hat{\mathcal{P}}(\hat{x})} = \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}$ and that $\pi_* \hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})} \ll \text{Leb}_{\pi(\hat{\mathcal{P}}(\hat{x}))}$ if, and only if $\hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})} \ll \hat{m}_{\hat{\mathcal{P}}(\hat{x})}$. Again, by the Radon-Nikodym theorem there is a measurable function $\hat{\rho} : \hat{\mathcal{P}}(\hat{x}) \rightarrow \mathbb{R}$ that is positive in $\hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})}$ almost every point such that $\hat{\mu}_{g, \hat{\mathcal{P}}(\hat{x})}(\hat{B}) = \int_{\hat{B}} \hat{\rho} d\hat{m}_{\hat{\mathcal{P}}(\hat{x})}$, for every measurable subset $\hat{B} \subset \hat{\mathcal{P}}(\hat{x})$.

We are now in a position to prove Theorem B on the statistical stability of these SRB measures. The proof, strongly inspired by [49], relies on a uniform control of the structures used in the construction of SRB measures for these partially hyperbolic endomorphisms. A brief sketch of the proof is as follows. First we prove that if μ_g is an SRB measure for $g \in \mathcal{V}$ then there exists a family of disjoint unstable disks whose size is bounded away from zero. This bound is uniform for every $g \in \mathcal{V}$. So, given a sequence $(g_n)_{n \in \mathbb{N}}$ in \mathcal{V} converging to $g \in \mathcal{V}$ and a sequence $(\mu_n)_{n \in \mathbb{N}}$, where μ_n is a SRB measure for g_n , and an accumulation point μ of $(\mu_n)_{n \in \mathbb{N}}$ the family of unstable disks \mathcal{K}_n associated to g_n and μ_n accumulate in a family \mathcal{K} of unstable disks for g with respect to μ . Using that each μ_n is absolutely continuous with respect to Lebesgue measure on the disks of \mathcal{K}_n with uniform constants) we will then show that μ is absolutely continuity of with respect to Lebesgue on the disks of \mathcal{K} . A further argument will show that almost all ergodic components of μ are absolutely continuous on the disks of \mathcal{K} and, consequently, are SRB measures, proving that μ is an convex combination of the latter.

Given a disk $\Delta \subset M$ denote $C_r^g(\Delta) := \bigcup_{y \in \Delta} W_r^s(g, y)$, where $W_r^s(g, y)$ is the disk of radius $r > 0$ centered at y in the local stable manifold $W_{loc}^s(g, y)$. We say that a disk $\tilde{\Delta}$ of the same dimension of Δ crosses $C_r^g(\Delta)$ if the stable holonomy $H^s : \tilde{\Delta} \rightarrow \Delta$ is a diffeomorphism. Denote by $\mathcal{K}_r^g(\Delta)$ the family of local unstable manifolds that cross $C_r^g(\Delta)$, and by $K_r^g(\Delta)$ the union of its elements.

Lemma 6.1. *There are $\delta > 0$, $r > 0$ and $\eta > 0$ so that: for every $g \in \mathcal{V}$ and for every SRB measure μ_g for $g|_{\Lambda_g}$, there exists a disk $\Delta := \Delta(g, \mu_g)$ of radius $\delta > 0$ such that $\mu_g(K_r^g(\Delta)) > \eta$.*

Proof. Fix $g \in \mathcal{V}$ and an SRB measure μ_g for $g|_{\Lambda_g}$. By Corollary 5.1 and Proposition 5.5, μ_g is absolutely continuous with respect to Lebesgue on unstable disks, and $\text{supp}(\mu_g)$ is contained in the union of cylinders $C_r^g(\Delta)$, where Δ is an unstable disk of radius $\delta(g) > 0$. As the constant $\delta > 0$ is the size of hyperbolic disks of the non uniformly hyperbolic endomorphism, this size depends only on the size of invertibility domain of the exponential map, the size of the domain of inverse branches and the uniform continuity of the map and these quantities vary continuously with $g \in \mathcal{V}$ (recall Proposition 5.2), δ can be taken uniformly in the neighborhood \mathcal{V} . Moreover, since the stable manifolds vary continuously with the dynamics, we can take $r_g := r > 0$ for every $g \in \mathcal{V}$. In particular there exists $b_0 \in \mathbb{N}$ such that $\text{supp}(\mu_g)$ is contained in at most b_0 cylinders of the form $C_r^g(\Delta)$, with Δ unstable disks of radius $\delta > 0$. This assures that there exists a disk $\Delta = \Delta(g, \mu_g)$ of radius δ such that $\mu_g(C_r^g(\Delta)) \geq \eta$, where $\eta = \frac{1}{b_0} > 0$. This proves the lemma. \square

Given $g \in \mathcal{V}$ and μ_g , let $C_g := C_r^g(\Delta)$ and $\mathcal{K}_g := \mathcal{K}_r^g(\Delta)$ be the family of local unstable manifolds that cross $C_r^g(\Delta)$, where Δ is given by the previous lemma. Let $\hat{\mathcal{K}}_g$ be the family of the lifts of the elements of \mathcal{K}_g . We have the following result:

Proposition 6.1. *There exists $\tau > 0$ such that if $g \in \mathcal{V}$, μ_g is an SRB measure for g , $\hat{\mu}_g$ is the lift of μ_g and $\{\hat{\mu}_{g,\hat{\Delta}}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_g}$ is a disintegration of $\hat{\mu}_g$ with respect to $\hat{\mathcal{K}}_g$ then*

$$\tau^{-1} \text{Leb}_{\Delta}(B) \leq \pi_* \hat{\mu}_{g,\hat{\Delta}}(B) \leq \tau \text{Leb}_{\Delta}(B) \quad (6.2)$$

for every measurable subset $B \subset \Delta$ and almost every $\Delta \in \mathcal{K}_g$.

Proof. Combining Corollary 5.1 and Theorem 5.1, for every $g \in \mathcal{V}$ there exists $\tau(g) > 1$ such that (6.2) holds. Observe that the constant $\tau(g)$ depends on the constant of bounded distortion along the images of the disk D_g (which can be used to recover μ_g as in Corollary 5.1) by iterates of g , and the regularity of the stable holonomy. The first constant comes from the regularity of the Jacobian given at Proposition 5.1, which can be taken uniformly in \mathcal{V} . Moreover, the stable holonomy varies continuously in \mathcal{V} because the stable manifolds vary continuously with $g \in \mathcal{V}$. Hence there exists a uniform constant $\tau > 1$ so that (6.2) holds for every $g \in \mathcal{V}$. \square

Now, fix a sequence $\{g_n\}_{n \in \mathbb{N}}$ in \mathcal{V} and assume that it converges to $g \in \mathcal{V}$ in the $C^{1+\alpha}$ -topology. Denote $\mathcal{K}_n := \mathcal{K}_{g_n}$, $\hat{\mathcal{K}}_n := \hat{\mathcal{K}}_{g_n}$ and $C_n := C_{g_n} = C_r^{g_n}(\Delta_n)$ for each $n \in \mathbb{N}$. Assume that for every $n \in \mathbb{N}$ the probability measure μ_n is an SRB measure for g_n and that μ is an accumulation point of $(\mu_n)_{n \in \mathbb{N}}$. It is clear that μ is a g -invariant probability.

Recall that each C_n is a cylinder formed by the union of stable manifolds passing through points of a disk Δ_n of radius δ tangent to the cone field \mathcal{C} . By compactness of M and the Arzela-Ascoli theorem we may assume, up to consider some subsequence, that $\{\Delta_n\}_n$ converges to a disk Δ_∞ of uniform radius δ . Define $C_\infty := \cup_{y \in \Delta_\infty} W_r^s(y, g)$, where $r > 0$ is given by the definition of cylinders C_n , and

$$\mathcal{K}_\infty := \left\{ D \subset M : D = \lim_{k \rightarrow \infty} D_{n_k} \text{ where } D_{n_k} \in \mathcal{K}_{n_k}, D \subset C \right\}.$$

In rough terms, \mathcal{K}_∞ is the family of disks tangent to the cone field \mathcal{C} , obtained as the limit of unstable disks for some subsequence $\{g_{n_k}\}_k$. As every disk $D_n \in \mathcal{K}_n$ is tangent to the cone field \mathcal{C} , then the same property holds for the elements of \mathcal{K}_∞ . Moreover:

Lemma 6.2. *Every $D \in \mathcal{K}_\infty$ is an unstable disk (with respect to some pre-orbit) for g . Moreover, if $\eta > 0$ is given by Lemma 6.1 then $\mu(\mathcal{K}_\infty) \geq \eta > 0$.*

Proof. Fix $D \in \mathcal{K}_\infty$ and, for notational simplicity, assume $D = \lim_{n \rightarrow \infty} D_n$. If $x \in D$ is so that $D \subset B(x, \delta)$ then $D_n \subset B(x, \delta)$ for n sufficiently large and, consequently, every g_n has well defined inverse branches in $B(x, \delta)$. Moreover, by definition, there exists $0 < \sigma < 1$ so that for every $n \in \mathbb{N}$ there exists an inverse branch $g_{n,i_n}^{-1} : B(x, \delta) \rightarrow V_{i_n}$ for which $\text{dist}(g_{n,i_n}^{-1}(z), g_{n,i_n}^{-1}(y)) \leq \sigma \text{dist}(z, y)$ for every $z, y \in D_n$. By the construction of the disks D we have that the sequence $\{D_n^{-1}\}_n$ of disks defined by $D_n^{-1} := g_{n,i_n}^{-1}(D_n)$ are tangent to the cone field \mathcal{C} . Thus, extracting a subsequence if necessary, the Arzela-Ascoli theorem assures that $\{D_n^{-1}\}_n$ is convergent to some disk D^{-1} . Moreover, $g(D^{-1}) = D$, the disk D^{-1} has radius smaller or equal to $\sigma\delta$ and the corresponding inverse branch of g from D to D^{-1} is a contraction by σ . Applying this argument

recursively we obtain contractive inverse branches for g^{-j} on D for every $j \in \mathbb{N}$, which implies that each $D \in \mathcal{K}$ is a unstable disk for some pre-orbit of g .

We are left to prove that $\mu(\mathcal{K}_\infty) \geq \eta > 0$. Every $D \in \mathcal{K}_\infty$ is liftable to a set $\hat{D} \subset M^g$ for which $\pi = \pi^g|_{\hat{D}}: \hat{D} \rightarrow D$ is a bijection. Moreover, these sets are pairwise disjoint and admit contracting and expanding directions (recall Proposition 5.7 and Lemma 5.6). By Lemma 6.1, $\mu_n(K_n) \geq \eta > 0$ for every $n \geq 0$. Moreover, if $V_\varepsilon(K_\infty)$ denotes the ε -neighborhood of K_∞ and $\varepsilon > 0$ is chosen such that $\mu(\partial V_\varepsilon(K_\infty)) = 0$ then $\mu(V_\varepsilon(K_\infty)) = \lim_{n \rightarrow \infty} \mu_n(V_\varepsilon(K_\infty)) \geq \liminf_{n \rightarrow \infty} \mu_n(K_n) \geq \eta > 0$, because $K_n \subset V_\varepsilon(K_\infty)$ for every large n . Therefore, we conclude that $\mu(K_\infty) = \liminf_{\varepsilon \rightarrow 0} \mu(V_\varepsilon(K_\infty)) \geq \eta > 0$. This proves the lemma. \square

Denote by $\hat{\mathcal{K}}_\infty$ the family of lifts of disks in \mathcal{K}_∞ and by \hat{K}_∞ the union of its elements. Since the proof of the following lemma is identical to the one of Theorem 5.1 for \mathcal{K}_∞ we shall omit it.

Lemma 6.3. *There exists $\tau_0 > 1$ such that if $\{\hat{\mu}_{\hat{\Delta}}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$ is a disintegration of $\hat{\mu}$ with respect to $\hat{\mathcal{K}}_\infty$ then $\pi_*\hat{\mu}_{\hat{\Delta}}$ is absolutely continuous with respect to Leb_Δ , where $\Delta = \pi(\hat{\Delta})$. Moreover, $\tau_0^{-1}Leb_\Delta(B) \leq \pi_*\hat{\mu}_{\hat{\Delta}}(B) \leq \tau_0 Leb_\Delta(B)$ for every measurable set $B \subset \Delta$.*

We need the following instrumental result:

Lemma 6.4. [1, Lemma 6.2]. *Let ν be a finite measure on a measure space Z , with $\nu(Z) > 0$. Let \mathcal{X} be a measurable partition of Z , and $(\nu_z)_{z \in Z}$ be a family of finite measures on Z such that:*

- (a) *the function $z \mapsto \nu_z(A)$ is measurable and it is constant on each element of \mathcal{X} , for any measurable subset $A \subset Z$.*
- (b) *$\{w : \nu_z = \nu_w\}$ is a measurable subset with full ν_z -measure, for every $z \in Z$.*

Assume that $\nu(A) := \int l(z)\nu_z(A)dv$ for some measurable function $l : Z \rightarrow \mathbb{R}_+$ and any measurable subset $A \subset Z$. Let $\{\nu_\gamma : \gamma \in \mathcal{X}\}$ and $\{\nu_{z,\gamma} : \gamma \in \mathcal{X}\}$ be disintegrations of ν and ν_z , respectively, into conditional probability measures along the elements of the partition \mathcal{X} . Then $\nu_{z,\gamma} = \nu_\gamma$ for ν -almost every $z \in Z$ and $\tilde{\nu}_z$ -almost every $\gamma \in \mathcal{X}$, where $\tilde{\nu}_z$ is the quotient measure induced by ν_z on \mathcal{X} .

We now prove that almost every ergodic component of $\hat{\mu}$ is an SRB measure. The ergodic decomposition theorem (see e.g. [29, Theorem II.6.4]) implies that there is a measurable subset $\hat{\Sigma}(\hat{g}) \subset M^g$, with full $\hat{\mu}$ -measure, such that the limit $\hat{\mu}_{\hat{x}} := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\hat{g}^j(\hat{x})}$ exists and defines an ergodic measure, for every $\hat{x} \in \hat{\Sigma}(\hat{g})$. Moreover, for every bounded and measurable observable $\hat{\phi} : M^g \rightarrow \mathbb{R}$: (i) the map $\hat{x} \mapsto \int \hat{\phi} d\hat{\mu}_{\hat{x}}$ is measurable and $\int \hat{\phi} d\hat{\mu} = \int (\int \hat{\phi} d\hat{\mu}_{\hat{x}}) d\hat{\mu}(\hat{x})$, and (ii) $\int \hat{\phi} d\hat{\mu}_{\hat{x}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^j(\hat{x}))$ for $\hat{\mu}$ almost every $\hat{x} \in M^g$. As before, if $\{\hat{\mu}_{\hat{\Delta}}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$ is a disintegration of $\hat{\mu}$ then $\pi_*\hat{\mu}_{\hat{\Delta}} \ll Leb_\Delta$ if and only if $\hat{\mu}_{\hat{\Delta}} \ll \hat{m}_{\hat{\Delta}}$, where $\hat{m}_{\hat{\Delta}} := (\pi|_{\hat{\Delta}})^* Leb_\Delta$. Fix $\hat{B} \subset M^g$ such that $\hat{m}_{\hat{\Delta}}(\hat{B} \cap \hat{\Delta}) = 0$ for every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$ and $\hat{\mu}(\hat{B})$ is maximal among all sets with this property. By Lemma 6.3 we have that $\hat{\mu}_{\hat{\Delta}} \ll \hat{m}_{\hat{\Delta}}$. Therefore, if $\tilde{\mu}$ is the quotient measure given by Rokhlin's disintegration theorem, we conclude that $\hat{\mu}(\hat{B}) = \int \int \hat{\mu}_{\hat{\Delta}}(\hat{B}) d\tilde{\mu}(\hat{\Delta}) = 0$.

Recall that the set of generic points $R(\hat{g})$ is defined by as the set of points $\hat{y} \in M^g$ such that the limits $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^j(\hat{y}))$ and $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^{-j}(\hat{y}))$ exist and coincide for every $\hat{\phi} \in C(M^g)$. Consider also the set $\hat{Z} := [\hat{K}_\infty \cap \Sigma(\hat{g}) \cap R(\hat{g})] \setminus \hat{B}$. The previous lemma allow us to compare the disintegration of $\hat{\mu}$ with respect to $\hat{\mathcal{K}}_\infty$ with the disintegration of the ergodic components of $\hat{\mu}$ with respect to $\hat{\mathcal{K}}_\infty$. In what follows we prove that almost every ergodic component of μ is an SRB measure.

Proposition 6.2. *Let $\{\hat{\mu}_{\hat{x},\hat{\Delta}}\}_{\hat{\Delta} \in \hat{\mathcal{K}}_\infty}$ be a disintegration of $\hat{\mu}_{\hat{x}}$ along $\hat{\mathcal{K}}_\infty$. Then $\hat{\mu}_{\hat{x},\hat{\Delta}} = \hat{\mu}_{\hat{\Delta}}$, hence absolutely continuous with respect to Lebesgue, for $\hat{\mu}$ -almost every $\hat{x} \in \hat{K}_\infty$ and $\tilde{\mu}$ -almost every $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$.*

Proof. We apply Lemma 6.4 with $Z = \hat{K}_\infty$, $\nu = \hat{\mu}|_{\hat{K}_\infty}$, $\mathcal{X} = \hat{\mathcal{K}}_\infty$ and $\nu_z = \hat{\mu}_z$, for each $\hat{z} \in \hat{K}_\infty$. Since $R(\hat{g})$ and $\Sigma(\hat{g})$ are full $\hat{\mu}$ -measure sets and $\hat{\mu}(\hat{B}) = 0$ then $\hat{\mu}(\hat{Z}) = \hat{\mu}(\hat{K}_\infty \cap R(\hat{g}) \cap \Sigma(\hat{g})) = \hat{\mu}(\hat{K}_\infty) \geq \eta > 0$. Given a fixed measurable subset $\hat{A} \subset M^g$, the map $\hat{Z} \ni \hat{x} \mapsto \hat{\mu}_{\hat{x}}(\hat{A})$ is measurable, by the ergodic decomposition theorem. Fix $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$ and $\hat{x}, \hat{y} \in \hat{Z} \cap \hat{\Delta}$. Given $\hat{\phi} \in C(M^g)$ we have that $\int \hat{\phi} d\hat{\mu}_{\hat{x}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^j(\hat{x})) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^{-j}(\hat{x}))$ and $\int \hat{\phi} d\hat{\mu}_{\hat{y}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^j(\hat{y})) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \hat{\phi}(\hat{g}^{-j}(\hat{y}))$ because $\hat{x}, \hat{y} \in R(\hat{g}) \cap \Sigma(\hat{g})$. But, since \hat{x}, \hat{y} belong to the same unstable set $\hat{\Delta}$ then $\int \hat{\phi} d\hat{\mu}_{\hat{x}} = \int \hat{\phi} d\hat{\mu}_{\hat{y}}$ and, since $\hat{\phi} \in C(M^g)$ was taken arbitrary, $\hat{\mu}_{\hat{x}} = \hat{\mu}_{\hat{y}}$. This proves that $\hat{x} \mapsto \hat{\mu}_{\hat{x}}(\hat{A})$ is constant on each $\hat{\Delta} \in \hat{\mathcal{K}}_\infty$. Moreover, it is a standard in ergodic theory to show that the set $E_{\hat{z}} := \{\hat{w} : \hat{\mu}_{\hat{w}} = \hat{\mu}_{\hat{z}}\}$ is a full $\hat{\mu}_{\hat{z}}$ -measure subset in M^g .

We are left to show that there is a measurable function $l : Z \rightarrow \mathbb{R}_+$ such that $\hat{\mu}(\hat{A}) = \int l(\hat{x}) \cdot \hat{\mu}_{\hat{x}}(\hat{A}) d\hat{\mu}(\hat{x})$ for every measurable subset $\hat{A} \subset \hat{Z}$. Denote by $\mathbf{1}_{\hat{A}}$ the characteristic function in the set \hat{A} . Then $\hat{\mu}_{\hat{x}}(\hat{A}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{\hat{A}}(\hat{g}^j(\hat{x}))$, which is positive if and only if \hat{x} has positive frequency of visits to $\hat{A} \subset \hat{Z}$. Given $\hat{z} \in \hat{Z}$ define $l(\hat{z}) := \min\{r > 0 : \hat{g}^{-r}(\hat{z}) \in \hat{Z}\}$. By Poincaré's Recurrence theorem the map $l(\cdot)$ is well defined $\hat{\mu}$ -almost everywhere in \hat{Z} . Then, following [1, Lemma 6.1], $\hat{\mu}(\hat{A}) = \int_{\Sigma} \hat{\mu}_{\hat{x}}(\hat{A}) d\hat{\mu}(\hat{x}) = \int_{\hat{Z}} l(\hat{z}) \hat{\mu}_{\hat{z}}(\hat{A}) d\hat{\mu}(\hat{z})$. So, we can apply Lemma 6.4 to conclude the proof of this proposition. \square

Consider $K = \pi(\hat{K}_{\infty})$. The following completes the proof of Theorem B.

Corollary 6.2. *For μ -almost every $x \in K$ the measure μ_x is an SRB measure for g . In particular, μ is a convex combination of SRB measures for g .*

Proof. Observe that for each $\hat{y} \in M^g$ we have that $\pi_* \delta_{\hat{y}} = \delta_{y_0}$ and, from the linearity and continuity of the projection $\pi_* : \mathcal{M}(M^g) \rightarrow \mathcal{M}(M)$, we have that $\pi_* \hat{\mu}_{\hat{x}} = \mu_{x_0}$ for every $\hat{x} \in \Sigma(\hat{g})$, where $\mu_{x_0} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x_0)}$. Moreover if $\hat{B} \subset M^g$ is $\hat{\mu}$ -full measure set then $\pi(\hat{B}) \subset M$ is a μ -full measure set, where $\mu = \pi_* \hat{\mu}$. Therefore we conclude that for μ -almost every $x \in K_{\infty}$, μ_x is such that its lift $\hat{\mu}_x$ has disintegration that coincides with the disintegration of $\hat{\mu}$ along the unstable disks of \hat{K}_{∞} almost everywhere. But, as the disintegration of $\hat{\mu}$ is absolutely continuous with respect to the Lebesgue measure in the unstable disks in \hat{K}_{∞} , we get that μ_x is an SRB measure. \square

7. PROOF OF THE COROLLARIES

7.1. Proof of Corollary A. Let \mathcal{U} be a open set of transitive $C^{1+\alpha}$ non-singular endomorphisms that satisfy (H1)-(H4) with uniform constants and, for any $f \in \mathcal{U}$ let μ_f be its unique SRB measure. By ergodicity of μ_f , Pesin's entropy formula for endomorphisms ([37, Theorem VII.1.1], after[24]), assures that

$$h_{\mu_f}(f) = \int \sum_{\lambda_i(f, \mu_f) > 0} \lambda_i(f, \mu_f) \dim E_i d\mu_f.$$

Therefore, by Oseledets theorem,

$$\begin{aligned} \int \log |\det Df| d\mu_f &= \int \sum_{\lambda_i(f, \mu_f) > 0} \lambda_i(f, \mu_f) \dim E_i d\mu_f + \int \sum_{\lambda_i(f, \mu_f) < 0} \lambda_i(f, \mu_f) \dim E_i d\mu_f \\ &= h_{\mu_f}(f) + \int \sum_{\lambda_i(f, \mu_f) < 0} \lambda_i(f, \mu_f) \dim E_i d\mu_f. \end{aligned} \quad (7.1)$$

Now, recall that μ_f is a hyperbolic measure and that all Oseledets subspaces associated to negative Lyapunov exponents are contained in the stable subbundle E^s . In consequence, applying the Oseledets theorem to the continuous cocycle $Df|_{E^s}$ we conclude that

$$\int \log |\det Df| d\mu_f = h_{\mu_f}(f) + \int \log |\det Df|_{E^s} d\mu_f$$

or, equivalently,

$$h_{\mu_f}(f) = \int \log \frac{|\det Df|}{|\det Df|_{E^s}} d\mu_f. \quad (7.2)$$

Since the stable subbundle E^s for f varies continuously with the non-singular endomorphism f (with the Grassmannian topology) and the function $\mathcal{U} \rightarrow C^0(M, \mathbb{R})$ given by $f \mapsto \log \frac{|\det Df|}{|\det Df|_{E^s}}$ is continuous, the continuity of the entropy function follows from the latter together with the continuity of the SRB measure $\mathcal{U} \ni f \mapsto \mu_f$ (in the weak* topology). This proves the corollary.

7.2. Proof of Corollary C. Let $\mathcal{U} \subset \text{End}^{1+\alpha}(M)$ be the (C^1 -open) set of non-singular endomorphisms on M satisfying (A1)-(A4). The corollary is an immediate consequence of Theorems A and B once we prove that assumptions (H1)-(H4) hold with uniform constants. Given $g \in \mathcal{U}$ condition (A1) implies that for every $x \in U$

$$E_x^s(g) = \bigcap_{n \geq 0} Dg^n(x)^{-1}(C_a^-(g^n(x))) \subsetneq C_a^-(x)$$

defines an Dg -invariant contracting subbundle $E^s(g)$ with contraction rate λ . Moreover, if $v \in E_x^s$, $w \in Df(x)(C_a(x))$ and $x \in M$ then $\|Dg(x) \cdot v\| \cdot \|Dg(x)^{-1} \cdot w\| \leq \lambda \max_{x \in M} L(x)^{-1} \cdot \|v\| \cdot \|w\| \leq \sqrt{\lambda} \|v\| \cdot \|w\|$ provided that

$$L := \min_{x \in M} L(x) \geq \sqrt{\lambda} \quad (7.3)$$

This implies (H1), (H2) and (H4) hold with the uniform constant $\tilde{\lambda} = \sqrt{\lambda} \in (0, 1)$.

It remains to prove that assumption (H3) holds. It is enough to prove that there exists $c > 0$ such that, for every disk D tangent to the cone field $C(\cdot)$, $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Dg(g^j(x)))|_{C(g^j(x))}^{-1}\| \leq -2c < 0$ for Lebesgue almost every $x \in D$. Fix any disk D tangent to $C(\cdot)$. Condition (A3) together with the positive Dg -invariance of the cone field C implies that $\det|Dg(x)|_{T_x D} > \sigma$ for every $x \in D$ and that this property holds for the positive iterates of the disk D . Now, given $n \geq 1$ and $\gamma \in (0, 1)$, consider the set

$$R(D, n, \gamma) := \{x \in D : \#\{0 \leq j \leq n-1 : g^j(x) \in \mathcal{O}\} \geq \gamma n\}$$

of points that spends a fraction of time larger than γ in the region \mathcal{O} .

Lemma 7.1. *There are $\gamma_0 \in (\frac{1}{2}, 1)$, $\varepsilon > 0$ and $K > 0$ (depending only on p, q and σ) such that $\text{Leb}_D(R(D, n, \gamma_0)) \leq K e^{-\varepsilon n}$ for every large $n \geq 1$.*

Proof. The proof is entirely analogous to [1, Lemma A.1]. \square

We are now in a position to determine L in terms of the frequency of visits described in the previous lemma. Assume that L is sufficiently close to 1 so that

$$L^{\gamma_0} \cdot \lambda_u^{(1-\gamma_0)} \geq e^c > 1 \quad (7.4)$$

for some $c > 0$. By this choice, if one partitions the iterates $g^j(x)$ of a point $x \in D$ depending on if these belong to \mathcal{O} or not, and set $r(n, x) := \#\{0 \leq j \leq n-1 : g^j(x) \in \mathcal{O}\}$ and $s(n, x) := \{0, 1, \dots, n-1\} \setminus r(n, x)$. Now, using Lemma 7.1, the Borel-Cantelli lemma implies that for Leb_D -a.e. x there is $n(x)$ such that $x \notin R(D, n, \gamma_0)$ for every $n \geq n(x)$. In other words, we have that $\#r(n, x) < \gamma_0 n$ for Lebesgue almost every $x \in D$ and all $n \geq n(x)$. Therefore, by Dg -invariance of the cone field C ,

$$\begin{aligned} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Dg(g^j(x)))|_{C(g^j(x))}^{-1}\| &\leq \frac{1}{n} \#r(n, x) \log L^{-1} - \frac{1}{n} \#s(n, x) \log \lambda_u \\ &\leq \gamma_0 \log L^{-1} + (1 - \gamma_0) \log \lambda_u \\ &= -(\log(L^{\gamma_0} \cdot \lambda_u^{(1-\gamma_0)})) \leq -c < 0 \end{aligned} \quad (7.5)$$

for Lebesgue almost every $x \in D$ and every large $n \geq 1$. Since C is a cone field preserved by all elements $g \in \mathcal{U}$ and the constants are uniform we conclude that all elements $g \in \mathcal{U}$ satisfy (H1)-(H4) with uniform constants, as claimed.

8. FINAL COMMENTS: SINGULAR ENDOMORPHISMS AND FUTURE DIRECTIONS

Let us make some final comments concerning singular endomorphisms. In [46], Tsujii considered the space \mathcal{PH}^u of C^r -endomorphisms on surfaces that admit a continuous splitting $TM = E^c \oplus E^u$ with $\dim E^c = \dim E^u = 1$ and for which there exist $C, \lambda > 0$ such that $\|Df^n(x)|_{E_x^u}\| \geq C e^{\lambda n}$ and $\|Df^n(x)|_{E_x^c}\| \leq C e^{-\lambda n} \|Df^n(x)|_{E_x^c}\|$ for all $x \in M$ and $n \geq 0$. The space PH^u forms a C^1 -open set of the space of endomorphisms and C^r -generic endomorphisms f in PH^u admit a finite number of physical measures, where $r \geq 19$ (cf. [46]). In particular these results hold for the class of robust transitivity endomorphisms with persistence of critical points considered in [23] (the kernel of Df at critical points is contained in the subbundle E^c).

In the present paper we consider a complementary situation. We consider partially hyperbolic endomorphisms (more generally attractors) \mathcal{PH}^s that admit a continuous splitting $TM = E^s \oplus E^c$, which is dominated, such that E^s is uniformly contracting and there exists an invariant cone field with a non-uniform expansion condition. These form an open set of partially hyperbolic non-singular endomorphisms. On the one hand, the existence of a (invariant) stable subbundle and an invariant cone field seems unnatural in the presence of critical points (e.g. the endomorphism $f : \mathbb{T}^2 \times [0, 1] \rightarrow \mathbb{T}^2 \times [0, 1]$ given by $f(x, y, z) = (3x + y, x + y, h^n(z))$, where $h(z) = 4z(1 - z^2)$ does not admit such a decomposition is $n \geq 1$ is large). On the other hand, the lack of domination is itself a very hard obstacle even in the case of diffeomorphisms. Indeed, the Hénon-like maps ([33]) given by $h_{a,b}(x, y) = (1 - ax^2, 0) + r(a, x, y)$ with $\|r\|_{C^3} \leq b$ (for b small enough) are C^3 -perturbations

of the singular endomorphism $h_{a,0}(x, y) = (1 - ax^2, 0)$ which admits a non-strict positively invariant cone field. Since this is not robust, inspired by [33, 48] we pose the following question:

Question 1: *Assume that f is an endomorphism with C^1 -persistence of singularities on a surface M (there exists a C^1 -open neighborhood of f formed by singular endomorphisms). If $f \notin \mathcal{PH}^u$ does there exist a C^1 -perturbation such that f admits a sink? Can f be approximated by C^1 endomorphisms with infinitely many sinks (hence infinitely many physical measures)?*

In view of the previous discussion it seems only reasonable to consider families of singular endomorphisms for which singular behavior is isolated from the domination. In view of [46] it is not reasonable to expect an affirmative answer to the following question:

Question 2: *Assume that $A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is an Anosov endomorphism and take $F_0 : \mathbb{T}^2 \times \mathbb{S}^1 \rightarrow \mathbb{T}^2 \times \mathbb{S}^1$ be given by $F_0 = A \times Id$. If \mathcal{S}_ε denotes the space of skew-products $F(x, y) = (A(x, y), f(x, y, z))$ such that $\|F - F_0\|_{C^2} < \varepsilon$ and ε is small, does there exist a C^r -residual subset $\mathcal{R} \subset \mathcal{S}_\varepsilon$ of singular partially hyperbolic endomorphisms such that every $G \in \mathcal{R}$ has a finite number of physical measures?*

A different question concerns the existence of singularities. Piecewise smooth partial hyperbolic endomorphisms with singularities arise naturally e.g. as Poincaré maps of Lorenz attractors. If $f \in \mathcal{PH}^u$ is a partially hyperbolic endomorphism with singularities whose derivative behaves like a power of the distance to the singular region satisfying an extra slow recurrence condition (see [1] for precise definition) it is most likely the ideas in the proof of Theorem A can be carried out to prove the existence and finitude of SRB measures. Finally, it is interesting to understand if this construction can probably made to work (maybe with some extra difficulties) if one removes the hypothesis of an everywhere invariant cone field and assume that the set of points that admit infinitely many iterates at which cone fields are in an invariant position and backward contraction holds simultaneously has positive Lebesgue measure. This seems to be the case of Hénon-like maps for the parameters obtained by the parameter exclusion process in [33].

Finally, it would be interesting to describe when these SRB measures are absolutely continuous with respect to the Lebesgue measure on the ambient space. On the one hand, in our context any absolutely continuous invariant measure is a convex combination of SRB measures (see Corollary VII.1.1.1 of [37]). In the context of Theorem A, it follows from [27] any SRB measure μ is absolutely continuous with respect to volume if and only if the entropy production e_μ is zero or, equivalently, the folding entropy $F(\mu)$ would satisfy $F(\mu) = \int \log |\det Df| d\mu_f$, in which case the folding entropy would vary continuously with the dynamics (we refer the reader to [27] for the definitions). It follows from [42] that this is equivalent to say that μ is absolutely continuous with respect to the stable foliation. This motivates the following:

Question 3: *In the context of Theorem B, are SRB measures of C^2 -generic endomorphisms singular with respect to the volume? If so, do these partially hyperbolic endomorphisms admit inverse SRB measures (in the sense of [31, 32])?*

Finally, as the existence of SRB measures relies on explicit geometric constructions, we expect to obtain further statistical properties of these hyperbolic SRB measures. In particular, it is interesting to establish the smooth dependence of these measures (and their entropies) with respect to the endomorphism. These problems are usually referred as linear response problems (see e.g. [5, 6] and references therein).

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