QUANTITATIVE RECURRENCE FOR FREE SEMIGROUP ACTIONS

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Abstract. We consider finitely generated free semigroup actions on a compact metric space and obtain quantitative information on Poincaré recurrence, average first return time and hitting frequency for the random orbits induced by the semigroup action. Besides, we relate the recurrence to balls with the rates of expansion of the semigroup's generators and the topological entropy of the semigroup action. Finally, we establish a partial variational principle and prove an ergodic optimization for this kind of dynamical action.

1. Introduction

The research on partially hyperbolic dynamics brought to the stage iterated systems of functions modeling the behavior within the central manifold. This circumstance led to the study of random dynamical systems and a thorough understanding of the dynamical and ergodic properties of these systems has already been achieved [24]. On the other hand, sequential dynamical systems have been introduced to model physical phenomena: instead of iterating the same dynamics, one allows the system that describes the real events to readjust with time, and one may work with a family of randomly chosen transformations in a way that matches the inevitable experimental errors [15]. However, it is not yet clear how the classical results on first hitting or return times may be generalized to stationary and non-autonomous sequences of maps.

In this work we aim at an extension of the quantitative analysis of Poincaré recurrence to the realm of finitely generated free semigroup actions. In this context, a first important contribution was obtained in [17], where the authors proved that, for rapidly mixing systems, the quenched recurrence rates are equal to the pointwise dimensions of a stationary measure. One should also refer [4, 22, 23] on the distribution of hitting times and extreme laws for random dynamical systems. Equally significant are the recent advances obtained in [1, 13, 24, 11]. Ultimately, we are concerned with the description of the fastest return time when considering all the semigroup elements instead of a single dynamical system. In a recent work [21], it has been introduced a notion of topological entropy and pressure for finitely generated continuous free semigroup actions on a compact metric space. Later, in [10], it has been shown that a free semigroup action of either C^1 expanding maps, or, more generally, Ruelle-expanding transformations, has a unique measure of maximal entropy which is linked to annealed equilibrium states for random dynamical systems [5]. The main strategy to deal with such a system has been the codification of the random orbits by a true dynamics, namely the skew product based on a full shift with finitely many symbols. Keeping this approach in mind, here we address a few questions regarding recurrence, first return or hitting time maps, and the connection between the rate of frequency of visits to a set, its size, the entropy of the semigroup action and the Lyapunov exponents of the generators.

We will start proving that almost every point is recurrent either by random dynamical systems or by stationary sequential dynamics. Then we will establish a Kac-like property for such return times and estimate an upper bound for the Poincaré recurrence to balls, linking the latter to the quenched pressure of random dynamical systems. As return times are strongly related to other dynamically significant quantities, like entropy and Lyapunov exponents, we will also show that, in the case of random dynamical systems generated by expanding maps, the shortest fibred return time to dynamic balls grows linearly, which implies that typical fibred return times to balls may

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be expressed in terms of the random Lyapunov exponents of the dynamics and, consequently, are independent of the point. Moreover, we shall study the connection between the maximum hitting frequency/fastest mean return time to a set with its size when estimated by different invariant measures, extending the ergodic optimization obtained in [14] to the random context we are considering. Finally, we will introduce the notion of measure-theoretic entropy of a semigroup action and obtain a partial variational principle which improves the estimate in [16] and complements [6, 7]. We refer the reader to Subsection 2.2 for the precise statements of the main results.

2. Main results

In this section we describe the free semigroup actions we are interested in and state our major conclusions on the quantitative recurrence within this context. The concepts and results we will consider in this work depend on the fixed set of generators G_1 but, to improve the general readability of the paper, we will omit this data in the notation.

2.1. **Setting.** Given a compact metric space (X, d), a finite set of continuous maps $g_i : X \to X$, $i \in \mathcal{P} = \{1, 2, \dots, p\}$, and the finitely generated semigroup $(G, \circ)$ with the finite set of generators $G_1 = \{id, g_1, g_2, \dots, g_p\}$, we write $G = \bigcup_{n \in \mathbb{N}_0} G_n$, where $G_0 = \{id\}$ and $\underline{g} \in G_n$ if and only if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$, with $g_{i_j} \in G_1$ (for notational simplicity's sake we will use $g_j\, g_i$ instead of the composition $g_j \circ g_i$). We note that a semigroup may have multiple generating sets. In what follows, we will assume that the generator set G_1 is minimal, meaning that no function g_j, for $j = 1, \dots, p$, can be expressed as a composition of the remaining generators. Observe also that each element $\underline{g}$ of G_n may be seen as a word which originates from the concatenation of n elements in G_1. Yet, different concatenations may generate the same element in G. Nevertheless, in most of the computations to be done, we shall consider different concatenations instead of the elements in G they create. One way to interpret this statement is to consider the itinerary map $\iota : \mathbb{F}_p \to G$ given by

$$\underline{i} = i_n \dots i_1 \quad \mapsto \quad \underline{g}_{\underline{i}} := g_{i_n} \dots g_{i_1}$$

where $\mathbb{F}_p$ is the free semigroup with p generators, and to regard concatenations on G as images by ι of paths on $\mathbb{F}_p$.

Set $G_1^* = G_1 \setminus \{id\}$ and, for every $n \geq 1$, let G_n^* denote the space of concatenations of n elements in G_1^*. To summon each element $\underline{g}$ of G_n^*, we will write $|\underline{g}| = n$ instead of $\underline{g} \in G_n^*$. In G, one consider the semigroup operation of concatenation defined as usual: if $\underline{g} = g_{i_n} \dots g_{i_2} g_{i_1}$ and $\underline{h} = h_{i_m} \dots h_{i_2} h_{i_1}$, where $n = |\underline{g}|$ and $m = |\underline{h}|$, then $\underline{g}\,\underline{h} = g_{i_n} \dots g_{i_2} g_{i_1} h_{i_m} \dots h_{i_2} h_{i_1} \in G_{m+n}^*$. The finitely generated semigroup G induces an *action* in X, say

$$\begin{array}{rccc} \mathbb{S}: & G \times X & \to & X \\ & (g, x) & \mapsto & g(x). \end{array}$$

We say that $\mathbb{S}$ is a *semigroup action* if, for any $\underline{g}, \underline{h} \in G$ and every $x \in X$, we have $\mathbb{S}(\underline{g}\,\underline{h}, x) = \mathbb{S}(\underline{g}, \mathbb{S}(\underline{h}, x))$. The action $\mathbb{S}$ is continuous if the map $\underline{g} : X \to X$ is continuous for any $\underline{g} \in G$. As usual, $x \in X$ is a *fixed point* for $\underline{g} \in G$ if $\underline{g}(x) = x$; the set of these fixed points will be denoted by $\text{Fix}(\underline{g})$. A point $x \in X$ is said to be a *periodic point with period* n by the action $\mathbb{S}$ if there exist $n \in \mathbb{N}$ and $\underline{g} \in G_n^*$ such that $\underline{g}(x) = x$. Write $\text{Per}(G_n) = \bigcup_{|\underline{g}|=n} \text{Fix}(\underline{g})$ for the set of all periodic points with period n. Accordingly, $\text{Per}(G) = \bigcup_{n \geq 1} \text{Per}(G_n)$ will stand for the set of periodic points of the whole semigroup action. We observe that, when $G_1^* = \{f\}$, these definitions coincide with the usual ones for the dynamical system f.

The action of semigroups of dynamics has a strong connection with skew products which has been scanned in order to obtain properties of semigroup actions by means of fibred and annealed quantities associated to the skew product dynamics (see e.g. [10]). We recall that, if X is a compact metric space and one considers a finite set of continuous maps $g_i : X \to X$, $i \in \mathcal{P} = \{1, 2, \dots, p\}$, $p \geq 1$, we have defined a skew product dynamics

$$\begin{array}{rccc} \mathcal{F}_G : & \Sigma_p^+ \times X & \to & \Sigma_p^+ \times X \\ & (\omega, x) & \mapsto & (\sigma(\omega), g_{\omega_1}(x)) \end{array} \tag{1}$$

where $\omega = (\omega_1, \omega_2, \dots)$ is an element of the full unilateral space of sequences $\Sigma_p^+ = \mathcal{P}^{\mathbb{N}}$ and σ denotes the shift map on Σ_p^+. We will write $\mathcal{F}_G^n(\omega, x) = (\sigma^n(\omega), f_\omega^n(x))$ for every $n \geq 1$.

In what follows, we will denote by $\mathcal{M}_G$ the set of Borel probability measures on X invariant by g_i for all $i \in \{1, \cdots, p\}$. And $\mathbb{P}_{\underline{a}}$ will stand for the Bernoulli probability measure in Σ_p^+ which is the Borel product measure determined by a vector $\underline{a} = (a_1, \cdots, a_p)$ satisfying $0 < a_i < 1$ for every $i \in \{1, 2, \cdots, p\}$ and $\sum_{i=1}^p a_i = 1$.

2.2. **Statements.** As a semigroup action is not a classical dynamical system, but rather an action of several dynamics in the same ambient space which are selected randomly according to some probability measure, the possible notions of recurrence must be carefully chosen and one needs to guarantee that recurrence actually happens. In what follows, we shall examine recurrence either from the point of view of individual concatenations of maps (associated to individual infinite paths in the free semigroup) or by estimating the fastest return times (the smallest return time associated to any of the dynamics in the semigroup).

2.2.1. **Poincaré recurrence for sequences of stationary maps**. While using infinite concatenations of elements in G_1, it is natural to consider the shift space $\Sigma_p^+ = \{1, \dots p\}^{\mathbb{N}}$. Any sequence $\omega \in \Sigma_p^+$ determines a sequential dynamical system $(g_{\omega_i})_{i \in \mathbb{N}}$ and their compositions

$$n \geq 1 \;\mapsto\; f_\omega^n = g_{\omega_n} \dots g_{\omega_2} g_{\omega_1}.$$

For any random walk $\mathbb{P}$ on $\mathbb{F}_p$ one expects to find generic paths for which the dynamics in X exhibits recurrence, meaning that, if one disregards the first shift iterations of the sequence $\omega \in \Sigma_p^+$, then almost every point in X returns infinitely often by the shifted stationary sequence of maps (a notion that generalizes periodicity). Our first result asserts that this is indeed the case.

Theorem A. *Let G be a finitely generated free semigroup, $\mathbb{S}$ be the corresponding continuous semigroup action, ν be a Borel probability measure invariant by every generator in G_1^* and $\mathbb{P}$ be a σ-invariant Borel probability measure on Σ_p^+. Then, for any measurable subset $A \subset X$ the following properties hold:*

1. *For any $\omega \in \Sigma_p^+$, the set of points $x \in A$ for which there are positive integers $n \geq k$ satisfying $g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_k}(x) \in A$ has full ν-measure in A.*
2. *For $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, the set of the points $x \in A$ whose orbit $\big(f_\omega^k(x)\big)_{k \in \mathbb{N}}$ returns to A infinitely often has full ν-measure in A.*

2.2.2. **Kac expected return time**. Given a measurable map $f : X \to X$ preserving an ergodic probability measure ν, Kac's Lemma asserts that the expected first return time to a positive measure set $A \subset X$ is $\frac{1}{\nu(A)}$. More precisely, if $\nu(A) > 0$ and the first hitting time of x to A is defined by

$$n_A(x) = \begin{cases} \inf \big\{k \in \mathbb{N} \colon f^k(x) \in A\big\} & \text{if this set is nonempty} \\ +\infty & \text{otherwise} \end{cases}$$

then n_A is ν-integrable and

$$\int_A n_A(x)\, d\nu_A = \frac{1}{\nu(A)} \tag{2}$$

where $\nu_A = \frac{\nu}{\nu(A)}$ is the normalized probability in A. A version of Kac's Lemma for suspension flows may be found in [26].

In view of Theorem A(2), it is natural to define, for each measurable $A \subset X$ and $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, the *first return time to A of $x \in A$* by the dynamics $\big(f_\omega^k\big)_{k \in \mathbb{N}_0}$ as follows:

$$n_A^\omega(x) = \begin{cases} \inf \big\{k \in \mathbb{N} \colon f_\omega^k(x) \in A\big\} & \text{if this set is nonempty} \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

We say that the semigroup action $\mathbb{S}$ is *ergodic with respect to $\mathbb{P}$ and ν* if the measure $\mathbb{P} \times \nu$ is ergodic with respect to $\mathcal{F}_G$. This assumption is somehow demanding, implying, in particular, that $\mathbb{P}$ is ergodic with respect to σ. In most instances, however, we will only need to assume that $\mathbb{P}$ is ergodic and that, for any set $A \subset X$ such that $g_i^{-1}(A) = A$ for all $1 \leq i \leq p$, we have $\nu(A) \times \nu(X \setminus A) = 0$.

In Section 4 we will show that ergodic semigroup actions satisfy a Kac-like recurrence property: the average asymptotic behavior, as k tends to infinity, of the expected first return time to A by the sequence $\big(f^n_{\sigma^k(\omega)}\big)_{n\,\in\,\mathbb{N}} = \big(g_{\omega_n} \circ \cdots \circ g_{\omega_{k+1}} \circ g_{\omega_k}\big)_{n\,\in\,\mathbb{N}}$ is precisely $\frac{1}{\nu(A)}$. In particular:

Theorem B. *Let G be a finitely generated free semigroup endowed with a Bernoulli probability measure $\mathbb{P}_{\underline{a}}$ and $\mathbb{S}$ be the corresponding continuous semigroup action. Consider a Borel probability measure ν in X invariant by every generator in G and assume that $\mathbb{S}$ is ergodic with respect to $\mathbb{P}_{\underline{a}}$ and ν. Then, for $\mathbb{P}_a$-almost every ω, $\lim_{k\,\to\,+\infty} \int_A n_A^{\sigma^k(\omega)}(x)\, d\nu_A(x) = \frac{1}{\nu(A)}$. Moreover, there exists a Baire residual subset $\mathcal{R} \subset \Sigma_p^+$ such that, for every $\omega \in \mathcal{R}$,*

$$\int_A n_A^{\omega}(x)\, d\nu_A(x) = \frac{1}{\nu(A)}.$$

As a consequence of the last claim, there exists a dense set of values $\omega \in \Sigma_p^+$ for which the non-autonomous dynamics $(f^n_\omega)_{n\geq 1}$ satisfy Kac's formula (2). It is still an open question to determine whether this formula holds for $\mathbb{P}$-almost every ω.

2.2.3. **Partial variational principle**. The formula of Abramov and Rokhlin [2] for the measure theoretical entropy of the skew product $\mathcal{F}_G$ with respect to the product measure $\mathbb{P} \times \nu$ suggests a way to define a fibred notion of *metric entropy of a free semigroup action* with respect to a random walk on Σ_p^+ and an invariant measure on X, which we will denote by $h_\nu(\mathbb{S}, \mathbb{P})$. This will be done in Subsection 5.1, just before proving a partial variational principle which extends Theorem 1.2 of [16] to non-symmetric random walks. Meanwhile, recall that $P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P})$ stands for the quenched topological pressure of the skew product $\mathcal{F}_G$ with respect to the random walk $\mathbb{P}$ (see [5]), $h_{\text{top}}(\mathbb{S})$ is the topological entropy of the free semigroup action $\mathbb{S}$ (cf. definition in [21]) and $h_{\text{top}}(\mathbb{S}, \mathbb{P})$ is the relative topological entropy of the free semigroup action with respect to the random walk $\mathbb{P}$ (see [10]).

Theorem C. *Let $\mathbb{S}$ be a finitely generated free semigroup action with generators $G_1 = \{id, g_1, \ldots, g_p\}$ and consider a Borel σ-invariant probability measure $\mathbb{P}$ on Σ_p^+. Then*

$$\sup_{\nu\,\in\,\mathcal{M}_G} h_\nu(\mathbb{S}, \mathbb{P}) \leq h_{top}(\mathbb{S}) + (\log p - h_{\mathbb{P}}(\sigma))\,.$$

If, additionally, each generator g_i is C^2 expanding ($1 \leq i \leq p$) and $\mathbb{P} = \mathbb{P}_{\underline{a}}$, then

$$\sup_{\nu\,\in\,\mathcal{M}_G} h_\nu(\mathbb{S}, \mathbb{P}_{\underline{a}}) \leq P^{(q)}_{top}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{a}}) \leq h_{top}(\mathbb{S}, \mathbb{P}_{\underline{a}}). \tag{4}$$

We remark that the second inequality in (4) may be strict, as shown by Example 5.3.

2.2.4. **Poincaré recurrence of balls**. In this subsection we will refer to return times of a set to itself by concatenations f^n_ω of dynamics in G_1 associated to a fixed $\omega \in \Sigma_p^+$. Given $A \subset X$ and $\omega \in \Sigma_p^+$, the *ω-shortest return time of A to itself* is defined by

$$\mathcal{T}^\omega(A) = \begin{cases} \inf\{k \in \mathbb{N}\colon f^k_\omega(A) \cap A \neq \emptyset\} & \text{if this set is nonempty} \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

The shortest return time of the ball $B_\delta(x)$ by the semigroup action $\mathbb{S}$ is equal to

$$\mathcal{T}^{\mathbb{S}}(B_\delta(x)) = \inf\{k \in \mathbb{N}\colon \exists\ \underline{g} \in G_k^*\colon\ \underline{g}(B_\delta(x)) \cap B_\delta(x) \neq \emptyset\} \tag{6}$$

whenever this set is nonempty. Or, equivalently,

$$\mathcal{T}^{\mathbb{S}}(B_\delta(x)) = \inf_{\omega\,\in\,\Sigma_p^+} \mathcal{T}^\omega(B_\delta(x)).$$

Concerning this concept, the next result asserts that, for $\mathbb{P}$-typical infinite concatenations of dynamics, the minimal returns of dynamical balls grow linearly with the radius, similarly to what happens with a single dynamical system satisfying the orbital specification property and having positive entropy (cf. [3, Theorem 1] and [25, Theorem B] for the case of return times to cylinders and dynamic balls, respectively).

Theorem D. *Let G be the semigroup generated by $G_1 = \{id, g_1, \dots, g_p\}$, where the elements in G_1^* are C^1 expanding maps on a compact connected Riemannian manifold X preserving a common Borel probability measure ν. Consider the continuous semigroup action $\mathbb{S}$ induced by G and a σ-invariant probability measure $\mathbb{P}$ on Σ_p^+. If $h_\nu(\mathbb{S},\mathbb{P}) > 0$, then, for ν-almost every $x \in X$, one has*

$$\limsup_{\delta \to 0} \frac{\mathcal{T}^{\mathbb{S}}(B_\delta(x))}{-\log \delta} \leq \frac{1}{\log \lambda} \tag{7}$$

where $\lambda = \min_{1 \leq i \leq p} \|Dg_i\|$. If, in addition, all elements in G_1^* are conformal maps, $\mathbb{P} = \mathbb{P}_{\underline{a}}$ and the semigroup action is ergodic with respect to $\mathbb{P}_{\underline{a}}$ and ν, then, for $\mathbb{P}_{\underline{a}}$-almost every ω and ν-almost every $x \in X$,

$$\lim_{\delta \to 0} \frac{\mathcal{T}^{\omega}(B_\delta(x))}{-\log \delta} = \frac{\dim X}{\sum_{i=1}^p a_i \int \log |\det Dg_i| \, d\nu} \tag{8}$$

where $\dim X$ stands for the dimension of the manifold X.

We note that, in the special case of finitely generated semigroups of conformal expanding maps for which $|\det Dg_i(\cdot)|$ is constant for every $1 \leq i \leq p$, the expression in the denominator of the right hand-side of (8) coincides with the quenched pressure of the skew product $\mathcal{F}_G$ with respect to the null observable and the random walk $\mathbb{P}_{\underline{a}}$ (cf. definition in [5]), and this is bounded above by the topological entropy of the semigroup action with respect to $\mathbb{P}_{\underline{a}}$ (cf. definition in [10]). Consequently, in this setting, for $\mathbb{P}_{\underline{a}}$-almost every ω and ν-almost every $x \in X$, we obtain

$$\lim_{\delta \to 0} \frac{\mathcal{T}^{\omega}(B_\delta(x))}{-\log \delta} \geq \frac{\dim X}{h_{\text{top}}(\mathbb{S},\mathbb{P}_{\underline{a}})} > 0.$$

2.2.5. **Ergodic optimization**. Our last result, inspired by [14], deals with the relation between the maximum hitting frequency, the essential maximal mean return time and the size of a set when measured by different measures. For the required definitions and the proof we refer the reader to Section 7.

Theorem E. *Let $\mathbb{S}$ be a finitely generated free semigroup action with generators $G_1 = \{id, g_1, \dots, g_p\}$ and $\mathbb{P}$ be a Borel σ-invariant probability measure on Σ_p^+. For every closed set $A \subset X$ there exists a marginal ν on X such that*

$$\mathbb{P} - \textit{esssup} \sup_{x \in X} \limsup_{n \to +\infty} \frac{\sharp \{0 \leq i \leq n-1 : f_\omega^i(x) \in A\}}{n} = \nu(A).$$

3. Proof of Theorem A

Let $\mathbb{S} : G \times X \to X$ be a semigroup action generated by a finite set $\{g_1, g_2, \dots, g_p\}$ of $p \geq 2$ dynamics acting on a compact metric space X endowed with a Borel probability measure ν which is invariant by g_i for every $i \in \{1, 2, \dots, p\}$. Consider the shift map σ on the full unilateral space of sequences $\Sigma_p^+ = \{1, 2, \dots, p\}^{\mathbb{N}}$ and a σ-invariant Borel probability measure $\mathbb{P}$ in Σ_p^+. The corresponding skew product $\mathcal{F}_G : \Sigma_p^+ \times X \to \Sigma_p^+ \times X$ has been defined in (1) and preserves the probability measure $\mathbb{P} \times \nu$.

3.1. **Random Ergodic Theorem.** Let us recall a generalized ergodic theorem from [12]. Let Z and X be measure spaces with probability measures $\mathbb{P}$ and ν, respectively. Suppose that $U : Z \to Z$ is an $\mathbb{P}$ preserving transformation and denote by $Q : Z \times X \to Z \times X$ the skew product defined by $Q(z, x) = (U(z), T_z(x))$, where the family $(T_z)_{z \in Z}$ is assumed to be measurable and, for each $z \in Z$, $T_z : X \to X$ is a ν-measure preserving map. The skew product Q is measurable and preserves the probability measure $\mathbb{P} \times \nu$. Write $T_z^0 = T_z$ and, for $k \in \mathbb{N}$, $T_z^k = T_{U^k(z)} \dots T_{U(z)} T_z$. It is not hard to show, using Birkhoff's Ergodic Theorem for the skew product Q and $\mathbb{P} \times \nu$, that, if φ is a ν-integrable function in X, then there exists a $(\mathbb{P} \times \nu)$-full measure subset $E \subset Z \times X$ such that, for every $(z, x) \in E$, the averages

$$\left(\frac{1}{n} \sum_{j=0}^{n-1} \varphi(T_z^j(x)) \right)_{n \in \mathbb{N}}$$

converge to a ν-integrable function φ^*_z so that

$$\int \varphi(x)\, d\nu(x) \quad = \quad \int\int \psi(z,x)\, d\mathbb{P}\, d\nu = \int\int \psi^*(z,x)\, d\mathbb{P}\, d\nu = \int\int \varphi^*_z(x)\, d\mathbb{P}(z)\, d\nu(x).$$

Then, Fubini theorem ensures that, for $\mathbb{P}$-almost every $z \in Z$, the set E^z of points $x \in X$ whose averages $\left(\frac{1}{n}\sum_{j=0}^{n-1} \varphi(T_z^j(x))\right)_{n\in\mathbb{N}}$ converge to φ^*_z has full ν measure. If, moreover, $\mathbb{P}\times\nu$ is ergodic with respect to the skew product Q, then $\varphi^*_z(x) = \int \varphi d\nu$ for $\mathbb{P}$-almost every $z \in Z$ and ν-almost every $x \in X$.

3.2. **Recurrence *via* the skew product.** To get a version of Poincaré's Recurrence Theorem for a semigroup action, we will start deducing recurrent properties of stationary non-autonomous sequences of dynamical systems and fibred maps.

Proposition 3.1. *Consider the skew product $\mathcal{F}_G$, a σ-invariant probability measure $\mathbb{P}$ on Σ_p^+ and a Borel probability measure ν in X invariant by every generator in G. For any measurable subset $A \subset X$ the following properties hold:*

1. *For $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, the set of the points $x \in A$ whose orbit $\left(f_\omega^k(x)\right)_{k\in\mathbb{N}}$ returns to A infinitely often has full ν-measure in A.*
2. *For every $\omega \in \Sigma_p^+$, the set of points $x \in A$ for which there are positive integers $n \geq k$ satisfying $g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_k}(x) \in A$ has full ν-measure in A.*
3. *If ν is ergodic with respect to one of the generators, say g_1, then there exists a subset $\Omega \subset \Sigma_p^+$ with $\mathbb{P}(\Omega) > 0$ such that for every $\omega \in \Omega$ there is a set $Y_\omega \subset X$ with $\nu(Y_\omega) = 1$ so that, for any $x \in Y_\omega$, we may find $\ell = \ell(\omega, x) \in \mathbb{N}$ such that the orbit $\left(f_\omega^k(g_1^\ell(x))\right)_{k\in\mathbb{N}}$ of $g_1^\ell(x)$ enters infinitely many times in A.*
4. *If $\mathbb{P}\times\nu$ is ergodic with respect to $\mathcal{F}_G$, then for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$ the orbit $\left(f_\omega^k(x)\right)_{k\in\mathbb{N}}$ of ν-almost every $x \in X$ enters infinitely many times in A.*

Some comments are in order. Items (1) and (4) provide expected results on the recurrence of almost every point with respect to almost every random path. Item (2) indicates that, for any stationary sequence of maps, recurrence surely happens up to a convenient shifting of the orbits. Item (3) imparts a dual statement by replacing this shifting by a finite transient of some generator g_1, which is assumed to be ergodic with respect to ν. We also remark that, in the case of finitely generated free abelian semigroups, the generators commute and probability measures invariant by any generator do exist.

Proof. Given $k \in \mathbb{N}$ and $\omega = \omega_1\omega_2 \ldots \in \Sigma_p^+$, recall that we write $f_\omega^k = g_{\omega_k}\, g_{\omega_{k-1}} \dots g_{\omega_1}$. Let A be a measurable subset of X with $\nu(A) > 0$ and consider $\Sigma_p^+ \times A$. As the probability measure $\mathbb{P}\times\nu$ is invariant by the skew product $\mathcal{F}_G$ and $(\mathbb{P}\times\nu)(\Sigma_p^+ \times A) = \nu(A) > 0$, by Poincaré's Recurrence Theorem there is a subset $E \subset \Sigma_p^+ \times A$ with $(\mathbb{P}\times\nu)(E) = \nu(A) > 0$ such that every $(\omega, x) \in E$ returns to $\Sigma_p^+ \times A$ infinitely often by the iteration of $\mathcal{F}_G$. Observe now that

$$\mathcal{F}_G^k(\omega, x) = (\sigma^k(\omega), g_\omega^k(x)) \in \Sigma_p^+ \times A \quad \Leftrightarrow \quad f_\omega^k(x) \in A \tag{9}$$

so the property describing the set E informs that, for every $(\omega, x) \in E$, there are infinitely many values of $k \geq 1$ such that $f_\omega^k(x) \in A$. Besides, by Fubini-Tonelli's Theorem we have

$$\nu(A) = (\mathbb{P}\times\nu)(E) = \int_X \mathbb{P}(E^x)\, d\nu(x) = \int_{\Sigma_p^+} \nu(E^\omega)\, d\mathbb{P}(\omega),$$

where $E^x = \{\omega \in \Sigma_p^+ : (\omega, x) \in E\}$ and $E^\omega = \{x \in A : (\omega, x) \in E\}$. Thus, for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, we must have $\nu(E^\omega) = \nu(A)$. This completes the proof of item (1).

To prove item (2), we will pursue another argument without summoning up the skew product, aiming the recurrence by the (possibly non-generic) random orbits $\left(f_\omega^k\right)_{k\in\mathbb{N}}$. Given $\omega = \omega_1\omega_2\omega_3 \ldots \in \Sigma_p^+$, write

$$B_\omega = \bigcap_{k\geq 1}\bigcap_{n\geq k} \{x \in A : g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_k}(x) \notin A\}.$$

Points in B_ω are in A but never return to A by any concatenation of dynamics given by the sequences $(g_{\omega_j})_{j\,\geq\,k}$, for all $k \geq 1$. We claim that $\{(g_{\omega_j} \dots g_{\omega_1})(B_\omega)\}_{j\,\geq\,1}$ defines a family of pairwise disjoint subsets of X. Indeed, given positive integers $m > n$, if $(g_{\omega_m} g_{\omega_{m-1}} \dots g_{\omega_1})^{-1}(B_\omega) \cap (g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1})^{-1}(B_\omega) \neq \emptyset$, then there would exist $x \in X$ such that $z = g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1}(x) \in B_\omega$ as well as $g_{\omega_m} g_{\omega_{m-1}} \dots g_{\omega_{n+1}}(z) \in B_\omega$, which contradicts the definition of B_ω. As ν is invariant by g_i for every $i \in \mathcal{P}$, it is also invariant by $g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1}$ for every $n \in \mathbb{N}$. Therefore,

$$\sum_{n=1}^{\infty} \nu(B_\omega) = \sum_{n=1}^{\infty} \nu\left((g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1})^{-1}(B_\omega)\right) = \nu\left(\bigcup_{n=1}^{\infty} (g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_1})^{-1}(B_\omega)\right) \leq 1$$

and so $\nu(B_\omega) = 0$. Thus, for ν-almost every $x \in A$, there exists $n \geq k \geq 1$ such that $g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_k}(x) \in A$. It is not hard to adapt the previous argument to show that, for every $\omega \in \Sigma_p^+$, there exists a full ν-measure subset of points $x \in A$ which exhibit infinitely many returns to A (that is, which admit infinitely many values $n_\ell \geq k_\ell$ such that $g_{\omega_{n_\ell}} g_{\omega_{n_\ell - 1}} \dots g_{\omega_{k_\ell}}(x) \in A$).

We now focus on item (3). As the probability measure $\mathbb{P} \times \nu$ is invariant by the skew product $\mathcal{F}_G$, we may apply to $\varphi = \chi_A$ the Random Ergodic Theorem quoted in Subsection 3.1. This way, we conclude that, for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, the frequency of visits to A given by

$$\frac{1}{n} \sharp \left\{0 \leq j \leq n-1 : f_\omega^j(x) \in A\right\} = \frac{1}{n} \sum_{j=0}^{n-1} \chi_A(f_\omega^j(x))$$

is convergent for ν-almost every $x \in X$. As $\nu(A) > 0$, we may add that, for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$ and ν-almost every $x \in A$, those averages converge to the value at x of a ν-integrable function φ^*_ω that satisfies $\int \int \varphi^*_\omega(x)\, d\nu(x)\, d\mathbb{P}(\omega) = \int \varphi(x)\, d\nu(x) = \nu(A) > 0$. Consequently, the set $C \subset \Sigma_p^+ \times X$ of the points (ω, x) for which we have $\varphi^*_\omega(x) > 0$ satisfies $(\mathbb{P} \times \nu)(C) > 0$. Therefore, for every $(\omega, x) \in C$, the point $f_\omega^k(x)$ is in A for infinitely many choices of $k \in \mathbb{N}$. By Fubini-Tonelli's Theorem, we get $(\mathbb{P} \times \nu)(C) = \int_X \nu(C^\omega)\, d\mathbb{P}(\omega) > 0$ where $C^\omega = \{x \in X : (\omega, x) \in C\} = \{x \in X : f_\omega^k(x) \in A \text{ for infinitely many } k \in \mathbb{N}\}$. Thus, there must exist a subset $\Omega \subset \Sigma_p^+$ with $\mathbb{P}(\Omega) > 0$ such that, for every $\omega \in \Omega$, we have $\nu(C^\omega) > 0$. Notice, however, that, if $\nu(A) < 1$, the previous property is not enough for us to be sure whether $\nu(A \cap C^\omega) > 0$ for some relevant subset of elements in Ω. Nevertheless, under the assumption that ν is ergodic by one of the generators, say g_1, we may take for each $\omega \in \Omega$ the set

$$Y_\omega = \bigcup_{k\,\in\,\mathbb{N}} g_1^{-k}(C^\omega)$$

and conclude that, as $Y_\omega \subset g_1^{-1}(Y_\omega)$, we have $\nu(Y_\omega) = 1$. That is, for each $\omega \in \Omega$ and every $x \in Y_\omega$, there is $k \in \mathbb{N}$ such that $g_1^k(x) \in C^\omega$.

Concerning item (4), observe that, if $\mathbb{P} \times \nu$ is ergodic with respect to $\mathcal{F}_G$, then $\varphi^*_\omega = \int \varphi d\nu = \nu(A) > 0$ for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$ and ν-almost every $x \in X$. That is, $(\mathbb{P} \times \nu)(C) = 1$ and there exists a subset $\Omega \subset \Sigma_p^+$ with $\mathbb{P}(\Omega) = 1$ such that, for every $\omega \in \Omega$, we have $\nu(C^\omega) = 1$. So, without assuming the ergodicity of ν with respect to one of the generators, the first part of the argument in the previous paragraph shows that, for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$ and ν-almost every $x \in X$, the orbit $(f_\omega^m(x))_{m\,\in\,\mathbb{N}}$ of x returns infinitely many times to A. This completes the proofs of Proposition 3.1 and Theorem A. □

Remark 3.2. The full measure subset mentioned in Proposition 3.1 depends on the sequential dynamical system $((f_\omega^n)_{\omega\,\in\,\Sigma_p^+})_{n\,\geq\,1}$. Nevertheless, the argument used in its proof contains a stronger statement if the semigroup G is finite or countable (as, for instance $\mathbb{Z}_+^p$): *if A is a positive ν-measure subset of X, then there exists $B \subset A$ such that, for every $\omega \in \Sigma_p^+$, there are positive integers $n \geq k$ such that $g_{\omega_n} g_{\omega_{n-1}} \dots g_{\omega_k}(x) \in A$.*

Example 3.3. Let X be a compact connected Riemannian manifold, m stand for the volume measure in X, $A \subset X$ be an open set, $\mathrm{Diff}^1_m(X)$ denote the group of C^1 volume preserving diffeomorphisms on X and $G_1 \subset \mathrm{Diff}^1_m(X)$ be a finite set. Then, for any sequence $(f_n)_{n\,\in\,\mathbb{N}}$ in

$G_1^{\mathbb{N}}$, there exists an m-full measure subset of points $x \in A$ for which we may find infinitely many positive integers $k_i(x) < \ell_i(x)$ such that $f_{\ell_i} \circ \cdots \circ f_{k_i}(x) \in A$.

4. Proof of Theorem B

Throughout this section we will study recurrence properties for semigroup actions using ergodic information about the skew product $\mathcal{F}_G$ and the measure $\mathbb{P} \times \nu$. Take $\nu \in \mathcal{M}_G$ and a Borel σ-invariant probability measure $\mathbb{P}$. The corresponding skew product $\mathcal{F}_G : \Sigma_p^+ \times X \to \Sigma_p^+ \times X$ has been defined in (1) and preserves the probability measure $\mathbb{P} \times \nu$. The next result is a quenched version of the expected first return time and provides an averaged fibred Kac's Lemma, from which Theorem B is a direct consequence.

Proposition 4.1. *Assume that $\mathbb{P} \times \nu$ is ergodic with respect to the skew product $\mathcal{F}_G$. Then, given a measurable set $A \subset X$ with $\nu(A) > 0$, for $\mathbb{P}$-almost every ω in Σ_p^+ one has*

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_A n_A^{\sigma^j(\omega)}(x)\, d\nu_A(x) = \frac{1}{\nu(A)}.$$

If, in addition, $\mathbb{P}$ is mixing, then, for $\mathbb{P}$-almost every ω,

$$\lim_{k \to +\infty} \int_A n_A^{\sigma^k(\omega)}(x)\, d\nu_A(x) = \frac{1}{\nu(A)}$$

Proof. Firstly, the Proposition 3.1 ensures that, for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$, the set A_ω of points $x \in A$ whose orbit $\big(f_\omega^k(x)\big)_{k \in \mathbb{N}}$ returns to A infinitely often has full ν-measure in A. Therefore, we may consider the map $\varphi : \Sigma_p^+ \to \mathbb{R}$ defined by

$$\omega \in \Sigma_p^+ \mapsto \varphi(\omega) = \int_A n_A^\omega(x)\, d\nu(x)$$

where $n_A^\omega(\cdot)$ denotes the first hitting time to the set A by the sequence $(f_\omega^n)_{n \geq 1}$ (cf. definition in (3)). The map φ is measurable and, as we are assuming that $\mathbb{P} \times \nu$ is ergodic with respect to $\mathcal{F}_G$, then, by Kac's Lemma, φ belongs to $L^1(\mathbb{P})$ and

$$\int \varphi\, d\mathbb{P} = \int_{\Sigma_p^+} \int_A n_A^\omega(x)\, d\nu(x)\, d\mathbb{P}(\omega) = 1. \tag{10}$$

Besides, as $\mathbb{P}$ is ergodic (a consequence of the ergodicity of $\mathbb{P} \times \nu$), the application of Birkhoff's Ergodic Theorem to φ and $\mathbb{P}$ yields that, for $\mathbb{P}$-almost every ω,

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \int_A n_A^{\sigma^j(\omega)}(x)\, d\nu(x) = \int \int_A n_A^\omega(x)\, d\nu(x) d\mathbb{P}(\omega) = 1.$$

If $\mathbb{P}$ is mixing, then, for $\mathbb{P}$-almost every ω,

$$\lim_{k \to +\infty} \int_A n_A^{\sigma^k(\omega)} d\nu(x) = 1.$$

□

Lemma 4.2. *The map $\varphi : \Sigma_p^+ \to \mathbb{R}$ is lower semi-continuous.*

Proof. First of all, we notice that $n_A^\omega(x) < \infty$ for $\mathbb{P}$-almost every $\omega \in \Sigma_p^+$ and ν-almost every $x \in X$ (cf. Theorem A). For such an $\omega = \omega_1\, \omega_2\, \cdots \in \Sigma_p^+$ and $k \in \mathbb{N}$, let A_k be the set $\{x \in A \colon n_A^\omega(x) = k\}$ and $[\omega_1 \dots \omega_k]$ denote the set of sequences $\theta \in \Sigma_p^+$ such that $\theta_i = \omega_i$ for all $1 \leq i \leq k$. Observe also that, if $\theta \in [\omega_1 \dots \omega_k]$, then $n_A^\omega(x) = n_A^\theta(x)$ for any $x \in A_k$. Besides, as

$$\int_A n_A^\omega(x)\, d\nu(x) = \sum_{k=1}^{\infty} k\, \nu(A_k) < \infty$$

for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\sum_{k=N(\varepsilon)+1}^{\infty} k\,\nu(A_k) < \varepsilon$. Therefore,

$$\varphi(\omega) = \sum_{k=1}^{N(\varepsilon)} k\,\nu(A_k) + \sum_{k=N(\varepsilon)+1}^{\infty} k\,\nu(A_k) < \sum_{k=1}^{N(\varepsilon)} k\,\nu(A_k) + \varepsilon$$

and so

$$\varphi(\omega) - \varepsilon < \sum_{k=1}^{N(\varepsilon)} k\,\nu(A_k) < \sum_{k=1}^{N(\varepsilon)} k\,\nu(A_k) + \int_{A\setminus \cup_{k=1}^{N(\varepsilon)} A_k} n_A^\theta(x)\,d\nu(x) = \int_A n_A^\theta(x)\,d\nu(x) = \varphi(\theta).$$

□

As φ is lower semi-continuous, φ has a residual set $\mathcal{C}$ of points of continuity and there exists $\bar\omega$ in the support of the measure $\mathbb{P}$ where φ attains its minimum, that is,

$$\int_A n_A^{\bar\omega}(x)\,d\nu(x) = \min_{\omega\,\in\,\Sigma_p^+} \int_A n_A^{\omega}(x)\,d\nu(x).$$

Moreover, if $\mathbb{P} = \mathbb{P}_{\underline{a}}$, then it is positive on nonempty open sets and so we may take $\omega_0 \in \mathcal{C} \cap \operatorname{supp}\mathbb{P}$. As $\mathbb{P}_{\underline{a}}$ is mixing, there exists $\omega \in \Sigma_p^+$ and a sequence $n_k \to \infty$ such that $\lim_{k\to\infty} d\,(\sigma^{n_k}(\omega), \omega_0) = 0$ and $\lim_{k\,\to\,+\infty} \int_A n_A^{\sigma^{n_k}(\omega)} d\nu(x) = 1$. Consequently, as ω_0 is a continuity point of φ,

$$\int_A n_A^{\omega_0}(x)\,d\nu(x) = \varphi(\omega_0) = \lim_{k\,\to\,+\infty} \varphi(\sigma^{n_k}(\omega)) = \lim_{k\,\to\,+\infty} \int_A n_A^{\sigma^{n_k}(\omega)} d\nu(x) = 1.$$

5. Proof of Theorem C

Firstly we will introduce the definition of measure-theoretic entropy for a free semigroup action. Afterwards, we will deduce a partial variational principle.

5.1. **Measure-theoretic entropy of a free semigroup action.** Let $\mathbb{P}$ be a σ-invariant probability measure and ν a probability measure invariant by any generator in G_1^*. Given a measurable finite partition β of X, $n \in \mathbb{N}$ and $\omega = \omega_1\omega_2\cdots \in \Sigma_p^+$, define

$$\begin{aligned} \beta_1^n(\omega) &= g_{\omega_1}^{-1}\beta \bigvee g_{\omega_1}^{-1}g_{\omega_2}^{-1}\beta \bigvee \cdots \bigvee g_{\omega_1}^{-1}g_{\omega_2}^{-1}\cdots g_{\omega_{n-1}}^{-1}\beta \qquad (11)\\ \beta_0^n(\omega) &= \beta \bigvee \beta_1^n(\omega) \quad \text{and} \quad \beta_1^\infty(\omega) = \bigvee_{n=1}^{\infty} \beta_1^n(\omega). \end{aligned}$$

Then the conditional entropy of β relative to $\beta_1^\infty(\omega)$, denoted by $H_\nu(\beta|\beta_1^\infty(\omega))$, is a measurable function of ω and $\mathbb{P}$-integrable (cf. [19]). Let $h_\nu(\mathbb{S},\mathbb{P},\beta) = \int_{\Sigma_p^+} H_\nu(\beta|\beta_1^\infty(\omega))\,d\,\mathbb{P}(\omega)$. Proposition 1.1 of [19, §6] shows that

$$h_\nu(\mathbb{S},\mathbb{P},\beta) = \lim_{n\,\to\,+\infty} \frac{1}{n} \int_{\Sigma_p^+} H_\nu(\beta_0^n(\omega))\,d\,\mathbb{P}(\omega). \qquad (12)$$

where $H_\nu(\beta_0^n(\omega))$ is the entropy of the partition $\beta_0^n(\omega)$.

Definition 5.1. The *metric entropy of the semigroup action with respect to* $\mathbb{P}$ *and* ν is given by

$$h_\nu(\mathbb{S},\mathbb{P}) = \sup_\beta h_\nu(\mathbb{S},\mathbb{P},\beta).$$

For instance, if $\mathbb{P}$ is a Dirac measure $\delta_{\underline{j}}$ supported on a fixed point $\underline{j} = jj\cdots$, where $j \in \{1,\cdots,p\}$, then $h_\nu(\mathbb{S},\delta_{\underline{j}}) = h_\nu(g_j)$. If, instead, $\mathbb{P}$ is the symmetric random walk, that is, the Bernoulli $(\frac{1}{p},\cdots,\frac{1}{p})$-product probability measure $\mathbb{P}_{\underline{p}}$, then (compare with [16, Definition 4.1])

$$h_\nu(\mathbb{S},\mathbb{P}_{\underline{p}}) = \sup_\beta \lim_{n\,\to\,+\infty} \frac{1}{n}\left(\frac{1}{p^n}\sum_{|\omega|=n} H_\nu(\beta_0^n(\omega))\right).$$

Let us resume the proof of Theorem C. For every ν and $\mathbb{P}$ as prescribed before, Abramov and Rokhlin proved that

$$h_{\mathbb{P}\times\nu}(\mathcal{F}_G) = h_{\mathbb{P}}(\sigma) + h_\nu(\mathbb{S},\mathbb{P}). \tag{13}$$

If we now summon Bufetov's formula $h_{\text{top}}(\mathcal{F}_G) = \log p + h_{\text{top}}(\mathbb{S})$ from [9] then we conclude that, for every σ-invariant probability measure $\mathbb{P}$, we have

$$\begin{aligned} \sup_{\nu\,\in\,\mathcal{M}_G} \; h_\nu(\mathbb{S},\mathbb{P}) \;&\leq\; \sup_{\nu\,\in\,\mathcal{M}_G} \{\, h_{\mathbb{P}\times\nu}(\mathcal{F}_G) - h_{\mathbb{P}}(\sigma) \,\} = \sup_{\nu\,\in\,\mathcal{M}_G} \{\, h_{\mathbb{P}\times\nu}(\mathcal{F}_G) \,\} - h_{\mathbb{P}}(\sigma) \\ &\leq\; h_{\text{top}}(\mathcal{F}_G) - h_{\mathbb{P}}(\sigma) = h_{\text{top}}(\mathbb{S}) + \log p - h_{\mathbb{P}}(\sigma). \end{aligned}$$

When $\mathbb{P} = \mathbb{P}_{\underline{p}}$, as $h_{\mathbb{P}_{\underline{p}}}(\sigma) = \log p$, we obtain

$$\sup_{\nu\in\mathcal{M}_G} \; h_\nu(\mathbb{S},\mathbb{P}_{\underline{p}}) \leq h_{\text{top}}(\mathbb{S}).$$

If each generator g_i, for $i = 1,\cdots,p$ is C^2 expanding and $\mathbb{P}$ is a Bernoulli probability measure $\mathbb{P}_{\underline{a}}$ for some probability vector $\underline{a} = (a_1,\cdots,a_p)$, then

$$\begin{aligned} \sup_{\nu\,\in\,\mathcal{M}_G} \; h_\nu(\mathbb{S},\mathbb{P}_{\underline{a}}) \;&=\; \sup_{\nu\,\in\,\mathcal{M}_G} \{\, h_{\mathbb{P}_{\underline{a}}\times\nu}(\mathcal{F}_G) \,\} - h_{\mathbb{P}_{\underline{a}}}(\sigma) \leq \sup_{\mu:\;(\mathcal{F}_G)_*\mu=\mu,\;\pi_*\mu=\mathbb{P}_{\underline{a}}} \{\, h_\mu(\mathcal{F}_G) \,\} - h_{\mathbb{P}_{\underline{a}}}(\sigma) \\ &=\; P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{a}}) \leq P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{a}}) = h_{\text{top}}(\mathbb{S},\mathbb{P}_{\underline{a}}). \end{aligned}$$

Remark 5.2. Observe that, when $\underline{a} = \underline{p}$, we have (cf. [10])

$$h_{\text{top}}(\mathbb{S},\mathbb{P}_{\underline{p}}) = h_{\text{top}}(\mathbb{S}) \qquad \text{and} \qquad P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{p}}) < P^{(a)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{p}}).$$

So, in this case, we get $\sup_{\nu\,\in\,\mathcal{M}_G} \; h_\nu(\mathbb{S},\mathbb{P}_{\underline{p}}) < h_{\text{top}}(\mathbb{S})$.

Example 5.3. Let $g_1 : \mathcal{S}^1 \to \mathcal{S}^1$ and $g_2 : \mathcal{S}^1 \to \mathcal{S}^1$ be the unit circle expanding maps given by $g_1(z) = z^2$ and $g_2(z) = z^3$ and consider the free semigroup G generated by $G_1 = \{id, g_1, g_2\}$. Their topological entropies are $\log 2$ and $\log 3$, respectively. Let $\mathbb{S}$ be the corresponding semigroup action. According to [10, Section §8], we have $h_{\text{top}}(\mathcal{F}_G) = \log 5 \sim 1.609$, $h_{\text{top}}(\mathbb{S}) = \log(\frac{5}{2}) \sim 0.916$ and $P^{(q)}_{\text{top}}(\mathcal{F}_G, 0, \mathbb{P}_{\underline{2}}) = \frac{\log 3 + \log 2}{2} \sim 0.896$.

Remark 5.4. Each time we fix $\omega = \omega_1\omega_2\cdots \in \Sigma_p^+$, we restrict the semigroup action to a sequential dynamical system, we will denote by ω-SDS, whose orbits are the sequences $(f^n_\omega(x))_{n\,\in\,\mathbb{N}_0;\,x\,\in\,X}$. Given $\omega \in \Sigma_p^+$ and $\nu \in \mathcal{M}_G$, we may define the measure-theoretic entropy of the ω-SDS by $h_\nu(\omega\text{-SDS}) = \sup_\beta \; h_\nu(\omega\text{-SDS}, \beta)$, where β is any measurable finite partition of X,

$$h_\nu(\omega\text{-SDS}, \beta) = \lim_{n\,\to\,+\infty} \; \frac{1}{n} \, H_\nu(\beta_0^n(\omega)) \tag{14}$$

and $\beta_0^n(\omega)$, $\beta_1^n(\omega)$ are as in (11). Then, using the Dominated Convergence Theorem, it is not hard to prove that, for every probability measure $\nu \in \mathcal{M}_G$, we have

$$h_\nu(\mathbb{S},\mathbb{P}) \leq \int_{\Sigma_p^+} h_\nu(\omega\text{-SDS}) \, d\,\mathbb{P}(\omega).$$

6. Proof of Theorem D

We will start this section recalling the notion of orbital specification property introduced in [21] and a few facts about recurrence by the skew product associated to a free semigroup action. The reader acquainted with this preliminary information may omit the next two subsections.

6.1. **Orbital specification.** We say that the continuous semigroup action $\mathbb{S} : G \times X \to X$ associated to the finitely generated semigroup G satisfies the *weak orbital specification property* if, for any $\delta > 0$, there exists $q(\delta) > 0$ such that, for any $q \geq q(\delta)$, we may find a set $\tilde{G}_q \subset G^*_q$ satisfying $\lim_{q\to\infty} \sharp\, \tilde{G}_p / \sharp\, G^*_p = 1$ and for which the following shadowing property holds: for any $h_{q_j} \in \tilde{G}_{q_j}$ with $q_j \geq q(\delta)$, any points $x_1,\ldots,x_k \in X$, any natural numbers $n_1,\ldots,n_k$ and any concatenations $\underline{g}_{n_j,j} = g_{i_{n_j},j} \ldots g_{i_2,j}\, g_{i_1,j} \in G_{n_j}$ with $1 \leq j \leq k$, there exists $x \in X$ satisfying $\text{dist}(\underline{g}_{\ell,1}(x) \;,\; \underline{g}_{\ell,1}(x_1)) < \delta, \forall \ell = 1,\ldots,n_1$ and dist $(\underline{g}_{\ell,j}\, \underline{h}_{q_{j-1}} \;\cdots\; \underline{g}_{n_2,2}\, \underline{h}_{q_1}\, \underline{g}_{n_1,1}(x) \;,\; \underline{g}_{\ell,j}(x_j)) < \delta$

for all $j = 2, \dots, k$ and $\ell = 1, \dots, n_j$. If $\tilde{G}_p$ can be taken equal to G_p^*, we say that $\mathbb{S}$ satisfies the *strong orbital specification property*. If the point x can be chosen in $\mathrm{Per}(G)$, then we refer to this property as the *periodic orbital specification property*. For instance, it is true for finitely generated semigroups of topologically mixing Ruelle expanding maps (cf. [21, Theorem 16]).

6.2. **First return times.** Although the recurrence for a semigroup action $\mathbb{S}$ and for the random dynamical system modeled by the skew product $\mathcal{F}_G$ are not the same, they are nevertheless bonded. Given a measurable subset A of X and any $x \in A$, we may define *the first return of x to A by the semigroup action* as follows

$$n_A^{\mathbb{S}}(x) = \begin{cases} \inf \left\{ n_A^\omega(x) \colon \omega \in \Sigma_p^+ \right\} & \text{if this set is nonempty} \\ +\infty & \text{otherwise.} \end{cases} \tag{15}$$

Then $n_A^{\mathbb{S}}(x) = \inf \{k \geq 1 \colon \mathcal{F}_G^k(\Sigma_p^+ \times \{x\}) \cap (\Sigma_p^+ \times A) \neq \emptyset\}$. Moreover, given $B \subset \Sigma_p^+ \times X$, we may take *the shortest return time of B to itself by the skew product $\mathcal{F}_G$*, that is,

$$\mathcal{T}^{\mathcal{F}_G}(B) = \inf \{k \in \mathbb{N} \colon \mathcal{F}_G^k(B) \cap B \neq \emptyset\}.$$

In particular, if $B = \Sigma_p^+ \times A$, we obtain

$$\inf_{x \in A} n_A^{\mathbb{S}}(x) = \mathcal{T}^{\mathcal{F}_G}(\Sigma_p^+ \times A) = \inf_{\omega \in \Sigma_p^+} \mathcal{T}^\omega(A)$$

and (see Definition 6)

$$\mathcal{T}^{\mathbb{S}}(A) = \inf \{k \geq 1 \colon \mathcal{F}_G^k(\Sigma_p^+ \times A) \cap (\Sigma_p^+ \times A) \neq \emptyset\} = \mathcal{T}^{\mathcal{F}_G}(\Sigma_p^+ \times A). \tag{16}$$

The pointwise return time functions for the semigroup action $\mathbb{S}$ and the skew product $\mathcal{F}_G$ are also related: by (9), given a measurable set $A \subset X$, for every $x \in X$ and $\omega \in \Sigma_p^+$ we have

$$n_A^\omega(x) = n_{\Sigma_p^+ \times A}^{\mathcal{F}_G}(\omega, x) = \text{ first return time of } (\omega, x) \text{ to the set } \Sigma_p^+ \times A \text{ by } \mathcal{F}_G.$$

6.3. **Shortest returns of balls and Lyapunov exponents.** In the special case of semigroups of topologically mixing expanding maps, it is known that the skew product map $\mathcal{F}_G$ satisfies the periodic specification property (see e.g. [21, Theorem 28]). Moreover, if $\mathbb{P} \times \nu$ has positive entropy with respect to $\mathcal{F}_G$ then, using (16), for $\mathbb{P} \times \nu$-almost every (ω, x), one has (cf. [3, 25])

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{\mathcal{T}^{\mathcal{F}_G}(B_\delta((\omega, x), n))}{n} = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{\mathcal{T}^{\mathcal{F}_G}(B_\delta((\omega, x), n))}{n} = 1$$

where $B_\delta^\omega(x, n) = \{y \in X \colon d(f_\omega^j(x), f_\omega^j(y)) < \delta, \ \forall\, 0 \leq j \leq n-1\}$ stands for the dynamical ball with center x, radius δ and length n for the dynamics $(f_\omega^n)_{n \geq 1}$. The next result generalizes this statement, employing a notion of metric entropy of the semigroup action whose definition will be given in Subsection 5.1.

Proposition 6.1. *Let G be the semigroup generated by $G_1 = \{id, g_1, \dots, g_p\}$, where the elements in G_1^* are C^1 expanding maps on a compact connected Riemannian manifold X, satisfy the orbital specification property and preserve a Borel probability measure ν on X. Consider a σ-invariant Borel probability measure $\mathbb{P}$ on Σ_p^+ such that $h_\nu(\mathbb{S}, \mathbb{P}) > 0$. Assume also that $\mathbb{P} \times \nu$ is ergodic with respect to $\mathcal{F}_G$. Then, for $\mathbb{P}$-almost every ω and ν-almost every $x \in X$, we have*

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{\mathcal{T}^\omega(B_\delta^\omega(x, n))}{n} = \lim_{\delta \to 0} \liminf_{n \to +\infty} \frac{\mathcal{T}^\omega(B_\delta^\omega(x, n))}{n} = 1.$$

Proof. Firstly, observe that, as we are considering the product metric in $\Sigma_p^+ \times X$, then $B_\delta((\omega, x)) = B_\delta(\omega) \times B_\delta(x)$ for every $(\omega, x) \in \Sigma_p^+ \times X$ and any $\delta > 0$. Moreover, dynamical balls with respect to the skew product dynamics $\mathcal{F}_G$ are in fact dynamical balls for the random composition of dynamics; that is, for every $n \geq 1$,

$$B_\delta((\omega, x), n) = \bigcup_{\theta \in B_\delta(\omega, n)} \{\theta\} \times B_\delta^\theta(x, n). \tag{17}$$

Besides, if we take $\Lambda = \max_{1 \leq i \leq p;\, x \in X} \|Dg_i(x)\|$ and $\lambda = \min_{1 \leq i \leq p;\, x \in X} \|Dg_i(x)\|$, then clearly

$$B_\delta(\omega, n) \times B_{\delta\Lambda^{-n}}(x) \subset B_\delta((\omega, x), n) \subset B_\delta(\omega, n) \times B_{\delta\lambda^{-n}}(x) \tag{18}$$

for every $x \in X$ and $n \geq 1$, which implies that the corresponding first return times are in decreasing order.

The periodic orbital specification property of the skew-product guarantees that, for any $\delta > 0$, there exists $N_\delta \geq 1$ such that, given $n \geq 1$, we may find a periodic point $y \in B^\omega_\delta(x,n) \cap \text{Fix}\,(f_\omega^{n+N_\delta})$. In particular, $\mathcal{T}^\omega(B^\omega_\delta(x,n)) \leq n + N_\delta$ and, consequently,

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{\mathcal{T}^\omega(B^\omega_\delta(x,n))}{n} \leq 1.$$

To complete the proof we are left to show that, for $(\mathbb{P} \times \nu)$-almost every (ω, x),

$$\lim_{\delta \to 0} \liminf_{n \to +\infty} \frac{\mathcal{T}^\omega(B^\omega_\delta(x,n))}{n} \geq 1. \tag{19}$$

We will argue as in [25, pages 2372–2373]. Notice that, as $\mathbb{P} \times \nu$ is ergodic and $h_\nu(\mathbb{S}, \mathbb{P}) > 0$, Theorem 2.1 of [28] informs that, for $\mathbb{P} \times \nu$-almost every (ω, x),

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{n} \log \nu(B^\omega_\delta(x,n)) = \lim_{\delta \to 0} \liminf_{n \to +\infty} -\frac{1}{n} \log \nu(B^\omega_\delta(x,n)) > 0 \tag{20}$$

and that

$$h_\nu(\mathbb{S}, \mathbb{P}) = \int \Big[\lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{n} \log \nu(B^\omega_\delta(x,n)) \Big] \, d(\mathbb{P} \times \nu)(\omega, x).$$

Take now a finite measurable partition β of X satisfying $\nu(\partial\beta) = 0$ and $h_\nu(\mathbb{S}, \mathbb{P}, \beta) > 0$. Let $V_\delta(\partial\beta)$ stand for the neighborhood of size δ of $\Sigma_p^+ \times \partial\beta$ in $\Sigma_p^+ \times X$; notice that $(\mathbb{P} \times \nu)(V_\delta(\partial\beta)) = \nu(V_\delta(\partial\beta))$. The Random Ergodic Theorem (cf. Subsection 3.1) assures that, for any small $\gamma > 0$, there exists $\delta > 0$ such that, at $\mathbb{P} \times \nu$-almost everywhere, one has

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{V_\delta(\partial\beta)}(\mathcal{F}_G^j(\omega, x)) \leq 2\,(\mathbb{P} \times \nu)(V_\delta(\partial\beta)) < \gamma. \tag{21}$$

Fix $\omega \in \Sigma_p^+$ in the $\mathbb{P}$-full measure subset of Σ_p^+ so that (20) and (21) hold. As the semigroup action $\mathbb{S}$ is ergodic (cf. definition in Subsection 2.2.2), for any ξ, $\varepsilon > 0$ small enough there exist $N \in \mathbb{N}$ and a measurable set $E^\omega_\xi \subset X$ satisfying $\nu(E^\omega_\xi) > 1 - \xi$,

$$e^{-\,n\,(h_\nu(\mathbb{S}, \mathbb{P}, \beta) + \xi)} \leq \nu(\beta_0^n(\omega)(x)) \leq e^{-\,n\,(h_\nu(\mathbb{S}, \mathbb{P}, \beta) - \xi)} \tag{22}$$

and $\sum_{j=0}^{n-1} \delta_{\mathcal{F}_G^j(\omega,x)} \leq \gamma n$ for all $x \in E^\omega_\xi$ and $n \geq N$. Besides, by equation (22), there exists $K_\omega > 0$ such that

$$K_\omega^{-1} e^{-\,n\,(h_\nu(\mathbb{S}, \mathbb{P}, \beta) + \xi)} \leq \nu(\beta_0^n(\omega)(x)) \leq K_\omega e^{-\,n\,(h_\nu(\mathbb{S}, \mathbb{P}, \beta) - \xi)}$$

for every $n \geq 1$ and $x \in E^\omega_\xi$. As $\xi > 0$ was chosen arbitrary, in order to prove (19) for ν-almost every x, it is enough to show, using Borel-Cantelli Lemma, that $\nu\big(\{x \in E^\omega_\xi : \mathcal{T}^\omega(B^\omega_\delta(x,n)) \leq (1-\xi)\,n\}\big)$ is summable for every small δ.

We proceed covering the dynamical ball $B^\omega_\delta(x,n) \subset X$ by a collection $\tilde\beta_0^n(\omega)$ of partition elements in $\beta_0^n(\omega)$. If $\delta > 0$ is chosen small enough, then (21) implies that the piece of orbit $(f_\omega^j(x))_{j=0}^n$ enters the δ-neighborhood of $\partial\beta$ in at most $\gamma\,n$ iterates. The argument used in [25, Lemma 3.2] implies that, for any $\alpha > 0$, there exist $\gamma > 0$ and $\delta > 0$ (given by (21)) so that $B^\omega_\delta(x,n) \subset X$ is covered by a collection $\tilde\beta_0^n(\omega)$ of at most $e^{\alpha n}$ partition elements of $\beta_0^n(\omega)$, for every $x \in E^\omega_\xi$. Therefore,

$$\begin{aligned}
\nu\big(\{x \in E^\omega_\xi : \mathcal{T}^\omega(B^\omega_\delta(x,n)) \leq (1-\xi)\,n\}\big) &= \sum_{k=0}^{(1-\xi)n} \nu\big(\{x \in E^\omega_\xi : \mathcal{T}^\omega(B^\omega_\delta(x,n)) = k\}\big) \\
&\leq \sum_{k=0}^{(1-\xi)n} \sum_{\substack{Q \in \tilde\beta_0^n(\omega) \\ f_\omega^k(Q) \in \tilde\beta_0^n(\omega)}} \nu(E^\omega_\xi \cap Q).
\end{aligned}$$

Note that $B^\omega_\delta(x,n)$ is covered by at most $e^{\alpha n}$ elements of $\tilde\beta^n_0(\omega)$ and, among these, every $Q \in \tilde\beta^n_0(\omega)$ satisfying $f^k_\omega(Q) \in \tilde\beta^n_0(\omega)$ is determined by the first k elements of the partition β that are visited under the iterations of f^j_ω, $1 \le j \le k$. Thus

$$\begin{aligned}
\nu\big(\{x \in E^\omega_\xi \colon \mathcal{T}^\omega(B^\omega_\delta(x,n)) \le (1-\xi)\, n\}\big) \\
&\le K_\omega e^{-n\,(h_\nu(\mathbb{S},\mathbb{P},\beta)\,-\,\xi)}\, e^{\alpha n} \sum_{k=0}^{(1-\xi)n} \#\{Q \,\in\, \tilde\beta^n_0(\omega) : f^k_\omega(Q) \,\in\, \tilde\beta^n_0(\omega)\} \\
&\le K_\omega e^{-n\,(h_\nu(\mathbb{S},\mathbb{P},\beta)\,-\,\xi)}\, e^{\alpha n} \sum_{k=0}^{(1-\xi)n} K_\omega e^{k\,(h_\nu(\mathbb{S},\mathbb{P},\beta)\,+\,\xi)} \\
&\le K^2_\omega (1-\xi) n\, e^{-n\,(h_\nu(\mathbb{S},\mathbb{P},\beta)\,-\,\xi)}\, e^{\alpha n}\, e^{(1-\xi)\,n\,(h_\nu(\mathbb{S},\mathbb{P},\beta)\,+\,\xi)}
\end{aligned}$$

and so it is summable provided that α, ξ are small enough. □

If we restrict to either C^1 expanding or conformal maps on a Riemannian manifold, we obtain the following corollary and finish the proof of Theorem D.

Corollary 6.2. *Let G be the semigroup generated by $G_1 = \{id, g_1, \dots, g_p\}$, where the elements in G_1^* are C^1 expanding maps on a compact connected Riemannian manifold X preserving a common Borel probability measure ν on X, and $\mathbb{P}$ be a σ-invariant Borel probability measure on Σ_p^+. Assume that $\mathbb{P} \times \nu$ is ergodic with respect to $\mathcal{F}_G$ and that $h_\nu(\mathbb{S},\mathbb{P}) > 0$. Then, for $\mathbb{P}$-almost every ω and ν-almost every $x \in X$, we have*

$$\frac{1}{\log \Lambda} \le \liminf_{\delta\to 0} \frac{\mathcal{T}^\omega(B_\delta(x))}{-\log\delta} \le \limsup_{\delta\to 0} \frac{\mathcal{T}^\omega(B_\delta(x))}{-\log\delta} \le \frac{1}{\log\lambda} \tag{23}$$

where $\Lambda \ge \lambda > 1$ are, respectively, $\Lambda = \max_{1\le i\le p} \|Dg_i\|_\infty$ and $\lambda = \min_{1\le i\le p} \|Dg_i\|_\infty$.

Proof. As a consequence of [21, Theorem 16], we know that every finitely generated free semigroup action by C^1 expanding maps on compact connected manifolds satisfies the orbital specification property. Moreover, for every $x \in X$, $n \in \mathbb{N}$, $\omega \in \Sigma_p^+$ and $\delta > 0$, we have $B_{\Lambda^{-n}\delta}(x) \subset B^\omega_\delta(x,n) \subset B_{\lambda^{-n}\delta}(x)$. Thus,if ν is a probability measure invariant by all elements in G, $\mathbb{P}\times\nu$ is ergodic for $\mathcal{F}_G$ and $h_\nu(\mathbb{S},\mathbb{P}) > 0$, we conclude from Proposition 6.1 that

$$\limsup_{\delta\to 0} \frac{\mathcal{T}^\omega(B_\delta(x))}{-\log\delta} \le \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{\mathcal{T}^\omega(B_{\lambda^{-n}\delta}(x))}{-\log(\lambda^{-n}\delta)} \le \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{\mathcal{T}^\omega(B^\omega_\delta(x,n))}{n\log\lambda} = \frac{1}{\log\lambda}.$$

The lower bound estimate is obtained by a similar reasoning. □

It is a straightforward outcome of (23) that, in the case of semigroups of C^1-expanding maps, one has

$$\limsup_{\delta\to 0} \frac{\mathcal{T}^{\mathbb{S}}(B_\delta(x))}{-\log\delta} \le \frac{1}{\log\lambda}.$$

Although we believe that a similar lower bound holds, we have not been able to obtain it.

Remark 6.3. Notice that, if the generators are not expanding maps one expects larger return times. For instance, if the semigroup action is generated by circle rotations with rotation numbers in the interval $0 < \alpha_0 \le \alpha \le \alpha_1 < 1$ then it is not hard to check that $\mathcal{T}^{\mathbb{S}}(B_\delta(x)) \le (\frac{1}{\alpha_1}+1)\frac{1}{\delta}$ for every $\delta > 0$ and $x \in \mathcal{S}^1$. However, it is not clear whether this bound is optimal.

If, besides being expanding, all elements in G_1^* are conformal, then $Dg_i(x) = \|Dg_i(x)\| Id$ and $\det|Dg_i(x)| = \|Dg_i(x)\|^{\dim X}$ for every $x \in X$ and any $i \in \{1, 2, \cdots, p\}$. In particular, it follows from Oseledets' Theorem that all the Lyapunov exponents of the skew product $\mathcal{F}_G$ along the fiber direction are equal and coincide with

$$\begin{aligned}\chi := \frac{1}{\dim X} \lim_{n \to +\infty} \frac{1}{n} \log|\det Df^n_\omega(x)| &= \frac{1}{\dim X} \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \Big| \det \frac{\partial \mathcal{F}_G}{\partial x}(\mathcal{F}_G^j(\omega, x) \Big| \\ = \frac{1}{\dim X} \int \int \log \Big| \det \frac{\partial \mathcal{F}_G}{\partial x}(\omega, x) \Big| \, d\nu(x) \, d\mathbb{P}_{\underline{a}}(\omega) &= \frac{1}{\dim X} \sum_{i=1}^{p} \, a_i \int \log|\det Dg_i| \, d\nu\end{aligned}$$

(notice that, in the last but one estimate, we have used the ergodicity of $\mathbb{P} \times \nu$). Moreover, the expanding nature of the generators in G_1^* and the assumption that $\mathbb{P} = \mathbb{P}_{\underline{a}}$, for some probability vector $\underline{a}$, imply that $\chi > 0$. Observe also that as $\|Dg_i(x)\| = \det|Dg_i(x)|^{\frac{1}{\dim X}}$ then, as a consequence of the mean value theorem, given $\delta > 0$, for $\mathbb{P}_{\underline{a}} \times \nu$-almost every (ω, x) there exists $N = N(\omega, x) \geq 1$ such that, for all $n \geq N$,

$$B_{e^{-(\chi+\varepsilon)\, n}\, \delta}(x) \subset B^\omega_\delta(x, n) \subset B_{e^{-(\chi-\varepsilon)\, n}\, \delta}(x).$$

Then, an argument identical to the one used in the first part of this proof yields that

$$\frac{1}{\chi + \delta} \leq \limsup_{\delta \to 0} \frac{\mathcal{T}^\omega(B_\delta(x))}{-\log \delta} \leq \frac{1}{\chi - \delta}.$$

To obtain (8) it is enough to let δ go to 0. This completes the proof of Theorem D.

Remark 6.4. If the measure $\mathbb{P} \times \nu$ has positive entropy and $\mathcal{F}_G$ has the specification property, then it is a consequence of [8, 25] that, for $\mathbb{P} \times \nu$-almost every (ω, x),

$$\lim_{\delta \to 0} \limsup_{n \to +\infty} \frac{\mathcal{T}(B_\delta((\omega, x), n))}{n} = \lim_{\delta \to 0} \liminf_{n \to +\infty} \frac{\mathcal{T}(B_\delta((\omega, x), n))}{n} = 1.$$

This differs substantially from the fibred assertion provided by Theorem D.

Example 6.5. Consider the generators g_1 and g_2 of Example 5.3, the Lebesgue measure *Leb* on $\mathcal{S}^1$ (which is invariant by both dynamics) and the symmetric random walk corresponding to the Borel probability measure $\mathbb{P}_{\underline{2}}$. The maps g_1 and g_2 are conformal, C^1 expanding, $|\det Dg_1(\cdot)| = 2$ and $|\det Dg_2(\cdot)| = 3$. Besides, $\mathbb{P}_{\underline{2}} \times Leb$ is ergodic with respect to the skew product $\mathcal{F}_G$ (it is even weak Bernoulli; cf. [18]). Therefore, by Corollary 6.2 and Example 5.3, for $\mathbb{P}_{\underline{2}}$-almost every ω in Σ_2^+ and *Leb*-almost every $z \in \mathcal{S}^1$, we have

$$\lim_{\delta \to 0} \frac{\mathcal{T}^\omega(B(z, \delta))}{-\log \delta} = \frac{2}{\log 3 + \log 2} > \frac{1}{h_{\text{top}}(\mathbb{S})}.$$

7. Proof of Theorem E

In this section we examine the connection between maximum random hitting frequency and the size of sets when evaluated by different measures. We start reviewing these concepts. Denote by $\mathcal{M}_{\mathcal{F}_G}$ the set of all probability measures invariant by the skew product $\mathcal{F}_G$. For every $\mu \in \mathcal{M}_{\mathcal{F}_G}$ the marginal of μ on X is the probability measure $(\pi_X)_*(\mu) := \mu \circ \pi_X^{-1}$ where $\pi_X : \Sigma_p^+ \times X \to X$ is the natural projection.

7.1. **Random mean sojurns.** Let G be a finitely generated free semigroup, with corresponding action $\mathbb{S}$, and $\mathbb{P}$ a σ-invariant probability measure on Σ_p^+.

Definition 7.1. For a measurable subset $A \subset X$, the *maximum random hitting frequency* of A with respect to $\mathbb{P}$ is given by

$$\gamma_{\mathbb{P}}(A) = \mathbb{P} - \text{esssup} \sup_{x \in X} \; \gamma_{\omega,x}(A) \tag{24}$$

where

$$\gamma_{\omega,x}(A) = \limsup_{n \to +\infty} \frac{\sharp \, \{0 \leq i \leq n-1 : f^i_\omega(x) \in A\}}{n}. \tag{25}$$

The *absolute maximum hitting frequency* of A with respect to $\mathbb{P}$ is defined by

$$\gamma(A) = \sup_{(\omega,x)\,\in\,\Sigma_p^+\times X} \gamma_{\omega,x}(A).$$

Given a measurable set $A \subset X$, consider the upper bound of its sizes when estimated by all probability measures in $\mathcal{M}_{\mathcal{F}_G}$ which project on $\mathbb{P}$, that is,

$$\alpha_{\mathbb{P}}(A) = \sup_{\{\mu\,\in\,\mathcal{M}_{\mathcal{F}_G}\,:\,\pi_*\mu=\mathbb{P}\}} \mu(\Sigma_p^+ \times A).$$

Lemma 7.2. *If A is a closed subset of X, then there exists an ergodic probability measure $\mu_A \in \mathcal{M}_{\mathcal{F}_G}$ with $\pi_*\mu_A = \mathbb{P}$ and such that $\alpha_{\mathbb{P}}(A) = \mu_A(\Sigma_p^+ \times A)$. Moreover, the set of maximizing measures is compact.*

Proof. Firstly, endow the space $\mathcal{M}_{\mathcal{F}_G}$ with the weak* topology. Therefore, as A is closed, the functional

$$\Psi_A : \mu \in \mathcal{M}_{\mathcal{F}_G} \mapsto \mu(\Sigma_p^+ \times A)$$

is upper semi-continuous (cf. [27]). Moreover, $\mathcal{M}_{\mathcal{F}_G,\mathbb{P}} := \mathcal{M}_{\mathcal{F}_G} \cap \pi_*^{-1}(\mathbb{P})$ is a non-empty compact subset of $\mathcal{M}_{\mathcal{F}_G}$. Hence, Ψ_A attains a maximum in $\mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$.

Let $\mathrm{Erg}(\mathcal{F}_G,\mathbb{P})$ be the ergodic members of $\mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$, and consider a measure $\xi_A \in \mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$ where Ψ_A attains its maximum, whose ergodic decomposition (cf. [27, 20]) in $\mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$ is $\xi_A = \int_{\mathrm{Erg}(\mathcal{F}_G,\mathbb{P})} m\, d\tau(m)$. As ξ_A maximizes Ψ_A, we know that $m(\Sigma_p^+ \times A) \leq \xi_A(\Sigma_p^+ \times A)$ for every $m \in \mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$. Therefore, as $\xi_A(\Sigma_p^+ \times A) = \int_{\mathrm{Erg}(\mathcal{F}_G,\mathbb{P})} m(\Sigma_p^+ \times A)\, d\tau(m)$, we must have $m(\Sigma_p^+ \times A) = \xi_A(\Sigma_p^+ \times A) = \alpha_{\mathbb{P}}(A)$ for τ-almost every m. Thus, we may take an ergodic maximizing measure of Ψ_A, as claimed.

We observe that the upper semi-continuity of Ψ_A also implies that the set of maximizing probability measures is compact. Indeed, if a sequence of measures $\xi_{A,n} \in \mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$ satisfies $\lim_{n\,\to\,+\infty} \xi_{A,n} = \xi$ in the weak* topology and $\alpha_{\mathbb{P}}(A) = \xi_{A,n}(\Sigma_p^+ \times A)$ for every $n \in \mathbb{N}$, then $\xi \in \mathcal{M}_{\mathcal{F}_G,\mathbb{P}}$ and, as Ψ_A is upper semi-continuous, we conclude that

$$\begin{aligned}\alpha_{\mathbb{P}}(A) &= \lim_{n\,\to\,+\infty} \xi_{A,n}(\Sigma_p^+ \times A) = \lim_{n\,\to\,+\infty} \Psi_A(\xi_{A,n}) \leq \Psi_A(\xi) = \xi(\Sigma_p^+ \times A)\\ &\leq \sup_{\{\mu\,\in\,\mathcal{M}_{\mathcal{F}_G}\,:\,\pi_*\mu=\mathbb{P}\}} \mu(\Sigma_p^+ \times A) = \alpha_{\mathbb{P}}(A)\end{aligned}$$

so $\Psi_A(\xi) = \alpha_{\mathbb{P}}(A)$. □

We are now ready to compare the rates of visits with the size of the visited set.

Proposition 7.3. *Let $\mathbb{P}$ be a σ-invariant probability measure on Σ_p^+. Then:*

1. *$\alpha_{\mathbb{P}}(A) \leq \gamma_{\mathbb{P}}(A) \leq \gamma(A)$ for every measurable set $A \subset X$.*
2. *If $\mathbb{P}$ is ergodic, one has:*
   - (a) *For every $x \in X$, there exists an $\mathcal{F}_G$-invariant probability measure $\mu_{\mathbb{P},x}$ such that $\pi_*(\mu_{\mathbb{P},x}) = \mathbb{P}$ and, for every closed set $A \subset X$, $\gamma_{\omega,x}(A) \leq \mu_{\mathbb{P},x}(\Sigma_p^+ \times A)$.*
   - (b) *$\gamma_{\mathbb{P}}(A) = \alpha_{\mathbb{P}}(A)$ for every closed set $A \subset X$.*

Proof. Consider a measurable set $A \subset X$. The inequality $\gamma_{\mathbb{P}}(A) \leq \gamma(A)$ is immediate. Conversely, if μ is an $\mathcal{F}_G$-invariant and ergodic probability measure on $\Sigma_p^+ \times X$, then it follows from Birkhoff's Ergodic Theorem that

$$\lim_{n\,\to\,+\infty} \frac{\sharp\,\{0 \leq i \leq n-1 : f_\omega^i(x) \in A\}}{n} = \lim_{n\,\to\,+\infty} \frac{\sharp\,\{0 \leq i \leq n-1 : \mathcal{F}_G^i(\omega,x) \in \Sigma_p^+ \times A\}}{n} = \mu(\Sigma_p^+ \times A)$$

for μ-almost every (ω, x). Thus, taking the supremum and the essential supremum in the first term of the previous equalities, we conclude that

$$\gamma_{\mathbb{P}}(A) = \mathbb{P} - \operatorname{esssup}\, \sup_{x\,\in\,X}\, \limsup_{n\,\to\,+\infty} \frac{\sharp\,\{0 \leq i \leq n-1 : f_\omega^i(x) \in A\}}{n} \geq \mu(\Sigma_p^+ \times A).$$

This proves (1) since μ is an arbitrary ergodic measure and, as a consequence of the Ergodic Decomposition Theorem, we have

$$\sup_{\{\mu \in \mathcal{M}_{\mathcal{F}_G} \,:\, \pi_*\mu=\mathbb{P}\}} \mu(\Sigma_p^+ \times A) = \sup_{\{\mu \in \mathcal{M}_{\mathcal{F}_G} \,:\, \pi_*\mu=\mathbb{P} \;\&\; \mu \text{ is ergodic}\}} \mu(\Sigma_p^+ \times A).$$

We proceed to prove (2). Assume that $\mathbb{P}$ is ergodic and take a point $\omega \in \Sigma_p^+$ in the ergodic basin of the measure $\mathbb{P}$

$$\mathcal{B}(\mathbb{P}) = \{\omega \in \Sigma_p^+ \colon \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j(\omega)} = \mathbb{P}\}.$$

Fix $x \in X$ and a closed set $A \subset X$. From Definition 7.1 (25), we may find a subsequence $(n_k = n_k(x))_{k \in \mathbb{N}}$ going to $+\infty$ and such that $\gamma_{\omega,x}(A) = \lim_{k \to +\infty} \frac{1}{n_k} \sharp\{0 \le i \le n_k - 1 : f_\omega^i(x) \in A\}$. By compactness of the set of probability measures on the Borel subsets of $\Sigma_p^+ \times X$, the sequence of measures $(\mu_k)_{k \ge 1}$ given by $\mu_k := \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{\mathcal{F}_G^i(\omega,x)}$ admits a weak* convergent subsequence to some $\mathcal{F}_G$-invariant probability measure $\mu_{\mathbb{P},x}$. Assume, without loss of generality, that $\mu_{\mathbb{P},x} = \lim_{k \to +\infty} \mu_k$. By the continuity of the projection π_* and the choice of ω, one has

$$\pi_*(\mu_{\mathbb{P},x}) = \lim_{k \to +\infty} \pi_*\Big(\frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{\mathcal{F}_G^i(\omega,x)}\Big) = \lim_{k \to +\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{\sigma^i(\omega)} = \mathbb{P}.$$

Moreover, as $\Sigma_p^+ \times A$ is a closed set and $(\mu_k)_{k \in \mathbb{N}}$ is weak* convergent to $\mu_{\mathbb{P},x}$, we get

$$\gamma_{\omega,x}(A) = \lim_{k \to +\infty} \frac{\sharp\{0 \le i \le n_k - 1 : f_\omega^i(x) \in A\}}{n_k} = \lim_{k \to +\infty} \mu_k(\Sigma_p^+ \times A) \le \mu_{\mathbb{P},x}(\Sigma_p^+ \times A).$$

This ends the proof of the item (2(a)) and also implies that

$$\gamma_{\mathbb{P}}(A) = \mathbb{P} - \text{esssup} \sup_{x \in X} \gamma_{\omega,x}(A) \le \mu_{\mathbb{P},x}(\Sigma_p^+ \times A)$$

for every closed set $A \subset X$. Finally, notice that items (1) and (2(a)) together yield the equality $\gamma_{\mathbb{P}}(A) = \alpha_{\mathbb{P}}(A)$ for every closed set $A \subset X$. □

To complete the proof of Theorem E we have just to assemble the statements of Lemma 7.2 and (2(b)) of Proposition 7.3, and take the marginal $(\pi_X)_*(\mu_A)$.

Example 7.4. Let $g_1 : [0,1] \to [0,1]$ and $g_2 : [0,1] \to [0,1]$ be the maps given by $g_1(x) = 4x\,(1-x)$ and $g_2(x) = 2x \mod 1$ and consider the continuous semigroup G generated by $G_1 = \{Id, g_1, g_2\}$. Given $n \in \mathbb{N}$, take $\ell_1 \in \Big[\sin\frac{\pi}{2(2^n+1)}, \sin\frac{\pi}{2(2^{n-1}+1)}\Big)$ and $\ell_2 \in \Big[\frac{1}{2^n+1}, \frac{1}{2^{n-1}+1}\Big)$. By Theorems 3 and 5 of [14], if $A_{\ell_i} = [\frac{1-\ell_i}{2}, \frac{1+\ell_i}{2}]$, then the maximum hitting frequency for the map g_i is equal to $\gamma_i(A_{\ell_i}) = \frac{1}{n}$ $(i = 1,2)$. Moreover, there are periodic points $z_1, z_2 \in [0,1]$ with period n and $2n$ for g_1 and g_2, respectively, whose orbits hit A_{ℓ_i} with maximum frequency $(i = 1,2)$.

Let $\mathbb{P} = \frac{1}{3}\delta_{\bar{1}} + \frac{2}{3}\delta_{\bar{2}}$ and $\mu = \frac{1}{3n}\Big(\sum_{j=1}^{n} \delta_{\mathcal{F}_G^j(\bar{1},z_\alpha)} + \sum_{j=1}^{2n} \delta_{\mathcal{F}_G^j(\bar{2},z_\beta)}\Big)$. Then μ is $\mathcal{F}_G$-invariant and, although $\mathbb{P}$ is not ergodic, we have $\pi_*\mu = \mathbb{P}$. Besides, by [14], for all $\ell \in \Big[\frac{1}{2^n+1}, \frac{1}{2^{n-1}+1}\Big) \cap [\sin\frac{\pi}{2(2^n+1)}, \sin\frac{\pi}{2(2^{n-1}+1)})$, we obtain $\gamma_{\mathbb{P}}(A_\ell) = \frac{1}{n} = \alpha_{\mathbb{P}}(A_\ell)$.

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