MULTIFRACTAL ANALYSIS FOR WEAK GIBBS MEASURES: FROM LARGE DEVIATIONS TO IRREGULAR SETS

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Abstract. In this article we prove estimates for the topological pressure of the set of points whose Birkhoff time averages are far from the space averages corresponding to the unique equilibrium state that has a weak Gibbs property. In particular, if f has an expanding repeller and ϕ is an Hölder continuous potential we prove that the topological pressure of the set of points whose accumulation values of Birkhoff averages belong to some interval $I \subset \mathbb{R}$ can be expressed in terms of the topological pressure of the whole system and the large deviations rate function. As a byproduct we deduce that most irregular sets for maps with the specification property have topological pressure strictly smaller than the whole system. Some extensions to a non-uniformly hyperbolic setting, level-2 irregular sets and hyperbolic flows are also given.

1. Introduction

Let $f : M \to M$ be a measurable transformation and μ an f-invariant and ergodic probability measure. The celebrated Birkhoff's ergodic theorem asserts that for any given $\psi \in L^1(\mu)$ and for μ-almost every $x \in M$

$$\frac{1}{n} S_n\psi(x) := \frac{1}{n}\sum_{i=1}^{n-1} \psi \circ f^i(x) \to \int \psi \, d\mu$$

as n tends to infinity. On the other hand, despite the fact that from the ergodic point of view the set of points where the Birkhoff averages do not converge is negligible it can be a topologically large set or have full dimension. To illustrate this fact let us mention that if f is continuous and have the specification property then the set of points where the Birkhoff averages do not converge is either empty or has total topological pressure with respect to any continuous potential (we refer the reader e.g. [Tho10] for details).

The study of the topological pressure or dimension of the these level sets multifractal can be traced back to Besicovitch and this topic had contributions by many authors in the recent years (see e.g. [BPS97, DK01, BG06, Cli10, FH10, GR, JR, PW97, PW01, PW99, Shu, T, TV99, Tho10, Cli13, ZC13] and the references therein). Most commonly, given an observable ψ and the decomposition

$$M = \bigcup_{\alpha \in \mathbb{R}} M_\alpha \, \cup \, E_\psi$$

where $M_\alpha = \{x \in M : \lim_{n\to\infty} \frac{1}{n} S_n\psi(x) = \alpha\}$ and the irregular set E_ψ is the set of points for which the Birkhoff averages do not converge, one is interested in

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describing each of the previous sets from the topological, dimensional or ergodic point of view. Motivated by the aforementioned results by Thompson [Tho10] and the special ergodic theorem proved by Kleptsyn, Ryzhov and Minkov [KRM12] that prove that the Haussdorf dimension of the deviation set for SRB measure is smaller than the dimension of the manifold provided a large deviations property, one of our aims in this article is to provide a multifractal description of more general sets.

For simplicity let consider the sets

$$\overline{X}_{\mu,\psi,c} = \Big\{ x \in M : \limsup_{n\to\infty} \Big| \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) - \int \psi \, d\mu \Big| \geq c \Big\}$$

and

$$\underline{X}_{\mu,\psi,c} = \Big\{ x \in M : \liminf_{n\to\infty} \Big| \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) - \int \psi \, d\mu \Big| \geq c \Big\}$$

where μ is an f-invariant and ergodic probability measure μ, ψ is an observable and $c > 0$ and to study them from the topological pressure viewpoint (more general sets will be defined later on). Notice that we have the inclusion $\underline{X}_{\mu,\psi,c} \subset \overline{X}_{\mu,\psi,c}$ for all $c > 0$. In many cases we are interested in studying $f\mid_\Lambda$ we will consider the corresponding sets $\underline{X}_{\mu,\psi,c} \cap \Lambda$ and $\overline{X}_{\mu,\psi,c} \cap \Lambda$. When no confusion is possible, for notational simplicity we shall omit the dependence of the sets on ψ, μ and Λ, and write simply $\underline{X}_c$ and $\overline{X}_c$. If $J \subset \mathbb{R}$ is an interval we define $X(J)$ as the set of points x so that the following limit exists and $\lim \frac{1}{n} S_n\psi(x) \in J$ and let $X(c)$ denote the case when one considers the degenerate interval $J = [c, c]$. One first motivation is to consider the decomposition of the set of points whose time averages do not converge to the space average

$$M \setminus \Big\{ x \colon \frac{1}{n} \sum_{i=1}^{n-1} \psi \circ f^i(x) \to \int \psi \; d\mu \Big\} = \bigcup_{c>0} \overline{X}_c = \bigcup_{c>0} \underline{X}_c$$

and to study the continuity, monotonicity and concavity of the pressure functions

$$c \mapsto P_{\overline{X}_c}(f, \phi) \quad \text{and} \quad c \mapsto P_{\underline{X}_c}(f, \phi)$$

Our first main purpose here is to study the previous functions in a context of dynamical systems admitting equilibrium states that exhibit a weak Gibbs property. Roughly, we prove that the previous topological pressure functions are bounded from above by the topological pressure of the dynamical system with an error term given by an exponential large deviations rate (see Theorem A and Corollaries A and B for precise statements). Furthermore, in a context of uniform hyperbolicity, we prove both upper and lower bounds for that $P_{\overline{X}_c}(f, \phi) = P_{\underline{X}_c}(f, \phi)$ and that the pressure function $c \mapsto P_{\overline{X}_c}(f, \phi)$ is differentiable, concave and strictly decreasing in c and varies continuously along continuous parametrized families of dynamical systems (see Theorem B for the precise statement). Similar results for the multifractal analysis of level-2 sets, meaning the analysis of level and irregular sets for Birkhoff averages in the space of probability measures, are also obtained (see Theorem C). Hence, the connection between large deviations and multifractal analysis revealed to be very fruitful.

Our second main purpose was to provide a finer description of the irregular set E_ψ. If, on the one hand, the dynamical system admits some hyperbolicity and a unique equilibrium state (that has some weak Gibbs property) with exponential large deviations estimates then the topological pressure of the sets $\underline{X}_{\mu,\psi,c}$ and $\overline{X}_{\mu,\psi,c}$ is strictly smaller than the topological pressure of the system one cannot expect immediate estimates for the irregular set. In fact, the irregular set E_ψ may not be contained in neither of the sets above for some fixed c. This motivates the decompositions $E_\psi = \cup_{c>0}\overline{E}_{\mu,\psi,c}$ and $E_\psi = \cup_{c>0}\underline{E}_{\mu,\psi,c}$ with

$$\overline{E}_{\mu,\psi,c} = E_\psi \cap \overline{X}_{\mu,\psi,c} \quad \text{and} \quad \underline{E}_{\mu,\psi,c} = E_\psi \cap \underline{X}_{\mu,\psi,c}.$$

In other words, the set $\underline{E}_{\mu,\psi,c}$ consists of points $x \in M$ whose Birkhoff averages $\frac{1}{n}\sum_{j=0}^{n-1}\psi(f^j(x))$ not only do not converge as they remain at distance larger than c from the time average $\int \psi\, d\mu$ for all large n. Finally, the set $\overline{E}_{\mu,\psi,c}$ consists of points $x \in M$ whose Birkhoff averages $\frac{1}{n}\sum_{j=0}^{n-1}\psi(f^j(x))$ do not converge and have infinitely many values of n so that the Birkhoff averages $\frac{1}{n}\sum_{j=0}^{n-1}\psi(f^j(x))$ remain at distance larger than c. As a direct consequence of our results, even for maps with specification property irregular sets $\underline{E}_{\mu,\psi,c}$ and $\overline{E}_{\mu,\psi,c}$ have topological pressure strictly smaller than the topological pressure of the whole system. In fact, we can indeed prove some lower bounds for the topological pressure of these irregular sets and, consequently, to study its regularity as a function of the parameter c (c.f. Corollary C). Some other extensions to hyperbolic maps and flows, non-uniformly hyperbolic dynamics and level-2 irregular sets are also given (see Section 4 for precise statements). In particular, an example from [DGR11, DG12] is given that illustrates the case where the pressure function is discontinuous and not strictly decreasing. An extension of the current results in a context of non-additive sequences of observables and applications to the irregular set studied in [ZZC11] was carried out in [BV14], while we expect that these results can be also extended to the class of partially hyperbolic diffeomorphisms considered in [CN14].

This article is organized as follows. In Section 2 we provide some definitions and state the main results. Section 3 is devoted to the proof of the results. A large amount of applications and examples is given in Section 4.

2. Statement of the main results

In this section we introduce some necessary notions and state the main results.

2.1. (Weak) Gibbs measures.

In many cases equilibrium states arise as invariant measures absolutely continuous with respect to probability measures exhibiting some Gibbs property. Let us now describe this wide class of measures. Given $\varepsilon > 0$, $n \geq 1$ and $x \in M$ the (n,ε)-*dynamical ball* $B(x,n,\varepsilon)$ is the set of points $y \in M$ so that $d(f^j(x), f^j(y)) < \varepsilon$ for all $0 \leq j \leq n-1$.

Definition 2.1. Given a potential ϕ we say that a probability ν is a *weak Gibbs measures* with respect the ϕ on $\Lambda \subset M$ if there exists $\varepsilon_0 > 0$ so that for every $0 < \varepsilon < \varepsilon_0$ there exists $K(\varepsilon) > 0$, for ν-almost every x there exists a sequence $n_k(x) \to \infty$ such that

$$K(\varepsilon)^{-1} \leq \frac{\nu(B(x, n_k(x), \varepsilon))}{e^{-n_k(x)P + S_{n_k(x)}\phi(x)}} \leq K(\varepsilon),$$

where $S_n\phi = \sum_{j=0}^{n-1}\phi \circ f^j$ denotes the usual Birkhoff sum. If the later condition holds for all positive integers n (independently of x) we will say that ν is a *Gibbs measure* with respect the ϕ.

In the later notion of weak Gibbs one does not require the sequence of times to have positive density at infinity in the set of integers. Naturally, in applications it is most interesting case is when the value P in the previous expression coincides with the topological pressure $P_{\text{top}}(f, \phi)$. Such measures arise naturally in the context of (non-uniform) hyperbolic dynamics. Given a basic set Ω for a diffeomorphism f Axiom A (or Ω repulsor to f) it is known that every potential ϕ satisfying

$$\exists A, \delta > 0 : \sup_{n\in\mathbb{N}} \gamma_n(\phi, \delta) \le A, \tag{2.1}$$

where $\gamma_n(\phi, \delta) := \sup\{|S_n\phi(y) - S_n\phi(z)| : y, z \in B(x, n, \delta)\}$, admits a unique equilibrium state μ_ϕ and it is a Gibbs measure. This condition, introduced by Bowen [Bow74] to prove uniqueness of equilibrium states for expansive maps with the specification property it is known as *Bowen condition*.

2.2. **Statement of the main results.** This section is devoted to the statement of the main results. For that purpose we shall introduce some definitions and notations. The first result provides a topological counterpart to the special ergodic theorem. In that follows, given a continuous function $\psi : M \to \mathbb{R}$, a probability measure μ and a closed set $I \subset \mathbb{R}$ we denote

$$\overline{X}_I = \Big\{x \in M : \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \in I\Big\}$$

and analogously

$$\underline{X}_I = \Big\{x \in M : \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \in I\Big\}.$$

Moreover, given $\delta > 0$ we denote by I_δ the δ-neighborhood of the set I. Finally, given a probability measure ν let us define the large deviations upper bound

$$L_{I,\nu} := -\limsup_{n\to+\infty} \frac{1}{n} \log \nu\Big(\{x \in \Lambda : \frac{1}{n} S_n\psi(x) \in I\}\Big). \tag{2.2}$$

We are now in a position to state our first main result.

Theorem A. *Let M be a compact metric space, $f : M \to M$ be a continuous map, $\phi : M \to \mathbb{R}$ be a continuous potential, ν be a (not necessarily invariant) Gibbs measure on M and $\mu_\phi \ll \nu$ be the unique equilibrium state of f with respect the ϕ. Then, for any continuous $\psi : M \to \mathbb{R}$, any closed interval $I \subset \mathbb{R}$ and any small δ,*

$$P_{\underline{X}_I}(f, \phi) \le P_{\overline{X}_I}(f, \phi) \le P_{\text{top}}(f, \phi) - L_{I_\delta,\nu} \;\le\; P_{\text{top}}(f, \phi).$$

In fact, it follows from [Var12, Theorem 2.1] that, since ν is a (strong) Gibbs measure, if $\int \psi d\mu_\phi \notin I_\delta$ then the large deviations property that $L_{I_\delta} > 0$ holds and, consequently, the topological pressure of the sets $\underline{X}_I$ and $\overline{X}_I$ is strictly smaller than $P_{\text{top}}(f, \phi)$. When no confusion is possible we shall omit the dependence of $L_{I,\nu}$ on ν. Our result is applicable to the case of topological repellers.

Definition 2.2. Given a compact metric space (M, d) and a continuous open map $f : M \to M$ we say that an f-invariant set $\Lambda \subset M$ is a *repeller* for f if there exists $C, \lambda, \varepsilon > 0$ so that for all $d(f^n(x), f^n(y)) \ge Ce^{\lambda n} d(x, y)$ for all $y \in B(x, n, \varepsilon)$ and $n \ge 1$.

It is easy to check that if f is differentiable and expanding then it satisfies the conditions of the previous definition. Recall that an observable $\psi : M \to \mathbb{R}$ is *cohomologous to a constant* if there exists a constant c and a measurable function u so that $\psi = u \circ f - u + c$.

Definition 2.3. Given an observable $\psi : M \to \mathbb{R}$ and $t \in \mathbb{R}$ the *free energy* $\mathcal{E}_{f,\phi,\psi}$ is

$$\mathcal{E}_{f,\phi,\psi}(t) = \limsup_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\psi} \, d\mu_{f,\phi}.$$

In many cases, e.g. when the transfer operator associated to the potential ϕ has a spectral gap property, the expression in the right hand side does converge to

$$\mathcal{E}_{f,\phi,\psi}(t) := \lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\psi} \, d\mu_{f,\phi} = P_{\text{top}}(f, \phi + t\psi) - P_{\text{top}}(f, \phi).$$

If this is the case and the topological pressure is smooth then $t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is affine if ψ is cohomologous to a constant and otherwise $t \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is strictly convex in some interval $J = [t_-, t_+]$ and one can associate the *"local" Legendre transform* $I_{f,\phi,\psi}$ given by

$$I_{f,\phi,\psi}(s) = \sup_{t\in J} \; \big\{ s\, t - \mathcal{E}_{f,\phi,\psi}(t) \big\}$$

and well defined in the interval $[\mathcal{E}'_{f,\phi,\psi}(t_-), \mathcal{E}'_{f,\phi,\psi}(t_+)]$. Let us mention that the interval J may depend on f, ϕ and ψ and that $I_{f,\phi,\psi}(s)$ can often be proved to be a (local) level-1 large deviations *rate function* (see e.g. [You90, RY08, BCV13]): for all $[a, b] \subset J$

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_{f,\phi} \left(x \in M : \frac{1}{n} S_n\psi(x) \in [a, b] \right) \leq - \inf_{s\in[a,b]} I_{f,\phi,\psi}(s) \tag{2.3}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_{f,\phi} \left(x \in M : \frac{1}{n} S_n\psi(x) \in (a, b) \right) \geq - \inf_{s\in(a,b)} I_{f,\phi,\psi}(s) \tag{2.4}$$

As a byproduct of our previous result and the large deviations property and the fact that equilibrium states associated to Hölder continuous potentials satisfy the Gibbs property we deduce the following:

Corollary A. *Let $f : M \to M$ be a continuous map, $\Lambda \subset M$ be a transitive repeller, $\phi : M \to \mathbb{R}$ be an Hölder continuous potential and $\mu = \mu_{f,\phi}$ be the unique equilibrium for $f \mid_\Lambda$ with respect to ϕ. Then, for any continuous observable $\psi : M \to \mathbb{R}$ and $c > 0$*

$$P_{\underline{X}_c}(f, \phi) \leq P_{\overline{X}_c}(f, \phi) \leq P_{\text{top}}(f, \phi) - L_{c-\delta} < P_{\text{top}}(f, \phi)$$

for every small δ, where $L_c := L_{I_c}$ is defined as in (2.2) *with respect to $I_c = (-\infty, \int \psi d\mu_\phi - c] \cup [\int \psi d\mu_\phi + c, +\infty)$.*

Since in the previous results the topological pressure is strictly smaller than the topological pressure $P_{\text{top}}(f, \phi)$, this has particularly interesting applications in connection with the specification property. Recall that a system satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that

$$d\Big(f^j(x), f^j(x_1)\Big) \leq \varepsilon, \quad \forall\, 0 \leq j \leq n_1$$

and

$$d\Big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\Big) \le \varepsilon$$

for every $2 \le i \le k$ and $0 \le j \le n_i$. We also obtain the following result:

Corollary B. *Let $f: M \to M$ be a continuous map admitting a transitive repeller $\Lambda \subset M$ and $\psi: M \to \mathbb{R}$ be such that the set of of irregular points satisfies $E_\psi \neq \emptyset$. Then, for every $c > 0$*

$$P_{\text{top}}(f,\phi) = P_{E_\psi}(f,\phi) > P_{\overline{X}_c}(f,\phi).$$

In fact, it follows from Thompson [Tho10] that a dynamical system with the specification property is such that irregular sets are either empty or have full topological pressure with respect to any continuous potential. Since the dynamical systems restricted to the transitive repeller satifsfies the specification property then the first equality follows from [Tho10]. In particular, using

$$E_\psi = \bigcup_{n\ge 1} [E_\psi \cap \overline{X}_{1/n}]$$

and also $P_{E_\psi}(f,\phi) = \sup_{n\ge1} P_{E_\psi\cap\overline{X}_{1/n}}(f,\phi)$ the previous corollary roughly means that despite the set of irregular points having full topological pressure, the ones that give a larger contribution to the topological pressure are those with time averages which are infinitely often very close to the mean.

One could wonder if there could exist a strict inequality $P_{\underline{X}_c}(f,\phi) < P_{\overline{X}_c}(f,\phi)$ and what is the regularity of the topological pressure of those subsets. The next theorem provides an answer to these questions under the assumption of uniform expansion.

Theorem B. *Let $f: M \to M$ be a continuous map admitting a mixing repeller $\Lambda \subset M$, $\phi: M \to \mathbb{R}$ be a continuous potential so that μ_ϕ is the unique equilibrium state for f with respect to ϕ and $\mu_\phi \ll \nu$ where ν is a Gibbs measure. If ϕ, ψ satisfy the Bowen condition, ψ is not cohomologous to a constant and $\int \psi\, d\mu_{f,\phi} = 0$ then*

$$P_{\overline{X}_c}(f,\phi) \le P_{\text{top}}(f,\phi) - \min\{I_{f,\phi,\psi}(-c)\ ,\ I_{f,\phi,\psi}(c)\}$$

where $I_{f,\phi,\psi}$ is the large deviations rate function. If $0 \notin [c_1, c_2]$ and $c = \min\{|c_1|, |c_2|\}$ then either $\overline{X}_c = \emptyset$ or

$$P_{\overline{X}_c}(f,\phi) = P_{\underline{X}_c}(f,\phi) = P_{X(c_*)}(f,\phi) = P_{\text{top}}(f,\phi) - I_{f,\phi,\psi}(c_*).$$

where

$$c_* = \begin{cases} c, & \text{if } I_{f,\phi,\psi}(c) < I_{f,\phi,\psi}(-c) \\ -c, & \text{otherwise.} \end{cases} \tag{2.5}$$

In particular $\mathbb{R}_0^+ \ni c \mapsto P_{\overline{X}_c}(f,\phi)$ is differentiable, concave and strictly decreasing. Furthermore, the right hand side expression varies continuously with c and also varies continuously with ϕ, ψ in the C^α-topology. Moreover, if V is a compact metric space and $V \ni v \mapsto (f_v)_v$ is a continuous (in the C^1-topology) family of expanding maps on M then $v \mapsto P_{\overline{X}_c}(f_v,\phi)$ is also a continuous function.

Under the previous assumptions we can provide a more detailed description of the irregular sets $\overline{E}_c$ as follows.

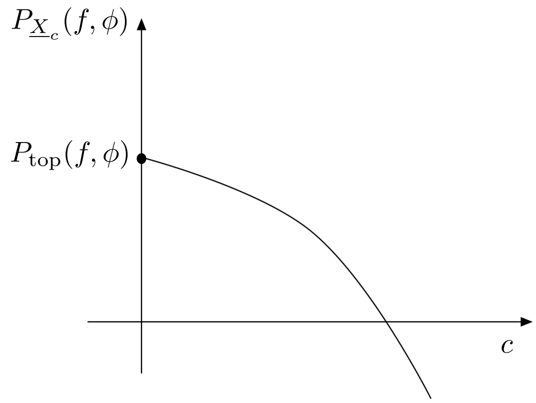

FIGURE 1. Continuity, monotonicity and concavity of the pressure function

Corollary C. *Let $f : M \to M$ be a continuous map admitting a mixing repeller $\Lambda \subset M$, $\phi : M \to \mathbb{R}$ be a continuous potential so that μ_ϕ is the unique equilibrium state for f with respect to ϕ and $\mu_\phi \ll \nu$ where ν is a Gibbs measure. Set $\overline{E}_c = \overline{X}_c \cap E_\psi$ the irregular set contained in $\overline{X}_c$. If $E_\psi \neq \emptyset$ then for every $c > 0$:*

1. $P_{\text{top}}(f,\phi) = P_{E_\psi}(f,\phi) > P_{\overline{X}_c}(f,\phi) \geq P_{\overline{E}_c}(f,\phi)$,
2. *if $\overline{E}_c \neq \emptyset$ then $P_{\overline{X}_c}(f,\phi) = P_{\overline{E}_c}(f,\phi)$ and $c \mapsto P_{\overline{E}_c}(f,\phi)$ is differentiable, concave and strictly decreasing.*

Actually, in this setting we can provide also estimates for irregular sets corresponding to empirical measures $\delta_{x,n} := \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}$. Let $\mathcal{M}_1$ denote the set of probability measures on M and let d be any metric compatible with the weak* topology (e.g. $d(\mu,\nu) = \sum_{k\geq 1}\frac{1}{2^k\|g_k\|_0}|\int g_k\,d\mu - \int g_k\,d\nu|$ for some countable and dense subset $(g_k)_k$ of continuous observables). We say that a *level-2 large deviations principle* holds for ν if there exists a lower semicontinuous function $Q : \mathcal{M}_1 \to [0,+\infty]$ so that

$$\limsup_{n\to\infty}\frac{1}{n}\log \nu_{f,\phi}\,(x\in M : \delta_{x,n}\in U) \leq -\inf_{\eta\in U} Q(\eta)$$

for every closed set $U \subset \mathcal{M}_1$ and

$$\liminf_{n\to\infty}\frac{1}{n}\log \nu_{f,\phi}\,(x\in M : \delta_{x,n}\in V) \geq -\inf_{\eta\in V} Q(\eta)$$

for every open set $V \subset \mathcal{M}_1$. Level-2 large deviations principles in dynamical systems have been obtained e.g. in [Lo90, CRL98, Chu11, CTY13]. Consider

$$\overline{Y}_{\mu,c} = \{x\in M : \limsup_{n\to\infty} d(\delta_{x,n},\mu) \geq c\} \quad \text{and} \quad \underline{Y}_{\mu,c} = \{x\in M : \liminf_{n\to\infty} d(\delta_{x,n},\mu) \geq c\}$$

and, for $C \subset \mathcal{M}_1$, define $Y(C) := \{x \in M : \lim_{n\to+\infty}\delta_{x,n}\in C\}$.

Part of the strategy can be used to estimate the topological pressure of points with specified behaviour of the empirical measures for dynamical systems that have the g-almost product structure and uniform separation property. These notions, introduced by C. Pfister and W. Sullivan [PS05], are strictly weaker than the specification property and the positive expansive property, respectivelly. In fact, the uniform separation property is true even for asymptotically entropy-expansive maps. Let us recall these notions.

Definition 2.4. Let M be a compact metric space and $f : M \to M$ be continuous. A nondecreasing unbounded map $g : \mathbb{N} \to \mathbb{N}$ is a *blow-up function* if $g(n) < n$ for all n and $\lim_{n\to+\infty} g(n)/n = 0$.

For any subset of integers $\Lambda \subset [0, N]$, we will use the family of distances in the metric space X given by $d_\Lambda(x, y) = \max\{d(f^i x, f^i y) : i \in \Lambda\}$ and consider the balls $B_\Lambda(x, \varepsilon) = \{y \in X : d_\Lambda(x, y) < \varepsilon\}$. Given a blow-up function g, $\varepsilon > 0$ and $n \geq 1$, the *g-mistake dynamical ball* $B_n(g; x, \varepsilon)$ of radius ε and length n associated to g is defined by

$$B_n(g; x, \varepsilon) = \{y \in X \mid y \in B_\Lambda(x, \varepsilon) \text{ for some } \Lambda \in I(g; n, \varepsilon)\} = \bigcup_{\Lambda\in I(g;n,\varepsilon)} B_\Lambda(x, \varepsilon)$$

where $I(g; n, \varepsilon) = \{\Lambda \subset [0, n-1] \cap \mathbb{N} \mid \#\Lambda \geq n - g(n)\}$. We are now in the position to define the g-almost product property.

Definition 2.5. Let g be a blow-up function. The continuous map $f : M \to M$ has the *g-almost product property* if there exists a nonincreasing function $m : \mathbb{R}^+ \to \mathbb{N}$, such that for any $k \in \mathbb{N}$, any points $x_1, x_2, \dots, x_k$, any positive $\varepsilon_1, \dots \varepsilon_k$ and any integers $n_i \geq m(\varepsilon_1)$ for $i = 1 \dots k$ it holds that $\bigcap_{j=1}^k f^{-M_{j-1}} B_{n_j}(g; x_j, \varepsilon_j) \neq \emptyset$. where $M_0 = 0$ and $M_i = n_1 + n_2 + \cdots + n_i, i = 1, 2, \cdots, k-1$.

Given $\delta, \varepsilon > 0$ and $n \geq 1$ we say that two points $x, y \in X$ are (δ, n, ε)*-separated* if $\#\{0 \leq j \leq n-1 : d(f^j(x), f^j(y)) > \varepsilon\} \geq \delta n$. In addition, a set $E \subset X$ is (δ, n, ε)-separated if all pairs of distinct points in E are (δ, n, ε)-separated. This means that the moments at which the two pieces of orbit are ε-separated form a δ-proportion.

Definition 2.6. A continuous map $f : M \to M$ has the *uniform separation property* if for any η there exists $\delta > 0$ and $\varepsilon > 0$ so that for any ergodic probability measure μ and any neighborhood F of μ in the space of all probability measures $\mathcal{M}_1$ there exists $n_{F,\mu,\eta} \geq 1$ such that

$$N(F; \delta, n, \varepsilon) \geq \exp\left[n(h_\mu(f) - \eta)\right]$$

for all $n \geq n_{F,\mu,\eta}$, where $N(F; \delta, n, \varepsilon)$ is the maximal cardinality of a (δ, n, ε)-separated subset of the set $\{x \in M : \delta_{x,n} \in F\}$.

Taking these notions in account we also obtained the following result.

Theorem C. *Let $f : M \to M$ and $\phi : M \to \mathbb{R}$ be continuous, ν be a (not necessarily invariant) Gibbs measure and assume $\mu = \mu_{f,\phi} \ll \nu$ is the unique equilibrium state for f with respect to ϕ. Assume the metric d on $\mathcal{M}_1$ has the following properties:*

i. $d(\eta_1 + \eta, \eta_2 + \eta) = d(\eta_1, \eta_2), \forall \eta_1, \eta_2, \eta \in \mathcal{M}_1$;
ii. $d(t\eta_1, t\eta_2) = t d(\eta_1, \eta_2), \forall \eta_1, \eta_2 \in \mathcal{M}_1$ *and* $t > 0$,

If a level-2 large deviations principle holds for ν then for every $c > 0$

$$P_{\overline{Y}_{\mu,c}}(f, \phi) \leq P_{\text{top}}(f, \phi) - \inf_{d(\eta,\mu)\geq c} Q(\eta) \leq P_{\text{top}}(f, \phi).$$

In addition, if f satisfies the almost product and uniform separation properties and $0 < c_1 < c_2$ then either $\overline{Y}_{\mu,c_1} = \emptyset$ or

$$\begin{aligned}P_{\overline{Y}_{\mu,c_1}}(f,\phi) &= P_{\underline{Y}_{\mu,c_1}}(f,\phi) = P_{Y(\partial B(\mu,c_1))}(f,\phi) = P_{Y(B(\mu,c_1))}(f,\phi) \\ &= P_{Y\left(\overline{B(\mu,c_1,c_2)}\right)}(f,\phi) = P_{Y(B(\mu,c_1,c_2))}(f,\phi) = P_{\text{top}}(f,\phi) - \inf_{d(\eta,\mu)=c_1} Q(\eta),\end{aligned}$$

where $B(\mu, c_1)$ denotes the ball of radius c_1 around μ and $B(\mu, c_1, c_2)$ denotes the annulus $\{\eta \in \mathcal{M}(X) : c_1 < d(\eta,\mu) < c_2\}$.

Further information can be extracted if one knows the behaviour of the rate function Q, in which case one can prove the topological pressure of the level sets is strictly smaller than the topological pressure $P_{\text{top}}(f,\phi)$. This is the case for repellers as we now detail.

Corollary D. *Let $f : M \to M$ be a continuous map admitting a transitive repeller $\Lambda \subset M$, $\phi : M \to \mathbb{R}$ is a continuous potential and there exists a unique equilibrium state μ_ϕ for f with respect to ϕ and it is a Gibbs measure under Λ. Then, for all $0 < c_1 < c_2$ either $\overline{Y}_{\mu,c} = \emptyset$ or*

$$P_{\overline{Y}_{\mu,c_1}}(f,\phi) = P_{\underline{Y}_{\mu,c_1}}(f,\phi) = P_{\text{top}}(f,\phi) - \inf_{d(\eta,\mu)=c_1} Q(\eta) < P_{\text{top}}(f,\phi)$$

where $Q(\eta) = P_{top}(f,\phi) - h_\eta(f) + \int \psi \, d\eta$.

The previous result implies that the set of irregular points whose range of values of Birkhoff averages are far from the corresponding value associated to the equilibrium state have topological pressure smaller than $P_{\text{top}}(f,\phi)$. In particular, this shows that in order to build an irregular set of points with large topological pressure one needs to use some specification property and points whose empirical measures are arbitrarily close to the equilibrium state. In some sense this means the classical construction of irregular sets with large topological pressure is optimal. Our next results apply for weak Gibbs measures.

Theorem D. *Let M be a compact metric space, $f : M \to M$ be a continuous map, $\phi : M \to \mathbb{R}$ be a continuous potential, ν be a (not necessarily invariant) weak Gibbs measure and $\mu_\phi \ll \nu$ be the unique equilibrium state of f with respect to ϕ. For any continuous $\psi : M \to \mathbb{R}$ and closed interval $I \subset \mathbb{R}$ it holds that*

$$P_{\underline{X}_I}(f,\phi) \le P_{\text{top}}(f,\phi) - L_{I_\delta}.$$

for every small δ. If, in addition, $L_{I_\delta} < 0$ then $P_{\underline{X}_I}(f,\phi) < P_{\text{top}}(f,\phi)$.

Estimates for L_{I_δ} will depend on the weak Gibbs property and some can be found in [Var12]. Actually we can indeed prove a version of the previous results in the non-uniformly expanding setting. Given $\sigma, \delta > 0$ we define $H = H(\sigma,\delta)$ as the set of points in Λ with infinitely many (σ,δ)-hyperbolic times (see e.g. [ABV00] for a precise definition). We will say that an f-invariant probability measure μ is *expanding* if $\mu(H(\sigma,\delta)) = 1$ for some positive constants σ, δ. Moreover, given ψ continuous, we will say that we have an *exponential large deviations upper bound* if

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mu_\Phi\Big(\{x \in M : \Big|\frac{1}{n} S_n\psi(x) - \int \psi \, d\mu_\phi\Big| \ge c\}\Big) < 0 \qquad (2.6)$$

for all $c > 0$. A direct consequence of the previous abstract result is as follows.

Corollary E. *Let $f : M \to M$ be a $C^{1+\alpha}$-smooth map on a compact manifold M and $\phi : M \to \mathbb{R}$ be a continuous potential so that μ_ϕ is the unique equilibrium state for f with respect to ϕ. If μ_ϕ is an expanding measure and $J_{\mu_\phi} f$ is Hölder continuous and has exponential large deviations upper bound then for any continuous $\psi : M \to \mathbb{R}$ and $c > 0$*

$$P_{\underline{X}_c}(f, \phi) \le P_{\mathrm{top}}(f, \phi) - L_{c-\delta} < P_{\mathrm{top}}(f, \phi)$$

for every small $\delta > 0$.

The key ingredient used in the proof of Corollary E is that hyperbolic times are instants at which the Gibbs property holds provided the Jacobian of the measure has enough regularity to deduce bounded distortion. One should also point out that if a local large deviations principle holds as in equations (2.3) and (2.4) (e.g. [You90, RY08, BCV13]) then it is not hard to see that an upper bound for $P_{\underline{X}_c}(f, \phi)$ can be taken as

$$P_{\mathrm{top}}(f, \phi) - \min \Big\{ I_{f,\phi,\psi}\Big(\int \psi d\mu_\phi + c \Big) \, , \, I_{f,\phi,\psi}\Big(\int \psi d\mu_\phi - c \Big) \Big\}.$$

There are examples where the right hand side term above can also be shown to vary continuously with the data even in the non-uniformly expanding context (see e.g. [BCV13]). Since we only estimated the topological pressure of the sets $\underline{X}_c$ in the non-uniformly expanding context one question that arises naturally is the following

Question: Are there examples of transitive non-uniformly expanding maps under the conditions of the previous theorem where $P_{\overline{X}_c}(f, \phi)$ differs from $P_{\underline{X}_c}(f, \phi)$ and coincides with the topological pressure $P_{\mathrm{top}}(f, \phi)$?

We include some examples where we give partial answers to this question in Section 4 by proving that these sets may have different upper Carathéodory capacities.

3. Proof of the main results

3.1. **Proof of Theorem A.** Our purpose it to estimate $P_{\underline{X}_I}(f, \phi)$ and $P_{\overline{X}_I}(f, \phi)$. Consider the sets $X_{I,n} = \{x \in M : \frac{1}{n} S_n \psi(x) \in I\}$. Let us first prove a preliminary lemma.

Lemma 3.1. *Let $I \subset \mathbb{R}$ be a closed set. For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ and $N = N_\delta \in \mathbb{N}$ so that $B(x, n, \varepsilon) \subset X_{I_\delta, n}$ for all $0 < \varepsilon < \varepsilon_\delta$, $n \ge N$ and $x \in X_{I,n}$.*

Proof. Let $\delta > 0$ be given. Since ψ is uniformly continuous then there is $\varepsilon = \varepsilon_\delta > 0$ and a large $N = N_\delta \in \mathbb{N}$ so that $\gamma_n(\psi, \varepsilon) \le \delta n$ for all $0 < \varepsilon < \varepsilon_\delta$ and $n \ge N$. So, if $n \ge N$, $x \in X_{I,n}$, $y \in B(x, n, \varepsilon)$ and $0 < \varepsilon < \varepsilon_\delta$ then

$$\frac{S_n\psi(x)}{n} - \frac{\gamma_n(\psi, \varepsilon)}{n} \le \frac{S_n\psi(y)}{n} \le \frac{S_n\psi(x)}{n} + \frac{\gamma_n(\psi, \varepsilon)}{n}$$

and, consequently,

$$\frac{S_n\psi(x)}{n} - \delta \le \frac{S_n\psi(y)}{n} \le \frac{S_n\psi(x)}{n} + \delta$$

meaning that $y \in X_{I_\delta, n}$. This finishes the proof of the lemma. □

Proof of Theorem A. Let $I \subset \mathbb{R}$ be a closed interval and assume $\overline{X}_I$ is non-empty. Let $\delta > 0$ be fixed and consider L_{I_δ} as defined in equation (2.2). For any positive integer n consider the set $\mathcal{I}_n \subset M \times \mathbb{N}$ of pairs (x, n) with $x \in M$. Recalling the notion of topological pressure for invariant sets introduced by Pesin and Pitskel (see e.g. [Pes97]), in order to prove that $P_{\overline{X}_I}(f,\phi) \le P_{\rm top}(f,\phi) - L_{I_\delta}$ it is enough to prove that for all $\alpha > P_{\rm top}(f,\phi) - L_{I_\delta}$, every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists a subset $\hat{\mathcal{G}}_N \subset \bigcup_{n \ge N} \mathcal{I}_n$ so that

$$\overline{X}_I \subset \bigcup_{(x,n)\in\hat{\mathcal{G}}_N} B(x,n,\varepsilon) \quad \text{and} \quad \sum_{(x,n)\in\hat{\mathcal{G}}_N} e^{-\alpha n + \phi_n(x)} \le a(\varepsilon) < \infty$$

independently of N.

Let $\alpha > P_{\rm top}(f,\phi) - L_{I_\delta}$ and $0 < \varepsilon < \varepsilon_\delta$ be fixed. Notice that if $x \in \overline{X}_I$ then there exists a sequence of positive integers $(m_j(x))_{j\in\mathbb{N}}$ converging to infinite with so that $x \in X_{I_\delta, m_j(x)}$ for all $j \in \mathbb{N}$. Thus $\overline{X}_I \subset \bigcap_{\ell\ge1}\bigcup_{j\ge\ell} X_{I_\delta,j}$. Given $N \ge 1$ and $x \in \overline{X}_I$ pick $m(x) \ge N$ in such a way that $x \in X_{I_{\frac{\delta}{2}}, m(x)}$ and consider $\mathcal{G}_N := \{(x, m(x)) : x \in \overline{X}_I\}$. Now, let $\hat{\mathcal{G}}_N \subset \mathcal{G}_N$ be a maximal set with a property of separation, namely, that if (x,l) and (y,l) belong to $\hat{\mathcal{G}}_N$ then $B(x,l,\frac{\epsilon}{2}) \cap B(x,l,\frac{\epsilon}{2}) = \emptyset$. So, for $0 < \varepsilon < \delta$ given by Lemma 3.1 using the Gibbs property for ν we deduce that

$$\begin{aligned}\sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{-\alpha m(x) + S_{m(x)}\phi(x)} &= \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{(P-\alpha)m(x)} e^{-Pm(x)+S_{m(x)}\phi(x)} \\ &\le \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{(P-\alpha)m(x)} K(\varepsilon)\nu(B(x,m(x),\varepsilon))\end{aligned}$$

Now, we write $\hat{\mathcal{G}}_N = \cup_{\ell\ge1}\hat{\mathcal{G}}_{\ell,N}$ with the level sets $\hat{\mathcal{G}}_{\ell,N} := \{(x,\ell) \in \hat{\mathcal{G}}_N\}$ and pick $\zeta > 0$ small such that $\alpha > P_{\rm top}(f,\Phi) - L_{I_\delta} + \zeta$ and $\mu\Big(\{x \in \Lambda : \frac{1}{n}S_n\psi(x) \in I_\delta\}\Big) \le e^{-(L_{I_\delta}-\zeta)n}$ for all $n \ge N$ large. By Lemma 3.1 each dynamical ball $B(x,\ell,\varepsilon)$ is contained in $X_{I_\delta,\ell}$. Therefore, using that $\nu(B(x,m(x),\varepsilon)) \le K(\varepsilon)K(\varepsilon/2)\nu(B(x,m(x),\varepsilon/2)$ then

$$\begin{aligned}\sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{-\alpha m(x)+S_{m(x)}\phi(x)} &\le K(\varepsilon) \sum_{(x,m(x))\in\hat{\mathcal{G}}_N} e^{(P-\alpha)(m(x))}\nu(B(x,m(x),\varepsilon)) \\ &= K(\varepsilon)\sum_{\ell\ge N} e^{(P-\alpha)\ell} \sum_{x\in\hat{\mathcal{G}}_{N,\ell}} \nu(B(x,\ell,\varepsilon)) \\ &\le K(\varepsilon)K(\frac{\varepsilon}{2})\sum_{\ell\ge N} e^{(P-\alpha)\ell} \sum_{x\in\hat{\mathcal{G}}_{N,\ell}} \nu(B(x,\ell,\varepsilon/2)) \\ &\le K(\varepsilon)K(\frac{\varepsilon}{2})\sum_{\ell\ge N} e^{(P-\alpha)\ell}\nu(X_{I_\delta,\ell}) \\ &\le K(\varepsilon)K(\frac{\varepsilon}{2})\sum_{\ell\ge N} e^{(P-\alpha-L_{c-\delta}+\zeta)\ell}\end{aligned}$$

which is finite and independent by the choice of α. This proves that $P_{\overline{X}_I}(f,\phi) \le P_{\rm top}(f,\phi) - L_{I_\delta}$. Since $P_{\underline{X}_I}(f,\phi) \le P_{\overline{X}_I}(f,\phi)$ this finishes the proof of the theorem. □

3.2. **Proof of Theorem B.** Let us assume that both ϕ, ψ satisfy the Bowen condition and ψ is not cohomologous to a constant. Assume without loss of generality that $\int \psi\, d\mu_{f,\phi} = 0$. Our first purpose is to prove

$$P_{\overline{X}_c}(f,\phi) \le P_{\rm top}(f,\phi) - \min\Big\{ I_{f,\phi,\psi}\Big(\int \psi d\mu_\phi + c\Big)\ ,\ I_{f,\phi,\psi}\Big(\int \psi d\mu_\phi - c\Big)\Big\},$$

where $I_{f,\phi,\psi}$ is the rate function of the large deviations function. Since $f\mid_\Lambda$ satisfies the specification property and ψ is not cohomologous to a constant it follows that (see e.g. [Tho09])

$$\Big\{\alpha\in\mathbb{R} : \exists\, x\in M \,\text{s. t.}\, \lim_{n\to\infty}\frac{1}{n}S_n\psi(x)=\alpha\Big\}=\Big\{\int\psi d\mu : \mu \,\text{is a probability}\, f\text{-invariant}\Big\}$$

is a non-empty compact interval. By the level-1 large deviations principle for uniformly hyperbolic dynamics of Young [You90] equations (2.3) and (2.4) hold with the rate function $I_{f,\phi,\psi}(s)=\sup\{-P_{\text{top}}(f,\phi)+h_\eta(f)+\int\phi\, d\eta : \int\psi\, d\eta=s\}$. Moreover, it follows from the functional analytic approach using transfer operators and the differentiability of the free energy function that $I_{f,\phi,\psi}$ is the Legendre transform of the free energy. On the one hand, using Theorem A and the previous upper bound

$$P_{\overline{X}_c}(f,\phi)\le P_{\text{top}}(f,\phi)-L_{c-\delta}\le P_{\text{top}}(f,\phi)-\min\{I_{f,\phi,\psi}(c-\delta),I_{f,\phi,\psi}(c+\delta)\}$$

for all positive δ. Now, assume for simplicity that $0<c=c_1<c_2$ and $c_*=-c$ is defined by equation (2.5) (the other cases are analogous). We claim that it follows from the continuity and convexity of the rate function that if $X_c\neq\emptyset$ then

$$\begin{aligned}P_{\overline{X}_c}(f,\phi)&=P_{\underline{X}_c}(f,\phi)=P_{X(-c)}(f,\phi)=P_{X([-c_2,-c_1])}(f,\phi)\\&=P_{X(-c_2,-c_1)}(f,\phi)=P_{\text{top}}(f,\phi)-I_{f,\phi,\psi}(-c_1)\\&=P_{\text{top}}(f,\phi)-I_{f,\phi,\psi}(c_*).\end{aligned}$$

In fact, using [Tho09] the topological pressure of the set $\{x\in M : \lim\frac{1}{n}S_n\psi(x)=c\}$ coincides with $\sup\{h_\eta+\int\psi\, d\eta : \eta \text{ is } f\text{-invariant and } \int\psi d\eta=c\}$. Then

$$\begin{aligned}P_{\text{top}}(f,\phi)-I_{f,\phi,\psi}(-c_1)&=P_{X(-c_1)}(f,\phi)\le P_{X(-c_2,-c_1)}(f,\phi)\\&\le P_{X[-c_2,-c_1]}(f,\phi)\le P_{\underline{X}_{c_1}}(f,\phi)\\&\le P_{\overline{X}_{c_1}}(f,\phi)\le P_{\text{top}}(f,\phi)-I_{f,\phi,\psi}(c_1)\\&\le P_{\text{top}}(f,\phi)-I_{f,\phi,\psi}(-c_1).\end{aligned}$$

This proves the first part of the theorem. We proceed to prove the continuity results using that $P_{\underline{X}_{f,\phi,\psi,c}}(f,\phi)=P_{\text{top}}(f,\phi)-\min\{I_{f,\phi,\psi}(c),I_{f,\phi,\psi}(-c)\}$ whenever the set $\underline{X}_{f,\phi,\psi,c}$ is non-empty. On the one hand, it is well known that $\phi\mapsto P_{\text{top}}(f,\phi)$ is continuous in the C^0-topology. On the other hand, if $\Lambda=M$ then f is expanding and $P_{\text{top}}(f,\phi)$ varies continuously with f in the C^1-topology since it coincides with the logarithm of the spectral radius of the transfer operator $\mathcal{L}_{f,\phi} : C^\alpha(M)\to C^\alpha(M)$ given by

$$\mathcal{L}_{f,\phi}\, g(x)=\sum_{f(y)=x}e^{\phi(y)}\, g(y).$$

In fact, due to the existence of a spectral gap property for $\mathcal{L}_{f,\phi}$ the spectral radius does vary continuously with respect to perturbations of the potential and the Legendre transform varies continuously with respect to the potential. We will provide a sketch of proof now addressing also the continuity of these objects as function of the dynamics f and observable ψ.

Given f, ϕ and ψ fixed, the spectral gap property for $\mathcal{L}_{f,\phi}$ implies that free energy $\mathcal{E}_{f,\phi,\psi}(t):=\limsup_{n\to\infty}\frac{1}{n}\log\int e^{tS_n\psi}\, d\mu_{f,\phi}$ is well defined for all $t\in\mathbb{R}$ and in fact it verifies $\mathcal{E}_{f,\phi,\psi}(t)=P_{\text{top}}(f,\phi+t\psi)-P_{\text{top}}(f,\phi)$. In particular, if ψ is cohomologous to a constant then $t\mapsto\mathcal{E}_{f,\phi,\psi}(t)$ is affine and otherwise $t\mapsto\mathcal{E}_{f,\phi,\psi}(t)$ is real analytic, strictly convex. Note also that for every $t \in \mathbb{R}$ the function $(f,\phi,\psi) \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is differentiable, the function $(\phi,\psi) \mapsto \mathcal{E}_{f,\phi,\psi}(t)$ is analytic and also

$$(f,\phi,\psi) \mapsto \mathcal{E}'_{f,\phi,\psi}(t) = \int \psi \, d\mu_{f,\phi+t\psi}$$

is continuous (see e.g. [BCV13]). Let $t \in \mathbb{R}$ be fixed. In order to establish the regularity of the rate function $I_{f,\phi,\psi}$ in what follows we assume without loss of generality that ψ is not cohomologous to a constant and that $m_{f,\phi} = \int \psi \, d\mu_{f,\phi} = 0$. Using that $\mathbb{R} \ni t \to \mathcal{E}_{f,\phi,\psi}(t)$ is strictly convex it is well defined its Legendre transform $I_{f,\phi,\psi}$ by

$$I_{f,\phi,\psi}(s) = \sup_{t\in\mathbb{R}} \big\{ st - \mathcal{E}_{f,\phi,\psi}(t) \big\}.$$

This function is non-negative and strictly convex since $\mathcal{E}_{f,\phi,\psi}$ is also strictly convex, and $I_{f,\phi,\psi}(s) = 0$ if and only if $s = m_{f,\phi}$. Morever, using the differentiability of the free energy function it is not hard to check the variational property

$$I_{f,\phi,\psi}(\mathcal{E}'_{f,\phi,\psi}(t)) = t\,\mathcal{E}'_{f,\phi,\psi}(t) - \mathcal{E}_{f,\phi,\psi}(t)$$

whenever the expressions make sense and, consequently, the rate function $I_{f,\phi,\psi}$ varies continuously with ϕ and ψ in the C^α-topology.

Finally we study the regularity of the function $v \mapsto P_{\overline{X}_c}(f_v,\phi)$ when $V \ni v \mapsto (f_v)_v$ is a continuous family of expanding maps on M and V is a compact metric space. Let $J \subset \mathbb{R}$ be a compact interval in the domain of $I_{f_v,\phi,\psi}$. From the previous variational relation we get that for any $s \in J$ there exists a unique $t = t(s,v)$ such that $s = \mathcal{E}'_{f_v,\phi_v,\psi_v}(t)$ and

$$I_{f_v,\phi,\psi}(s) = s \cdot t(s,v) - \mathcal{E}_{f_v,\phi,\psi}(t(s,v)). \tag{3.1}$$

Now, notice that the skew-product

$$\begin{array}{rcc} F : V \times J & \to & V \times \mathbb{R} \\ (v,t) & \mapsto & (v, \mathcal{E}'_{f_v,\phi,\psi}(t)) \end{array}$$

is continuous and injective because it is strictly increasing along the fibers (using the strict convexity of the free energy function). Since $V \times J$ is a compact metric space then F is a homeomorphism onto its image $F(V \times J)$. In particular this shows that for every $(v,s) \in F(V \times J)$ there exists a unique $t = t(v,s)$ varying continuously with (v,s) such that $F(v,t(v,s)) = (v,s)$ and $s = \mathcal{E}'_{f_v,\phi_v,\psi_v}(t)$. Finally, relation (3.1) above yields that $(s,v) \mapsto I_{f_v,\phi,\psi}(s)$ is continuous on $J \times V$. This finishes the proof of the continuity.

3.3. **Proof of Corollary C.** Let $f : M \to M$ be a continuous map admitting a mixing repeller $\Lambda \subset M$, $\phi : M \to \mathbb{R}$ be a continuous potential so that μ_ϕ is the unique equilibrium state for f with respect to ϕ and $\mu_\phi \ll \nu$ where ν is a Gibbs measure. Given $c > 0$ consider $\overline{E}_c = \overline{X}_c \cap E_\psi \subset \overline{X_c}$. Taking into account Theorem B and Corollary B then part (1) is immediate and we are reduced to prove the lower bound: if $\overline{E}_c \neq \emptyset$ then $P_{\overline{E}_c}(f,\phi) \geq P_{\overline{X}_c}(f,\phi)$.

Assume $\overline{E}_c \neq \emptyset$ for some $c > 0$. Under our assumptions it is well known that f satisfies the specification property and that for any f-invariant probability measure μ there exists a sequence of f-invariant ergodic probability measures μ_n so that $\mu_n \to \mu$ in the weak* topology and $h_{\mu_n}(f) \to h_\mu(f)$ as $n \to \infty$ (c.f. Theorem B in

[1]). By Theorem B and the thermodynamical formulation of the large deviations rate function obtained by L.S. Young [You90] we know that

$$P_{\overline{X}_c}(f,\phi) = P_{\text{top}}(f,\phi) - I_{f,\phi,\psi}(c_*) = \sup_{\eta} \Big\{ h_\eta(f) + \int \phi \, d\eta \Big\} \tag{3.2}$$

where $|c_*| = |c|$ and the supremum is taken over all f-invariant probability measures η so that $|\int \psi \, d\eta - \int \psi \, d\mu_\phi| \geq c_*$. Assume for simplicity that $c_* = c$ (the case other is analogous). Observe that

$$\begin{aligned} &\sup\Big\{ h_\eta(f) + \int \phi \, d\eta : |\int \psi \, d\eta - \int \psi \, d\mu_\phi| \geq c \Big\} \\ &\quad = \sup \Big\{ h_\eta(f) + \int \phi \, d\eta : |\int \psi \, d\eta - \int \psi \, d\mu_\phi| > c \Big\} \end{aligned}$$

by the continuity of the rate function $c \mapsto I_{f,\phi,\psi}(c)$ (since it coincides with the Legendre transform of the free energy function). Together with the variational relation (3.2), this yields that for any $\gamma > 0$ one can take two f-invariant probability measures η_1, η_2 so that

(i) $|\int \psi \, d\eta_i - \int \psi \, d\mu_\phi| > c$
(ii) $h_{\eta_i}(f) + \int \phi \, d\eta_i \geq P_{\overline{X}_c}(f,\phi) - 2\gamma$
(iii) $\int \psi \, d\eta_1 \neq \int \psi \, d\eta_2$

for $i = 1, 2$. Taking the approximation in entropy by f-invariant and ergodic probability measures, there are distinct ergodic probability measures ν_1 and ν_2 satisfying $|\int \psi \, d\nu_i - \int \psi \, d\mu_\phi| > c$, $\int \psi \, d\nu_1 \neq \int \psi \, d\nu_2$ and $h_{\nu_i}(f) + \int \phi \, d\nu_i \geq P_{\overline{X}_c}(f,\phi) - \gamma$ for $i = 1, 2$. Observe that $\overline{X}_c$ is an f-invariant set and the ergodicity together with the first property above implies that $\nu_i(\overline{X}_c) = 1$.

Now the proof follows the same lines of the proof of Theorem 2.6 in [Tho10]. Consider a strictly decreasing sequence $(\delta_k)_{k\geq 1}$ of positive numbers converging to zero, a strictly increasing sequence of positive integers $(\ell_k)_{k\geq 1}$, so that the sets

$$Y_{2k+i} = \Big\{ x \in \overline{X}_c : |\frac{1}{n} S_n \psi(x) - \int \psi \, d\nu_i| < \delta_k \text{ for every } n \geq \ell_k \Big\}$$

satisfy $\nu_i(Y_{2k+i}) > 1 - \gamma$ for every k ($i = 1, 2$). Consider the fractal set F given *ipsis literis* by the construction of Subsection 3.1 with ν_i replacing μ_i, $P_{\overline{X}_c}(f,\phi)$ replacing C and ψ replacing φ. From the construction (c.f. Lemma 3.8) there is a sequence $(t_k)_{k\geq 1}$ so that

$$\lim_{k\to\infty} |\frac{1}{t_{2k+i}} S_{t_{2k+i}} \psi(x) - \int \psi \, d\nu_i| = 0 \quad \text{for every } x \in F$$

and $P_F(f,\psi) \geq C - 8\gamma$. In particular F is contained in the irregular set E_ψ. Since γ was chosen arbitrary and $F \subset E_\psi$, to complete the proof of the corollary it is enough to prove that $F \subset \overline{X}_c$. This actually follows from item (1) above since for any $x \in F$

$$\begin{aligned} \limsup_{k\to\infty} \Big| \frac{1}{t_{2k+2}} S_{t_{2k+2}} \psi(x) - \int \psi \, d\mu_\phi \Big| &\geq \limsup_{k\to\infty} \Big[\Big| \int \psi \, d\nu_2 - \int \psi \, d\mu_\phi \Big| \\ &\quad - \Big| \frac{1}{t_{2k+2}} S_{t_{2k+2}} \psi(x) - \int \psi \, d\mu_\phi \Big| \Big] \\ &\geq c \end{aligned}$$

This finishes the proof of the corollary.

3.4. **Proof of Theorem C.** For the proof of the theorem we will need the following auxiliary lemma that will play the same role of Lemma 3.1 in the proof of Theorem A. It is here that we need the metric on $\mathcal{M}_1$ to be translation invariant and affine.

Lemma 3.2. *Let $c > 0$ be given. For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ and $N = N_\delta \in \mathbb{N}$ so that $B(x, n, \varepsilon) \subset Y_{\mu,c-\delta,n}$ for all $0 < \varepsilon < \varepsilon_\delta$, $n \geq N$ and $x \in Y_{\mu,c,n}$.*

Proof. Since $M \in x \mapsto \delta_x \in \mathcal{M}_1$ is uniformly continuous then given $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that if $d(x, y) < \varepsilon_\delta$ we have $d(\delta_x, \delta_y) < \delta$. Hence, if $x \in Y_{c,n}$ and $y \in B(x, n, \varepsilon)$ we have: $d(\delta_{y,n}, \mu) \geq d(\delta_{x,n}, \mu) - d(\delta_{x,n}, \delta_{y,n}) \geq c - \frac{1}{n}\sum_{i=0}^{n-1} d(\delta_x, \delta_y) \geq c - \delta$, and thus $y \in Y_{\mu,c,n}$, which proves the lemma. □

We proceed with the proof of the theorem assuming that $\mu = \mu_{f,\phi}$ is the unique equilibrium state for the continuous map f with respect to the continuous potential ϕ and also $\overline{Y}_{\mu,c} \neq \emptyset$. In order to prove that $P_{\overline{Y}_{\mu,c}}(f, \phi) \leq P_{\text{top}}(f, \phi) - \inf_{d(\eta,\mu)=c} Q(\eta)$ is strictly smaller than the topological pressure $P_{\text{top}}(f, \phi)$ we proceed to cover $\overline{Y}_{\mu,c}$ by a properly chosen family of dynamical balls. Fix $\delta > 0$ small and $\alpha > P_{\text{top}}(f, \Phi) - \inf_{d(\eta,\mu)\geq c-\delta} Q(\eta)$. Given $\varepsilon > 0$ small and $N \in \mathbb{N}$, for any $x \in \overline{Y}_{\mu,c}$ pick $m(x) \geq N$ in such a way that $x \in Y_{\mu,c-\frac{\delta}{2},m(x)}$ and consider $\mathcal{G}_N := \{(x, m(x)) : x \in \overline{Y}_{\mu,c}\}$. Hence

$$\overline{Y}_{\mu,c} \subset \bigcup_{(x,n)\in\mathcal{G}_N} B(x, n, \varepsilon)$$

and also $B(x, n, \varepsilon) \subset \overline{Y}_{\mu,c-\delta,n}$, for all $x \in Y_{\mu,c-\frac{\delta}{2},n}$ and $n \geq N$ and ε small (by Lemma 3.2). Therefore we can proceed as in the proof of Theorem A and extract a subset $\hat{\mathcal{G}}_N \subset \mathcal{G}_N$ in such a way that if (x, l) and (y, l) belong to $\hat{\mathcal{G}}_N$ then $B(x, l, \frac{\varepsilon}{2}) \cap B(x, l, \frac{\varepsilon}{2}) = \emptyset$. If $\zeta = (-P + \alpha + \inf_{d(\eta,\mu)\geq c-\delta} Q(\eta))/2 > 0$, by the large deviations upper bound, for every closed set U one has $\mu\,(x \in M : \delta_{x,n} \in U) \leq \exp(-n[\inf_{\eta\in U} Q(\eta) - \zeta])$ provided that $n \geq N$ is large enough. This, together with the Gibbs property for ν, yields that

$$\begin{aligned}
\sum_{(x,n)\in\hat{\mathcal{G}}_N} e^{-\alpha n + S_n\phi(x)} &\leq K(\varepsilon) \sum_{n\geq N} \sum_{x\in\hat{\mathcal{G}}_{N,n}} e^{(P-\alpha)n} \nu(B(x, n, \varepsilon)) \\
&\leq K(\varepsilon)K(\frac{\varepsilon}{2}) \sum_{n\geq N} e^{(P-\alpha)n} \nu\Big(\bigcup_{x\in\hat{\mathcal{G}}_{N,n}} B(x, n, \frac{\varepsilon}{2})\Big) \\
&\leq K(\varepsilon)K(\frac{\varepsilon}{2}) \sum_{n\geq N} e^{(P-\alpha)n} \nu\big(\overline{Y}_{\mu,c-\delta,n}\big) \\
&\leq K(\varepsilon)K(\frac{\varepsilon}{2}) \sum_{n\geq N} \exp n\big(P - \alpha - \inf_{d(\eta,\mu)\geq c-\delta} Q(\eta) + \zeta\big) \\
&\leq K(\varepsilon)K(\frac{\varepsilon}{2}) \sum_{n\geq N} e^{-\zeta n}
\end{aligned}$$

which is finite and independent of N. This proves that $P_{\overline{Y}_{\mu,c}}(f, \phi) \leq P_{\text{top}}(f, \phi) - \inf_{d(\eta,\mu)\geq c-\delta} Q(\eta)$. Since Q is lower semicontinuous it follows that

$$P_{\overline{Y}_{\mu,c}}(f, \phi) \leq P_{\text{top}}(f, \phi) - \inf_{d(\eta,\mu)\geq c} Q(\eta).$$

For the proof of the second part of the theorem we make use of the level-2 large deviations principles obtained by Zhou and Chen [ZC13] under the assumptions of almost product structure and the uniform separation properties. Let $0 < c_1 < c_2$ be so that $\overline{Y}_{\mu,c_1} \neq \emptyset$. Using [ZC13], given a compact connected subset in $\mathcal{M}_1$ then the topological pressure of the set $Y(C) := \{x \in M : \lim_{n\to+\infty} \delta_{x,n} \in C\}$ coincides with $\inf\{h_\eta(f) + \int \psi \, d\eta : \eta \text{ is } f\text{-invariant and } \eta \in C\}$. On other hand, by [CTY13] $Q(\eta) = P_{\text{top}}(f,\phi) - h_\eta(f) - \int \phi d\eta$. Thus using that the metric entropy is linear convex and the choice of the metric on $\mathcal{M}_1$ we have

$$
\begin{aligned}
P_{\text{top}}(f,\phi) - \inf_{d(\eta,\mu)=c_1} Q(\eta) &= P_{Y(\partial B(\mu,c_1))}(f,\phi) \le P_{Y(B(\mu,c_1,c_2))}(f,\phi) \\
&\le P_{Y\left(\overline{B(\mu,c_1,c_2)}\right)}(f,\phi) \le P_{Y(B(\mu,c_1))}(f,\phi) \\
&\le P_{Y(\overline{B(\mu,c_1)})}(f,\phi) \le P_{\overline{Y}_{\mu,c_1}}(f,\phi) \\
&\le P_{\underline{Y}_{\mu,c_1}}(f,\phi) \le P_{\text{top}}(f,\phi) - \inf_{d(\eta,\mu)\ge c_1} Q(\eta) \\
&\le P_{\text{top}}(f,\phi) - \inf_{d(\eta,\mu)=c_1} Q(\eta),
\end{aligned}
$$

proving all quantities coincide. This finishes the proof of the theorem.

3.5. **Proof of Theorem D.** This section is devoted to the proof of Theorem D that claims that if the set $\underline{X}_c$ is non-empty then it has smaller topological pressure. The strategy for the proof is similar to the one of Theorem A with the difficulty that in the non-uniformly expanding setting the Gibbs property holds at a sequence of moments that does depend on the point. For that reason we shall give a sketch of the proof with the main ingredients. Since μ is a weak Gibbs measure then there exists $\varepsilon_0 > 0$ so that the following property holds: for every $0 < \varepsilon < \varepsilon_0$ there exists $K(\varepsilon) > 0$ and for μ-almost every x there exists a sequence $n_k(x) \to \infty$ such that

$$K(\varepsilon)^{-1} \le \frac{\mu(B(x, n_k(x), \varepsilon))}{e^{-n_k(x)P + S_{n_k(x)}\phi(x)}} \le K(\varepsilon).$$

Assume the weak Gibbs property holds for *all* points in the invariant set $\Lambda = H$ and in what follows consider $\underline{X}_I := \underline{X}_I \cap \Lambda$.

Let $\delta > 0$ be arbitrary. We proceed to prove that $P_{\underline{X}_I}(f,\phi) \le P_{\text{top}}(f,\phi) - L_{I_\delta}$ is strictly smaller than the topological pressure. Consider $\alpha > P_{\text{top}}(f,\Phi) - L_{I_\delta}$ be given and take $\varepsilon > 0$ arbitrarily small and $N \in \mathbb{N}$ arbitrarily large in what follows. One can write

$$\underline{X}_I \subset \bigcup_{\ell \ge 1} \bigcap_{j \ge \ell} X_{I_\delta, j}.$$

where as before $X_{I,n} = \{x \in M : \frac{1}{n} S_n \psi(x) \in I\}$. It is not hard to check that for any $x \in \underline{X}_I$ there exists a sequence of positive integers $(m_j(x))_{j\in\mathbb{N}}$ converging to infinite so that $x \in X_{I_\delta, m_j(x)}$ and $m_j(x)$ is a moment at which the Giibs property holds. Therefore, one can pick $m(x) \ge N$ in such a way that $x \in X_{I_\delta, m(x)-1}$ and consider $\mathcal{G} := \{(x, m(x)) : x \in \overline{X}_I\}$. Now the proof proceeds with the estimates used in the proof of Theorem A.

Remark 3.3. In the previous proof we did not require the times at which the Gibbs property hold to have positive density at infinity as in usual notions of non-lacunary Gibbs measures. In particular, this gives a wider range of applications.

Remark 3.4. Actually let us mention that we could not estimate the topological pressure of the larger set $\overline{X}_c$. In fact, for that purpose we would need to guarantee that for each point there would exist a sequence of instants at which simultaneously the Gibbs property and the time averages being far from the time average occurs. Nevertheless this can be verified in examples.

4. Examples and applications

4.1. **Hyperbolic diffeomorphisms.** Let now $f : M \to M$ be a diffeomorphism, and let $\Lambda \subset M$ be a compact $f-$invariant set. We recall that such a set Λ is called a hyperbolic set for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space $T_xM = E^s(x) \oplus E^u(x)$ such that

$$Df(x) \cdot E^s(x) = E^s(fx) \;\text{ and }\; Df(x) \cdot E^u(x) = E^u(fx),$$

and there exist constants $\lambda \in (0, 1)$ and $C > 0$ such that

$$||Df(x)^n_{|E^s(x)}|| \leq C\lambda^n \;\text{ and}||Df(x)^{-n}_{|E^u(x)}|| \leq C\lambda^n$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$. Any hyperbolic set has Markov partitions with arbitrarily small diameter (see for example [Bow75]). Given a Markov partition $\mathcal{P}$ and $x \in \Lambda$ we can define $\mathcal{P}_n(x) := \{y \in \Lambda : \; f^j(y) \in R(f^j(x)) \text{ for all } -n \leq j \leq n\}$ as the cilinder of size n, where $R(x)$ denotes the element of the Markov partition that contains x. Given a potential $\phi : M \to \mathbb{R}$ Holder continuous we know that there exists a unique equilibrium state μ for $f_{|\Lambda}$ with respect the ϕ. Furthermore, μ is Gibbs measure: there exists a contanst $K > 0$ such that

$$K^{-1} \leq \frac{\mu(\mathcal{P}_n(x))}{e^{-nP_{\text{top}}(f,\phi)+S_n\phi(x)}} \leq K$$

for all $x \in \Lambda$ and $n \in \mathbb{N}$. This can be proved via semi-conjugacy to a subshift of finite type (see e.g. [Bow75]). With this in mind it is not hard to check that Theorem A holds also for bilateral subshifts of finite type and *locally Hölder* observables. In fact, given such $g : \Lambda \to \mathbb{R}$ there exists ψ that is constant along local stable leaves (depends only on future coordinates of the shift) and such that $g = \psi + u \circ f - u$ for some continuous u. In particular $\left|\frac{1}{n}S_n g(x) - \frac{1}{n}S_n\psi(x)\right| \leq \frac{2\|u\|_0}{n}$ tends to zero (uniformly) as approaches infinity and, consequently, $X_I(g) = X_I(\psi)$, $\overline{X}_I(g) = \overline{X}_I(\psi)$ and $\underline{X}_I(g) = \underline{X}_I(\psi)$ for all intervals $I \subset \mathbb{R}$. Hence, using the Gibbs property and replacing dynamic balls by cylinders $\mathcal{P}_n$ associated to the Markov partition the same results as in Theorem A hold.

4.2. **Hyperbolic flows.** Let $Y \in \mathfrak{X}^1(M)$ be a smooth vector field and $(Y_t)_{t\in\mathbb{R}}$ be the associated flow. A compact invariant set $\Lambda \subset M$ is a *hyperbolic set for* $(Y_t)_t$ if there are constants $\lambda \in (0, 1)$ and $C > 0$ so that for all $x \in \Lambda$ there exists a splitting $T_xM = E^u_x \oplus E^0_x \oplus E^s_x$ so that $E^0_x =< Y(x) >$ is a one-dimensional subspace, $DY_t(x) \cdot E^u_x = E^u_{Y_t(x)}, DY_t(x) \cdot E^s_x = E^s_{Y_t(x)}$ and also

$$\|DY_t(x)^t_{|E^s(x)}\| \leq C\lambda^t \quad \text{and} \quad \|DY_t(x)^{-t}_{|E^u(x)}\| \leq C\lambda^t$$

for all $t \geq 0$. A flow is *Axiom A* if the non-wandering set is a hyperbolic set and periodic orbits are dense. It follows from the pioneering work of Bowen and Ruelle [BR75] that Axiom A flows $(Y_t)_t$ are semi-conjugate to suspension flows over subshifts of finite type. Recall that given a subshift of finite type $\sigma \colon \Sigma \to \Sigma$ and a ceiling function $h \colon \Sigma \to \mathbb{R}^+$ bounded away from zero and infinity the associated *suspension flow* $(S_t)_t$ is defined in $M_h = \{(x,t) \in \Sigma \times \mathbb{R}_+ : 0 \le t \le h(x)\}$ with the identification between the pairs $(x, h(x))$ and $(\sigma(x), 0)$. The semiflow defined on M_h by $S_t(x,r) = (\sigma^n(x), r + t - \sum_{i=0}^{n-1} h(\sigma^i(x)))$, where $n \in \mathbb{Z}+$ is uniquely defined by

$$\sum_{i=0}^{n-1} h(\sigma^i(x)) \le r + t < \sum_{i=0}^{n} h(\sigma^i(x)). \tag{4.1}$$

Since σ is invertible then $(S^t)_t$ is indeed a flow. Moreover, since h is bounded then $\eta \mapsto \frac{\eta \times \text{Leb}_1}{\int h d\eta}$ is a one-to-one correspondence between σ-invariant probability measures and S^t-invariant probability measures, where Leb_1 denotes the one dimensional Lebesgue measure and the probability measure $\tilde{\eta} := (\eta \times \text{Leb}_1)/\int h\, d\eta$ is defined on M_h by $\int g\, d\tilde{\eta} = \frac{1}{\int h\, d\eta} \int \left(\int_0^{h(x)} g(x,t) dt \right) d\eta(x), \quad \forall g \in C^0(M_h)$. Given $\psi : M_h \to \mathbb{R}$ we associate the observable $\bar{\psi}$ on Σ defined as $\bar{\psi}(x) = \int_0^{h(x)} \psi(x,t)\, dt$. In particular, given $T > 0$ large, $(x,s) \in M_h$ and $n = n(x, T+s)$ defined by equation (4.1) it follows that $n(x, T+s) \to \infty$ as $T \to \infty$ and

$$\frac{1}{n(x,T+s)} \sum_{i=0}^{n(x,T+s)-1} h(\sigma^i(x)) \le \frac{T+s}{n(x,T+s)} < \frac{1}{n(x,T+s)} \sum_{i=0}^{n(x,T+s)} h(\sigma^i(x)). \tag{4.2}$$

With this notation we can write

$$\begin{aligned} \frac{1}{T} \int_0^T \psi(S_t(x,s))\, dt &= \frac{n(x,T+s)}{T} \frac{1}{n(x,T+s)} \sum_{i=0}^{n(x,T+s)} \overline{\psi}(\sigma^i(x)) \\ &\quad + \frac{1}{T} \int_0^{T+s-\sum_{i=0}^{n(x,T+s)-1} h(\sigma^i(x))} \psi(S_t(\sigma^n(x),0))\, dt, \end{aligned}$$

where the second term is bounded from above by $\|\psi\|_0 \;\|h\|_0/T$ and converges uniformly to zero as T tends to infinity. Moreover, for any $\beta_1, \beta_2, \beta_3 > 0$ sufficiently small such that $\frac{\beta_1}{\int h d\mu_\Sigma} + ||\psi||_0 \cdot \beta_2 < c$ and $\frac{\beta_3}{\inf h \cdot \int h d\mu_\Sigma} < \beta_2$,

$$\overline{X}_c = \Big\{ (x,s) \in M_h : \limsup_{T\to\infty} \big| \frac{1}{T} \int_0^T \psi(S_t(x,s))\, dt - \int \psi\, d\mu \big| \ge c \Big\}$$

is contained in the union of the $(S_t)_t$-invariant sets

$$\begin{aligned} \overline{X}_{c,h} &= \big\{ (x,s) \in M_h : \limsup_{T\to\infty} \big| \frac{n(x,T+s)}{T+s} - \frac{1}{\int h\, d\mu_\Sigma} \big| \ge \beta_2 \big\} \\ &\subseteq \big\{ (x,s) \in M_h : \limsup_{T\to\infty} \big| \frac{1}{n(x,T+s)} \sum_{i=0}^{n(x,T+s)} h(\sigma^i(x)) - \int h\, d\mu_\Sigma \big| \ge \beta_3 \big\} \end{aligned}$$

and

$$\overline{X}_{c,\overline{\psi}} = \big\{ (x,s) \in M_h : \limsup_{T\to\infty} \big| \frac{1}{n(x,T)} \sum_{i=0}^{n(x,T+s)} \overline{\psi}(\sigma^i(x)) - \int \overline{\psi}\, d\mu_\Sigma \big| \ge \beta_1 \big\}$$

In consequence, using $P_{\overline{X}_c}((S_t)_t, \phi) \le \max\{P_{\overline{X}_{c,h}}((S_t)_t, \psi), P_{\overline{X}_{c,\overline{\phi}}}((S_t)_t, \phi)\}$ and the relations as in Theorem A we can also deduce by semi-conjugacy the following:

Corollary F. *Let $(X_t)_t$ be an Axiom A flow, $\phi : M \to \mathbb{R}$ be an Hölder continuous potential and $\mu = \mu_\phi$ be the unique equilibrium state for $(X_t)_t$ with respect to ϕ. Then, for any continuous observable $\psi : M \to \mathbb{R}$ and $c > 0$ it holds that*

$$P_{\overline{X}_c}((X_t)_t, \phi) < P_{\text{top}}((X_t)_t, \phi).$$

Actually it seems reasonable to expect similar property to hold for other classes of suspension flows for which a thermodynamical formalism is proved. See e.g. [IJT14] and references therein.

4.3. **Maneville-Pommeau maps.** If $\alpha \in (0,1)$, let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the local homeomorphism given by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

This map satisfies the specification property since it is topological conjugate to the double expanding map. Pollicott and Weiss [PW99] established a multifractal formalism for the Lyapunov spectrum associated to this class of transformations and proved precise formulas for the dimension of the level sets of points with same Lyapunov exponent. Clearly $\frac{1}{n}\log|(f_\alpha^n)'(x)| = \frac{1}{n}\sum_{j=0}^{n-1}\psi(f_\alpha^j(x))$ with $\psi(x) = \log|(f_\alpha)'(x)|$. For every $t \in (-\infty, 1)$ there exists a unique equilibrium state μ_t with respect to the Hölder continuous potential $\phi_t = -t\log|(f_\alpha)'(x)|$ and it is well known that there are two equilibrium states for f_α with respect to $-\log|(f_\alpha)'(x)|$ namely an absolutely continuous invariant probability measure μ_1 and the Dirac measure δ_0. Moreover, for every $t \le 1$ the equilibrium state μ_t for f with respect to the potential ϕ_t satisfies a weak Gibbs property: there exists a sequence K_n so that $\limsup_{n\to\infty}\frac{1}{n}\log K_n = 0$ so that

$$\frac{1}{K_n} \le \frac{\mu_t(\mathcal{P}^{(n)}(x))}{e^{-nP_t}|(f^n)'(x)|^t} \le K_n$$

for all $x \in [0,1]$ and $n \ge 1$, where $\mathcal{P}$ is the Markov partition for f, $\mathcal{P}^{(n)}(x)$ is the element of the partition $\mathcal{P}^{(n)} = \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$ that contains x and $P_t = P_{\text{top}}(f, \phi_t)$. By the Ruelle inequality all measures μ_t are expanding.

The SRB measure μ_1 is absolutely continuous w.r.t. Lebesgue, has with polynomial decay of correlations of order $\mathcal{O}(n^{\frac{1}{\alpha}-1})$ and polynomial upper and lower bounds for Hölder continuous observables have been established in [MN08, Mel09, PS09] and our results cannot apply.

We will address the question when $|t|$ is small. First, it follows from [Var12] that for every $c > 0$ and every *continuous* observable ψ

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\left(x \in M : \left|\frac{1}{n}S_n\psi(x) - \int\psi\,d\mu_t\right| \ge c\right) < 0$$

Finer results hold if one assumes Hölder regularity of the observables. Indeed, if $|t|$ is small it follows from [BCV13] that there exists an interval $J \subset \mathbb{R}$ such that the following local large-deviations principle holds:

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_t\left(x \in M : \frac{1}{n}S_n\psi(x) \in [a,b]\right) \le -\inf_{s\in[a,b]} I_{f,\phi_t,\psi}(s)$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\log\mu_t\left(x \in M : \frac{1}{n}S_n\psi(x) \in (a,b)\right) \ge -\inf_{s\in(a,b)} I_{f,\phi_t,\psi}(s)$$

every $[a,b] \subset J$. Continuity and smoothness of the rate function was also studied in [BCV13]. As a consequence we deduce from Theorem D that for all $c>0$ satisfying $[\int \psi d\mu_t - c, \int \psi d\mu_t + c] \subset J$ either $\underline{X}_c = \emptyset$ or else

$$P_{\overline{X}_c}(f,\phi_t) = P_{\underline{X}_c}(f,\phi_t) = P_{X(c)}(f,\phi_t) = P_{X([c_1,c_2])}(f,\phi_t)$$
$$= P_{X(c_1,c_2)}(f,\phi_t) = P_{\text{top}}(f,\phi_t) - I_{f,\phi_t,\psi}(c).$$

for $c = \max\{|c_1|,|c_2|\}$. Furthermore, the right hand side expression varies continuously with c and also varies continuously with f, ϕ_t and ψ. In particular, although the set of irregular points has full topological entropy $\log 2$ (see e.g. [Tho10]) the set of Lyapunov irregular points whose Birkhoff averages remain far from $\lambda(\mu_0) := \int \psi d\mu_0$ for all large iterates has topological entropy strictly smaller than $\log 2$.

4.4. **Quadratic maps.** Let us consider the class of quadratic maps f_a on the real line given by $f_a(x) = 1 - ax^2$. Benedicks and Carleson [BC85] proved the existence of a positive Lebesgue measure set of parameters $\Omega \in [0,2]$ such that for every $a \in \Omega$ the quadratic map f_a has positive Lyapunov exponent and an unique absolutely continuous invariant probability measure μ_a supported on $[f^2(0), f(0)]$. Since these maps are continuous and topologically mixing on $[f^2(0), f(0)]$ then it follows from Blokh's theorem that they satisfy the specification property.

Upper and lower large deviations estimates for one-dimensional non-uniformly expanding maps were obtained e.g. by Keller and Nowicki [KN92] for quadratic maps satisfying the Collet-Eckmann condition (large deviations principle), by Araújo and Pacífico [AP06] for the absolutely continuous invariant measure of non-uniformly expanding quadratic maps, by the second author [Var12] for equilibrium states satisfying a weak Gibbs property and by Chung and Takahasi [CT12] (full large deviation principle). Moreover by Pesin-Ruelle inequality it follows that the topological pressure for $-\log|f'|$ is zero. As a consequence of our results, the topological pressure of the set of points whose Lyapunov spectrum is far from $\lambda(\mu) := \int \log|f'| d\mu$ for all large n has topological pressure strictly smaller than zero although points that do not converge have full pressure.

4.5. **Interval maps.** A broad class of interval maps for which our results are the ones considered by Bruin and Todd in [BT09]. Assume that f is a transitive multimodal interval map with finitely many non-degenerate critical points with negative Schwarzian derivative. If there exists $C > 0$ and $\beta > 2\ell - 1$ so that $|Df^n(c)| \geq n^\beta$ for every critical point c and all $n \geq 1$ (where ℓ denotes the maximal order of the critical points) then it follows from [BT09, Theorem 1] that there exists $t_1 < 1$ so that for all $t \in (t_1, 1)$:

(i) there exists a unique equilibrium state μ_t for f with respect to the potential $\varphi_t = -t \log|Df|$, and
(ii) μ_t has a compatible inducing scheme with exponential tails, hence it has exponential decay of correlations
(iii) μ_t has positive Lyapunov exponent almost everywhere

Moreover, there exists a conformal probability measure ν_t so that

$$J_{\nu_t} f(x) = e^{P(t)} |f'(x)|^{-t} \quad \text{almost everywhere}$$

and $\mu_t \ll \nu_t$, where $P(t) = P_{\text{top}}(f, -t\log|f'|)$. In addition, since μ_t has only positive Lyapunov exponents then almost every point has infinitely many hyperbolic times. If n is a hyperbolic time for x then the Jacobian $J_{\nu_t} f^n$ has bounded distortion and, consequently, ν_t satisfies the weak Gibbs property. On the other hand, using property (ii) above with the results by [ALFV] that μ_t has exponential large deviation estimates and so satisfies the assumptions of Theorem D.

4.6. **Higher dimensional non-uniformly expanding maps.** A much larger class of local diffeomorphisms than those considered in Maneville-Pommeau can be proved to satisfy our results. Assume that M is a metric space where the Besicovitch covering lemma and let $f : M \to N$ be a local homeomorphism so that: there exists a bounded function $x \mapsto L(x)$ such that, for every $x \in M$ there is a neighborhood U_x of x so that $f_x : U_x \to f(U_x)$ is invertible and

$$d(f_x^{-1}(y), f_x^{-1}(z)) \le L(x)\, d(y,z), \quad \forall y,z \in f(U_x).$$

Assume also that every point has finitely many preimages and that the level sets for the degree $\{x : \#\{f^{-1}(x)\} = k\}$ are closed. Given $x \in M$ set $\deg_x(f) = \# f^{-1}(x)$. Define $h_n(f) = \min_{x\in M} \deg_x(f^n)$ for $n \ge 1$, and consider the limit

$$h(f) = \liminf_{n\to\infty} \frac{1}{n} \log h_n(f).$$

Up to consider the iterate f^N instead of f we will assume that every point in M has at least $e^{h(f)}$ preimages by f. Assume also f is uniformly expanding outside $\mathcal{A}$ and not too contracting inside $\mathcal{A}$ (see [VV10] for precise statements). Then it follows from [Var12, BCV13] that local large deviation estimates hold for all equilibrium states associated to Hölder continuous potentials with low variation and our results in Theorem D apply.

Example 4.1. A concrete example can be build on the torus as follows. Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ be a linear expanding map. Fix some covering $\mathcal{P}$ for f_0 and some $P_1 \in \mathcal{P}$ containing a fixed (or periodic) point p. Then deform f_0 on a small neighborhood of p inside P_1 by a pitchfork bifurcation in such a way that p becomes a saddle for the perturbed local homeomorphism f. By construction, f coincides with f_0 in the complement of P_1, where uniform expansion holds. Observe that we may take the deformation in such a way that f is never too contracting in P_1, which guarantees that (H1) holds, and that f is still topologically mixing.

4.7. **Bowen-eye like systems and a counter-example.**

4.7.1. *Distinction of $\overline{X}_c$ and $\underline{X}_c$.* We shall present a simple example of a discrete dynamical system f, potential ϕ, observable ψ and constant $c > 0$ so that $\overline{X}_c \neq \underline{X}_c$. The map f corresponds to the time-one map of a flow known as the Bowen eye. The map f has three fixed points p_1, p_2 and p_3 (labeled from the left in Figure 2 below) and is such that $\{p_1, p_2, p_3\} = \mathrm{Per}(f) = R(f)$ while the non-wandering set is formed by the fixed point p_2 and the closure D of the two separatrices corresponding to the singularities p_1 and p_3 of the original vector field.

Moreover, it is well known that for *every* x in inner region of the plane determined by D (except p_2) the empirical measures $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}$ have the Dirac measures δ_{p_1} and δ_{p_3} as accumulation points. If $\phi : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the x-coordinate then, by the variational principle,

$$P_{\mathrm{top}}(f,\phi) = \sup\{h_\mu(f) + \int \phi\, d\mu\} = \max_i \{\phi(p_i)\} = \phi(p_3)$$

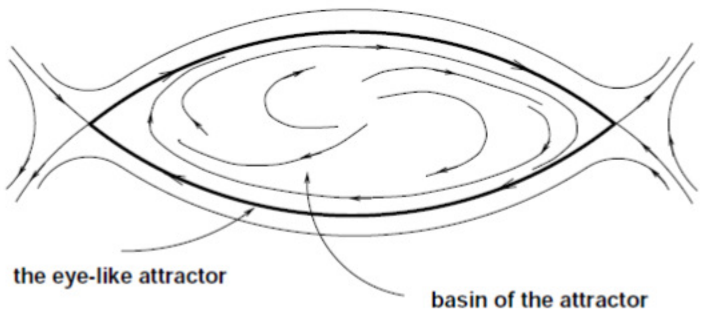

FIGURE 2. Bowen eye attractor

and δ_{p_3} is the unique equilibrium state for ϕ. On the other hand, for $0 < c < d(p_2, p_3)$ it is clear that $\underline{X}_c = W^s(p_1) \cup \{p_2\}$ and $\overline{X}_c = D \setminus (W^s(p_3) \cup \{p_2\})$. However, in this case one has $P_{\underline{X}_c}(f, \phi) = P_{\overline{X}_c}(f, \phi)$.

4.7.2. *A counter-example.* Despite the fact that the topological pressure of both sets $\underline{X}_c$ and $\overline{X}_c$ do coincide, the previous Bowen-eye construction gives some light on how to construct an example where $\overline{CP}_{\underline{X}_c}(f, \phi) < \overline{CP}_{\overline{X}_c}(f, \phi)$, where $\overline{CP}_\Lambda$ denotes the upper Carathéodory capacity of the set Λ (see e.g. [Pes97, Section 11]). The following can be realized as a non-compact invariant set of a horseshoe.

Let $\sigma : \Sigma_A \to \Sigma_A$, with $\Sigma_A \subset \{0, 1, 2, 3\}^{\mathbb{Z}}$, be the subshift of finite type associated to the transition matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, consider the σ-invariant subset $\Sigma \subset \Sigma_A$ that contains the four fixed points for the shift σ and (corresponding to the constant sequences) and be such that any $x = (x_n)_n \in \Sigma \setminus \{\underline{3}\}$ it holds that

$$\limsup_{n\to\infty} \frac{1}{2n} \#\big\{ |j| \le n : x_j \in \{1, 2\}\big\} = 1$$

and

$$\liminf_{n\to\infty} \frac{1}{2n} \#\big\{ -n \le j \le n : x_j \in \{1, 2\}\big\} = 0$$

Let ϕ be a continuous potential so that the unique equilibrium state is $\mu_\phi = \delta_{\underline{0}}$ (such a potential can be build non-negative following the ideas of Hofbauer [Hof77, Page 226, 239]) and consider the continuous observable $\psi = \chi_{[0]}$. Notice that $\int \psi \, d\mu_\phi = 1$ and for $c > 0$ small enough we get that $\underline{X}_c = \{\underline{3}\}$ while $\overline{X}_c = \Sigma \setminus \{\underline{0}\}$. Since $\phi \mid_{\underline{X}_c} \equiv 0$ and $\underline{X}_c = \{\underline{3}\}$ then $\overline{CP}_{\underline{X}_c}(f, \phi) = h_{\underline{X}_c}(f) = 0$. On the other hand, since ϕ is non-negative then $\overline{CP}_{\overline{X}_c}(f, \phi) \ge \overline{CP}_{\overline{X}_c}(f, 0)$ which we now claim to be strictly positive. In fact if $0 < \alpha < \log 2$ we will prove that $m_\alpha(f, \overline{X}_c) = +\infty$ and deduce that $\overline{CP}_{\overline{X}_c}(f, 0) > 0$. Recall that

$$m_\alpha(f, \overline{X}_c) = \lim_{\mathrm{diam}(\mathcal{U}) \to 0} m_\alpha(f, \overline{X}_c, \mathcal{U})$$

where $m_\alpha(f,\overline{X}_c,\mathcal{U}) = \lim_{N\to\infty} m_\alpha(f,\overline{X}_c,\mathcal{U},N)$, and

$$m_\alpha(f,\overline{X}_c,\mathcal{U},N) = \inf\Big\{ \sum_{U\in\mathcal{G}_N} e^{-\alpha N} : \mathcal{G}_N \text{ is subcover of } \vee_{0\le j\le N}\, \sigma^{-j}\mathcal{U}\Big\}.$$

Let $\varepsilon > 0$ be small and fixed (to be made precise later and depending only on α). For any $\ell \ge 1$, let us consider an open cover $\mathcal{U}_\ell$ of $\overline{X}_c$ formed by cylinders as follows: a $(2n+1)$-cylinder $U = [x_{-n}, \dots, x_n]$ belongs to $\mathcal{U}_\ell$ if and only if $n \ge \ell$ is the smallest positive integer such that

$$\#\big\{|j| \le n : x_j \in \{1,2\}\big\} \ge (2n+1)(1-\varepsilon). \tag{4.3}$$

In fact, given $x = (x_j)_j \in \underline{X}_c$ and defining $n(x) \ge \ell$ to be the first instant such that equation (4.3) holds it follows that $[x_{-n}, \dots, x_n]$ belongs to $\mathcal{U}_\ell$ and so $\mathcal{U}_\ell$ covers $\underline{X}_c$. Moreover, by construction, the elements of $\mathcal{U}_\ell$ are all disjoint, every such element contains at least one point of $\underline{X}_c$ and the diameter of $\mathcal{U}_\ell$ goes to zero as $\ell \to \infty$. Thus

$$m_\alpha(f,\overline{X}_c) = \lim_{\ell\to\infty} m_\alpha(f,\overline{X}_c,\mathcal{U}_\ell)$$

Observe also that $\# U_\ell \ge 2^{(2\ell+1)(1-\varepsilon)}$ which correspond to the number of disjoint cylinders of length $(2\ell+1)$ satisfying (4.3). Therefore, given any $N \gg 1$ and any subcover $\mathcal{G}_{N,\ell}$ of the space of cylinders $\vee_{0\le j\le N}\sigma^{-j}\mathcal{U}_\ell$ that covers $\overline{X}_c$ coincides with the space of all $(N+\ell)$-cylinders. If one writes $N+\ell = (2\ell+1)s+r$ with $s \ge 1$ and $0 \le r \le 2\ell$ then there are at least $2^{(N+\ell-r)(1-\varepsilon)}$ such cylinders (just by considering N-concatenations of $(2\ell+1)$-cylinders that satisfy equation (4.3)). Thus, if $\varepsilon > 0$ is chosen small then it follows that

$$m_\alpha(f,\overline{X}_c,\mathcal{U}_\ell) \ge \limsup_{N\to\infty} \sum_{U\in\mathcal{G}_{N,\ell}} e^{-\alpha N} \ge \limsup_{N\to\infty} e^{-\alpha N}\, 2^{(N-\ell)(1-\varepsilon)} = +\infty.$$

Hence $\overline{CP}_{\overline{X}_c}(f,0) \ge \log 2 > 0$ which proves our claim. Finally let us mention that since $P_\Lambda(f,\phi) \le \overline{CP}_\Lambda(f,\phi)$ then it is still a question to construct an example where $P_{\underline{X}_c}(f,\phi) < P_{\overline{X}_c}(f,\phi)$.

4.8. **Discontinuity and non-strict monotonicity of the pressure function.**

4.8.1. *Porcupine-like horseshoes.* To present an example where there is discontinuity and non-strict monotonicity of the pressure function $c \mapsto P_{\overline{X}_c}(f,\phi)$ we use the class of local diffeomorphisms f studied by Díaz, Gelfert and Rams that exhibit porcupine-like horseshoes. In fact, it follows from the analysis of the Lyapunov spectrum in the central direction (see [DG12, Remark 5.4] and [DGR11]) that there are constants $\lambda < 0 < \tilde\beta < \beta$ so that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n\mid_{E^c}(x)\| \in [\log\lambda, \log\tilde\beta] \cup \{\log\beta\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \|Df^n\mid_{E^c}(x)\| \in [\log\lambda, \log\tilde\beta] \cup \{\log\beta\},$$

and also that there exists a unique point Q (indeed it is a fixed point by f) so that the central Lyapunov exponent is $\log\beta > 0$. Let us consider the Hölder continuous potential $\phi_t = -t\log\|Df\mid_{E^c}\|$ for a large value of negative t and the observable $\psi = \log\|Df\mid_{E^c}\|$. It follows from [DG12, Proposition 5.6] that for all $t \ll 0$ the Dirac measure δ_Q is the unique equilibrium state for f with respect to ϕ_t and consequently $P_{\text{top}}(f,\phi_t) = -t\log\|Df(Q)\mid_{E^c}\| = -t\log\beta$.

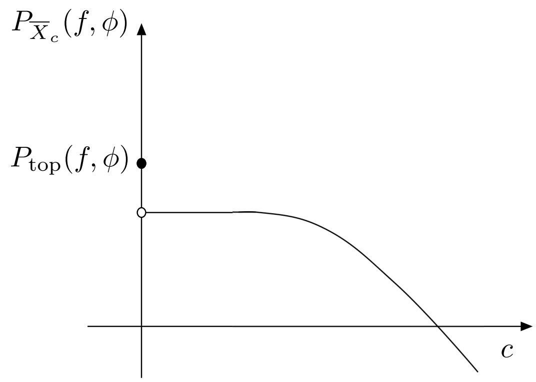

FIGURE 3. Discontinuity of the pressure function

On the other hand if $c \neq \log\beta$ then, using all invariant measures that have central Lyapunov exponent equal to c, we can estimate

$$\sup\Big\{h_\eta(f) + \int -t\log\|Df(x)\mid_{E^c}\|\,d\eta : \lambda(\eta) = c\Big\} \le h_{\text{top}}(f) - t\log\tilde\beta$$
$$< P_{\text{top}}(f,\phi_t)$$

which shows the discontinuity of the pressure function $c \mapsto P_{\overline{X}_c}(f,\phi_t)$ where $\overline{X}_c$ is associated to the observable ψ. Actually the same argument leads to prove that

$$\sup\Big\{h_\eta(f) + \int -t\log\|Df(x)\mid_{E^c}\|\,d\eta : \lambda(\eta) \in [\log\lambda, \log\tilde\beta]\Big\}$$
$$= \sup\Big\{h_\eta(f) + \int -t\log\|Df(x)\mid_{E^c}\|\,d\eta : \lambda(\eta) \in [\log\lambda, \log\beta)\Big\}$$
$$< P_{\text{top}}(f,\phi_t)$$

and so there exists an interval of constancy for this pressure function.

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