TOPOLOGICAL FEATURES OF FLOWS WITH THE REPARAMETRIZED GLUING ORBIT PROPERTY

THIAGO BOMFIM, MARIA JOANA TORRES AND PAULO VARANDAS

Abstract. The notions of shadowing, specification and gluing orbit property differ substantially for discrete and continuous time dynamical systems. In the present paper we continue the study of the topological and ergodic properties of continuous flows with the (reparametrized) periodic and nonperiodic gluing orbit properties initiated in [4]. We prove these flows satisfy a weak mixing condition with respect to balls and, if the flow is Komuro expansive, the topological entropy is a lower bound for the exponential growth rate of periodic orbits. Moreover, we show that periodic measures are dense in the set of all invariant probability measures and that ergodic measures are generic. Furthermore, we prove that irrational rotations and some minimal flows on tori and circle extensions over expanding maps satisfy gluing orbit properties, thus emphasizing the difference of this property with respect the notion of specification.

1. Introduction

The recent revived interest for specification properties in the last few years indicate that the original concept of specification introduced by Bowen [6] is far from generating an old fashioned mechanism to study the topological and ergodic features of the dynamical system. While the strong specification property fails to extend beyond uniformly hyperbolic diffeomorphisms and flows (cf. [26, 27]) many other non-uniform notions have been introduced to reflect non-uniform hyperbolicity (cf. [22, 20, 31, 21]). In the time-continuous setting the property of specification is not satisfied even among uniformly hyperbolic basic sets since these may fail to be topologically mixing. Indeed, any suspension flow obtained as the suspension of an Anosov diffeomorphism with a constant roof function is clearly an Anosov flow, hence it is expansive and satisfies the shadowing property, but it misses to be topologically mixing and therefore to satisfy the specification property. Since the specification property has proved to be a very useful tool to study multifractal formalism, thermodynamical formalism and large deviations it is important to create mechanisms that enable us to study these properties in the setting of flows with some weak forms of hyperbolicity.

Motivated by the common features of uniformly hyperbolic flows, in [4] the first and third authors introduced a concept of 'gluing orbit property' which is a topological invariant and much weaker than specification. Among the mechanisms to construct continuous flows with the gluing orbit property we mention: (i) suspension flows of homeomorphisms with the gluing orbit property or specification; and (ii) continuous flows with dense set of periodic orbits and satisfying the shadowing property (see [4, 2] for more details). In this way it is possible to provide vast classes of examples of continuous flows with this property, which includes e.g. a C^0-generic subset (hence dense) of Lipschitz vector fields [2].

It is well known that uniformly hyperbolic flows, hence strongly chaotic, have a rich structure on their simplex of invariant probability measures and their thermodynamic formalism is well established [7, 24]. The study of the ergodic properties of non-uniformly hyperbolic flows presents key difficulties either in the reduction to the analysis of the discrete-time dynamics of Poincaré return maps, which creates discontinuities for the discrete time dynamics, or by the presence of singularities. For that reason, an extension of such results for wider classes of non-uniformly hyperbolic flows is still a challenge. Some recent contributions in this direction include the fact that periodic measures are dense in the space of invariant probability measures for geodesic flows on non-positively curved manifolds ([9]) and that flows and Hölder continuous potentials for which

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obstructions to expansiveness and specification have small topological pressure have unique equilibrium states ([8]).

In this paper our purpose is to study some topological and ergodic features of continuous flows with the gluing orbit property and to establish their similitude and differences with respect to flows with specification and also with discrete time dynamical systems. Our first main result is that these systems satisfy some positive lower frequency of visits to balls and this frequency can be taken proportional to the radius of the balls. This condition resembles a weak mixing condition on balls, although the gluing orbit property does not imply on weak mixing (cf. Example 3.1). Moreover, despite the fact that the gluing orbit property need not imply on positive topological entropy (cf. Example 3.2), if the entropy is positive then it implies the number of periodic orbits to grow exponentially. Finally, if the flow is Komuro expansive (a weak notion of expansiveness that allows the presence of singularities) then topological entropy coincides with the exponential growth rate of periodic orbits and it can be computed in any open set. We refer the reader to Theorem A for the precise statements. From the ergodic theory viewpoint, the space of invariant probability measures contains a dense set of periodic measures and the ergodic measures are residual (Theorem B). In particular, we conclude this is the case for continuous flows with the shadowing property and displaying a dense subset of periodic points (Corollary 1). We also prove that the cohomology for continuous functions is determined by periodic orbits (Corollary 2). Finally in Section 3 we relate positive entropy, the existence of periodic orbits, the specification and the gluing orbit properties and include many examples (e.g. irrational flows on tori, suspension semiflows over the Maneville-Pomeau map, circle extensions over expanding maps and polygonal billiards satisfy the gluing orbit property). Indeed, although the product of dynamics with gluing may not satisfy the gluing orbit property (cf. Example 3.3), the later holds for products of a map with specification and other with the gluing orbit property. Furthermore, it is well known that every continuous flow with specification has positive entropy and is mixing. This is not the case for flows with the gluing orbit property (see Section 3 for examples).

This paper is organized as follows. In Section 2 we discuss the features of discrete-time dynamical systems with the specification property and state our main results for continuous flows with the gluing orbit property. Section 3 is devoted to show some main differences between the concepts of gluing orbit property and specification, and the relation of the former with the existence of periodic orbits and topological entropy. This is also accomplished by some examples. Finally, in Section 4 we recall some necessary tools that will be used in the proof of the main results, which appear in Section 5.

2. Statement of the results

This section is devoted to the statement of the main results. In order to establish a comparison between discrete-time and continuous-time dynamics, first we give a brief description on the topological notions of specification, shadowing, and mixing for discrete-time dynamical systems and then state our main results. Throughout the article, except if otherwise stated, we assume that M is a compact and connected Riemannian manifold and d is the distance on the manifold M induced by the Riemannian structure. Although a large part of the arguments hold on compact and connected metric spaces, assuming that M is a Riemmanian manifold allows some of the statements to become clearer and to deduce stronger consequences (see the discussion right after Theorem A).

2.1. **Discrete-time dynamical systems.** The classical notions of shadowing and specification are two important measurements of topological chaoticity in dynamics and its relation is rather well understood for discrete time dynamics. For instance, Bowen proved that specification implies on topological mixing and that any topologically mixing homeomorphism with the shadowing property satisfies the specification (we refer the reader e.g. to [10] for definitions and proofs of these results). Moreover, for continuous maps on compact metric spaces:

- specification implies the periodic probability measures are dense and full supported, non-atomic, zero entropy and ergodic measures forms a residual subset of the set of invariant measures (see e.g. [24])
- specification implies periodic probability measures are dense in the set of invariant measures that give full weight to the support of a hyperbolic measure (see e.g. [18, 19])

- specification implies the Birkhoff irregular points are either empty or carry full topological pressure [30]
- specification implies the space of invariant measures is the Poulsen simplex (see e.g. [12]) [1]
- shadowing implies that specification is equivalent to topologically mixing (see e.g. [16])
- weak shadowing implies on uniform positive entropy (see e.g. [16])

It is known that topologically mixing uniformly hyperbolic sets have the specification property, that C^1 open subsets of diffeomorphisms with the specification property consist of Anosov diffeomorphisms and that specification is rare among C^1 partially hyperbolic diffeomorphisms (see e.g. the introduction of [26] and references therein). For the later reason, weaker forms of specification should be considered to deal with dynamical systems that are not hyperbolic.

In [4] the first and third authors introduced a notion of gluing orbit property for homeomorphisms and flows (see Section 4 for the definition) which is weaker than specification. Afterwards, we learned that Sun and Tian [29] introduced an identical notion for C^1-diffeomorphisms which they called transitive specification property. Since the notion of specification implies topological mixing we opt to refer to these as gluing orbit properties (we refer to Section 4 for the definition).

2.2. **Statement of the main results.** A main purpose in this paper is describe the topological properties of continuous flows with the reparametrized gluing orbit property. In opposition to dynamics with specification, there are flows with the gluing orbit property that are not topologically mixing and have no positive topological entropy. Indeed, suspension flows over irrational rotations on the circle have zero entropy but satisfy the gluing orbit property (see Example 3.1). This shows that the gluing orbit property is much more embracing than previous notions of specification. Moreover, there are also flows with the shadowing property and dense set of periodic points that are not topologically mixing, hence cannot satisfy specification (e.g. every constant time suspension flow of a toral Anosov diffeomorphism), and have the reparametrized gluing orbit property (cf. [2, Theorem 1]). This illustrates that the gluing orbit properties are reasonably mild properties and that may not have such a strong connection with the shadowing property as for discrete-time dynamics. Our first result asserts that flows with the gluing orbit property still have some chaotic features (see Section 4 for definitions).

Theorem A. *Let M be a compact and connected Riemmanian manifold. Assume that $(X^t)_t$ is a continuous flow on M generated by a Lipschitz vector field $X : M \to TM$ and that M is not reduced to a periodic orbit of $(X^t)_t$. If $(X^t)_t$ satisfies the reparametrized gluing orbit property then:*

1. *$(X^t)_t$ has positive lower frequency of visits to balls;*
2. *$(X^t)_t$ has super-linear lower asymptotic mixing rates on the family of balls $\mathcal{B} = \{B(x, \varepsilon) : x \in Per((X^t)_t),\ \varepsilon > 0\}$;*

If the flow satisfies the periodic reparametrized gluing orbit property then

3. *$h_{\text{top}}((X^t)_t) \leq \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T)$ where $P((X^t)_t, T)$ denotes the number of periodic orbits of period smaller or equal to T; and*
4. *if, in addition, the flow $(X^t)_t$ is Komuro expansive then*
   (i) $h_{\text{top}}((X^t)_t) = \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T)$ and
   (ii) every point in M is an entropy point for the flow $(X^t)_t$.

We notice that if the reparametrized gluing orbit property assumption is replaced by the gluing orbit property then item (4.ii) in the theorem remains valid even if $(X^t)_t$ is not Komuro expansive. More precisely, every point is an entropy point for continuous flows with the gluing orbit property (cf. proof of item 4(ii) of the theorem and Remark 5.1). Reparametrizations are allowed in order to include larger classes of dynamical systems. Finally, the Riemannian structure and the Lipschitz regularity are only used to deduce precise estimates on some rates in items (1) and (2) above, while non-explicit bounds and items (3)-(4) can be proven for continuous flows on more general compact metric spaces.

[1] Poulsen (1961) found a simplex K whose extremal points $\mathcal{E}(K)$ are dense, a G_δ-set and arcwise connected. Lindenstrauss, Olsen, Sternberg (1987) proved that the Poulsen simplex is unique with respect to affine homeomorphisms.

Our second main result describes the space of invariant probability measures for flows with the gluing orbit property. The following result extends a classical result by Sigmund [24] from continuous maps with specification to continuous flows with the reparametrized gluing orbit property (see Section 4 for definitions). Given a continuous flow $(X^t)_t$ on M, let $\mathcal{M}_1((X^t)_t)$ stand for the space of $(X^t)_t$-invariant probability measures.

Theorem B. *Let (M, d) be a compact and connected metric space. If a continuous flow $(X^t)_t$ on M satisfies the periodic reparametrized gluing orbit property then periodic measures are dense in $\mathcal{M}_1((X^t)_t)$, and the ergodic measures form a residual subset of the set of invariant probability measures.*

This result was known to hold for for basic pieces of Axiom A flows (cf. [25]). One should mention that the statement and proof of the previous theorem extend trivially for the context of continuous maps with the gluing orbit property, which corresponds to the simpler case where the dynamics is not reparametrized. As an immediate consequence we relate the denseness of periodic points on the ambient space with the denseness of the periodic measures in the space of invariant probability measures.

Corollary 1. *Let $(X^t)_t$ be a continuous flow on a compact and connected manifold M satisfying the periodic shadowing property. If periodic orbits are dense in M then the set of periodic measures is dense in $\mathcal{M}_1((X^t)_t)$. Conversely, if periodic measures are dense in $\mathcal{M}_1((X^t)_t)$ then periodic points are dense in the union of the supports of ergodic measures.*

Finally, inspired by similar results for discrete-time dynamics (see e.g. [30]) we prove that cohomology is detected by invariant measures and periodic points.

Corollary 2. *Let $(X^t)_t$ be a continuous flow with the periodic reparametrized gluing orbit property and let $\phi : M \to \mathbb{R}$ be a continuous observable. Then, the following are equivalent:*

(1) $\inf_{\mu\in\mathcal{M}_1((X^t)_t)} \int \phi \, d\mu < \sup_{\mu\in\mathcal{M}_1((X^t)_t)} \int \phi \, d\mu$;
(2) *there exist periodic orbits $p, q \in M$ for $(X^t)_t$ so that $\int \phi \, d\mu_p < \int \phi \, d\mu_q$, where $\mu_p = \int_0^{\pi(p)} \delta_{X^s(p)} \, ds$ for every periodic orbit p for the flow with period $\pi(p) > 0$; and*
(3) *there exists $x \in M$ so that the time averages $\left(\frac{1}{t}\int_0^t \phi(X^s(x)) \, ds\right)_{t\geq 0}$ do not converge.*

Concerning the previous corollary, one should refer that the existence of periodic orbits in the gluing orbit process is needed only for item (2) above. Secondly, it is clear that the previous theorem opens the way to the study of multifractal analysis for flows with the gluing orbit property. In particular, under the previous conditions, one expects the Birkhoff irregular set to be either empty or a Baire generic subset of M.

Remark 2.1. Given an observable $\phi : M \to \mathbb{R}$ consider $\psi(x) := \int_0^1 \phi(X^s(x))ds$ and the time-1 map $f := X^1$. Note that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \phi(X^s(x)) = c \quad \text{if only if} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i) = c.$$

On the other hand, it follows from [30, Lemma 1.6] that the items of the previous corollary are equivalent to $\psi \notin \overline{Cob(M, f, c)}$ (the closure is considered in the C^0-topology) where $Cob(M, f, c)$ is the space of continuous functions $g : M \to \mathbb{R}$ so that there exists $u \in C(M, \mathbb{R})$ with $g = u - u \circ f + c$.

Remark 2.2. It is straightforward to deduce the analogous version of the main results for diffeomorphisms with the gluing orbit property. Indeed, computations are simpler and correspond in rough terms to the case where the reparametrization coincides with the identity map.

3. Examples

There are many evidences that relate specification and uniform hyperbolicity. The next example illustrates that the gluing orbit property is a much weaker condition than specification. It implies neither positive topological entropy nor the dynamics to be topologically mixing.

Example 3.1. Consider the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ let $f = f_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ denote the irrational rotation of angle α given by $f(x) = x + \alpha \pmod 1$ for every $x \in \mathbb{S}^1$. It well known that all orbits are dense in the circle and that f has zero topological entropy. We proceed to prove that f satisfies the gluing orbit property: for every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ so that for any points $x_1, \dots, x_k \in \mathbb{S}^1$ and times $n_1, \dots, n_k \geq 0$ there are $p_1, \dots, p_{k-1} \leq K(\varepsilon)$ and a point $y \in \mathbb{S}^1$ so that

$$y \in B(x_1, n_1, \varepsilon) := \{x \in \mathbb{S}^1 : d(f^j(x), f^j(x_1)) \leq \varepsilon \text{ for all } 0 \leq j \leq n_1 - 1\}$$

and $f^{\sum_{j=1}^i p_j + n_j}(y) \in B(x_{i+1}, n_{i+1}, \varepsilon)$, for all $i = 1, \dots, k-1$.

Since f is an isometry, for every $x \in \mathbb{S}^1$, $n \geq 1$ and $\varepsilon > 0$ the dynamical ball $B(x, n, \varepsilon)$ coincides with the ball $B(x, \varepsilon)$ of radius ε around x, and also $f^j(B(x, n, \varepsilon)) = B(f^j(x), \varepsilon)$ for every $j \in \mathbb{Z}$. In order to prove the gluing orbit property it is enough to show the following:

Claim: *For every $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ so that for all $z, w \in \mathbb{S}^1$ there exists $p \leq K(\varepsilon)$ such that $f^p(z) \in B(w, \varepsilon)$.*

Indeed, assuming the claim, given $x_1, \dots, x_k \in \mathbb{S}^1$, integers $n_1, \dots, n_k \geq 0$ and $\varepsilon > 0$, choose $y = x_1$. Obviously $y \in B(x_1, n_1, \varepsilon)$. Using that f is an isometry and the claim (with $z = f^{n_1}(y)$ and $w = x_2$), there exists $p_1 \leq K(\varepsilon)$ such that $f^{p_1+n_1}(y) \in B(x_2, \varepsilon) = B(x_2, n_2, \varepsilon)$. Using the claim once more with $z = f^{n_2+p_1+n_1}(y)$ and $w = x_3$, there exists $p_2 \leq K(\varepsilon)$ so that $f^{p_2+n_2+p_1+n_1}(y) \in B(x_3, n_3, \varepsilon)$. Using this argument recursively we conclude that f satisfies the gluing orbit property. Thus, we are left to prove the claim.

Proof of the Claim. Given $\varepsilon > 0$, let $n \geq 0$ be the unique integer determined by $n\varepsilon/2 \leq 1 < (n+1)\varepsilon/2$. Then the closed balls of radius $\varepsilon/2$ centered at the points $x_i := i\varepsilon/2$, for $i = 0, \dots n$, cover the circle $\mathbb{S}^1$. By transitiveness, for every $0 \leq i \leq n$ there exists $n_i \geq 0$ with $f^{n_i}(0) \in B(i\varepsilon/2, \varepsilon/2)$. Since $B(f^{n_i}(0), \varepsilon) \supset \overline{B(i\varepsilon/2, \varepsilon/2)}$ and $\mathbb{S}^1 = \bigcup_{i=0}^n \overline{B(i\varepsilon/2, \varepsilon/2)}$ it follows that $\bigcup_{j=0}^{K(\varepsilon)} B(f^j(0), \varepsilon) = \mathbb{S}^1$ for $K(\varepsilon) := \max_{0 \leq i \leq n}\{n_i\}$. Since f is an isometry and order preserving a simple argument shows that $\bigcup_{j=0}^{K(\varepsilon)} B(f^j(x), \varepsilon) = \mathbb{S}^1$, for every $x \in \mathbb{S}^1$. Then, it is clear that for any $z, w \in \mathbb{S}^1$ there exists $p \leq K(\varepsilon)$ such that $w \in B(f^p(z), \varepsilon)$, which proves the claim. □

We now prove that every irrational flow on the torus $\mathbb{T}^2$ has the gluing orbit property.

Example 3.2. Given $\alpha \notin \mathbb{Q}$ consider the irrational translation flow on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ defined by the ordinary differential equation $\frac{dx}{dt} = (1, \alpha)$. This flow admits a global cross section and is modeled as a suspension flow over the irrational rotation $x \mapsto x + \alpha$ on the circle with constant roof function (see e.g. [11]). Since the irrational rotation satisfies the gluing orbit property then so does the irrational flow (cf. [4]). However, these flows have zero topological entropy as a consequence of the variational principle and Abramov's formula.

The next example illustrates that the product of dynamics with the gluing orbit property need not have the same property. By [4], suspension flows of maps with the gluing orbit property also inherit this property. For that reason we shall focus on discrete time dynamical systems.

Example 3.3. Let f_α be an irrational rotation on the circle $\mathbb{S}^1$ and consider the diffeomorphism F on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $F(x, y) = (f_\alpha(x), f_\alpha(y))$. Since f_α is an isometry then for any $\varepsilon > 0$ the set $\Delta = \{(x, y) \in \mathbb{T}^2 : |x - y| \leq \varepsilon\}$ is F-invariant. This implies that F is not transitive, hence it does not satisfy the gluing orbit property.

In what follows we give a criterium to produce examples of dynamics with the gluing orbit property.

Example 3.4. Let (M, d_M) and (N, d_N) be compact metric spaces, and let $f : M \to M$ and $g : N \to N$ be continuous maps such that f satisfies the specification property and g has the gluing orbit property. We claim that the product map $F : M \times N \to M \times N$ given by $(x, y) \mapsto (f(x), g(y))$ has the gluing orbit property. Indeed, given $\varepsilon > 0$ let $L = L(\varepsilon) \geq 1$ be given by the specification property of f and $K = K(\varepsilon) \geq 1$ be given by the gluing orbit property for g. Set $T = T(\varepsilon) = K + L \geq 1$. Given $(x_1, y_1), \dots, (x_k, y_k) \in M \times N$ and $n_1, \dots, n_k \geq 0$, as g has gluing orbit property, there exists $y \in N$ and $p_1, \dots, p_{k-1} \leq K(\varepsilon)$ such that

$$d_N(g^n(y)), g^n(y_1)) < \varepsilon \quad \forall 0 \leq n \leq n_1 + L$$

and

$$d_N(g^{n+\sum_{j=1}^{i-1} p_j+n_j}(y), g^n(y_i)) < \varepsilon \quad \forall 0 \le n \le n_i + L$$

for every $2 \le i \le k$ (in other words, y shadows pieces of orbits of size n_i+L for the point y_i). By the specification property of f, using gap sizes $(L + p_i)_{i=1^k}$, there exists $x \in M$ such that

$$d_M(f^n(x)), f^n(x_1)) < \varepsilon \quad \forall 0 \le n \le n_1$$

and

$$d_M(f^{n+\sum_{j=1}^{i-1} L+p_j+n_j}(x), f^n(x_i)) < \varepsilon \quad \forall 0 \le n \le n_i$$

for every $2 \le i \le k$. Putting altogether, for each $2 \le i \le k$,

$$d(F^n(x, y)), F^n(x_1, y_1)) < \varepsilon \quad \text{for every } 0 \le n \le n_1$$

and

$$d(F^{n+\sum_{j=1}^{i-1}(L+p_j+n_j)}(x, y), F^n(x_i, y_i)) < \varepsilon \quad \text{for every } 0 \le n \le n_i$$

where we consider the distance $d((x_1, y_1), (x_2, y_2)) := \max\{d_M(x_1, x_2), d_N(y_1, y_2)\}$. Since each time $L + p_j$ is bounded above by $L + K(\varepsilon)$ we conclude that the product map F has the periodic gluing orbit property.

A question that arises naturally is wether dynamics with the gluing orbit property and positive topological entropy have the specification property. This is not the case as shown in the next example.

Example 3.5. Consider the torus $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear Anosov diffeomorphism and fix $\alpha \notin \mathbb{Q}$. The C^1 diffeomorphism $F : \mathbb{T}^3 \to \mathbb{T}^3$ given by $F(x, y, z) = (A(x, y), z+\alpha (\bmod 1))$ is a transitive strongly partially hyperbolic diffeomorphism (see e.g. [27] for the definition of partial hyperbolicity). Moreover, since transitive Anosov diffeomorphisms satisfy the specification property, the criterium given in Example 3.4 implies that F satisfies the gluing orbit property. Furthermore, it is clear that $h_{top}(F) = \log 2 > 0$ while there are no periodic points for F.

In what follows we provide examples of dynamics with positive entropy, dense periodic orbits and satisfying the gluing orbit property.

Example 3.6. Set $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and consider the positive entropy skew-product map $F : \mathbb{T}^2 \to \mathbb{T}^2$ given by $F(x, y) = (f(x), g(x, y)) = (2x \,(\bmod 1),\ y + x \,(\bmod 1))$ for all $(x, y) \in \mathbb{T}^2$. The periodic points for the doubling map f are dense in $\mathbb{S}^1$. If $x \in \mathbb{S}^1$ is a periodic point of period $\pi(x) \ge 1$ for the doubling map f (in particular x is rational) then $F^{\pi(x)}$ preserves the circle $\{x\} \times \mathbb{S}^1$ whose dynamics

$$F^{\pi(x)}(x, y) = \Big(x, y + \sum_{j=0}^{\pi(x)-1} 2^j x \,(\bmod 1)\Big)$$

corresponds to a rotation of rational angle in the circle. In consequence all points in the circle $\{x\} \times \mathbb{S}^1$ are periodic for F and the periodic points for F form a dense subset of $\mathbb{T}^2$. We claim that F satisfies the gluing orbit property. Using that F factors over the doubling map and the fiber dynamics is given by isometries, for any $(x, y) \in \mathbb{T}^2$ and $\varepsilon > 0$,

$$B_2((x, y), n, \varepsilon) = B_1(x, 2^{-n}\varepsilon) \times B_1(y, \varepsilon) \quad \text{and} \quad F^n(B_2((x, y), n, \varepsilon))) = \bigcup_{z \in B_1(f^n(x),\varepsilon)} B_1(z_*, \varepsilon)$$

where $z_* = F^n(\tilde{z}, y)$ and $\tilde{z} \in B_1(x, 2^{-n}\varepsilon)$ is uniquely determined by $f^n(\tilde{z}) = z$ (here B_1 denotes a ball with respect to the usual metric on $\mathbb{S}^1$). A simple modification of the argument used in the proof of the Claim in Example 3.1 also yields that for any $\varepsilon > 0$ that is rationally independent with the fiber rotations (e.g. $\varepsilon \in \sqrt{2}\mathbb{Q}^+$) there exists $N(\varepsilon) \ge 1$ such that

$$\bigcup_{j=0}^{N(\varepsilon)} B_2((x, y), n, \varepsilon))) = \mathbb{T}^2,$$

which ultimately implies that F satisfies the gluing orbit property.

If $h : \mathbb{T}^2 \to \mathbb{R}^+$ is any Hölder continuous roof function bounded away from zero so that $h(x,y) = h(x,z)$ for all $x, y, z \in \mathbb{S}^1$ then the suspension flow $(X^t)_t$ over F with roof function h satisfies the gluing orbit property (cf. [4, Theorem F]). Theorems A and B imply that

$$0 < h_{top}((X^t)_t) \le \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T),$$

and the set of all periodic probability measures is dense in $\mathcal{M}_1((X^t)_t)$.

4. Preliminaries

In the present section we introduce the necessary terminology and recall some results that will be needed. Throughout this section assume (M, d) is a compact and connected metric space.

4.1. **Invariant and empirical measures.** We write $\mathcal{M}_1(M)$ for the space of all Borel probability measures on a compact metric space (M, d). The space $\mathcal{M}_1(M)$ provided with the weak* topology is a complete metrizable topological space. We shall consider on $\mathcal{M}_1(M)$ the Prohorov metric ρ defined by:

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(A) \le \nu(A^\varepsilon) + \varepsilon, \text{ for every Borel set } A \subset M\},$$

where $A^\varepsilon = \{x \in M : d(x, A) < \varepsilon\}$ and $\mu, \nu \in \mathcal{M}_1(M)$ (see [23]). Let $(X^t)_t$ be a continuous flow on M. Let $\mathcal{M}_1((X^t)_t)$ stand for the space of $(X^t)_t$-invariant probability measures. We write $\mathcal{M}_1^e((X^t)_t)$ for the subset of all ergodic measures. As M is compact the space of invariant probability measures $\mathcal{M}_1((X^t)_t)$ is non-empty and is the closure of the convex hull of the ergodic measures $\mathcal{M}_1^e((X^t)_t)$. Thus, $\mathcal{M}_1((X^t)_t)$ is residual if, and only if, it is a dense G_δ. Invariant measures are often constructed using accumulation of empirical measures, that is, measures obtained by averaging along the orbit of a starting point. Since we deal with reparametrized flows we will consider the following two notions.

Definition 4.1. Given $x \in M$ and $T \in \mathbb{R}^+$, we shall consider the *T-empirical measure of* x, $m(x, T)$ defined, for each Borel set $A \subset M$, by

$$m(x, T)(A) = \frac{1}{T} \int_0^T \delta_{X^s(x)}(A)\, ds,$$

where δ_x stands for the Dirac measure supported on x. Given $x \in M$, $\varepsilon > 0$, $\tau \in \mathrm{Rep}(\varepsilon)$ and $T, T_0 \in \mathbb{R}^+$, we shall consider the *T-reparametrized empirical measure of* x, $m(x, T, T_0, \tau)$ defined, for each Borel set $A \subset M$, by

$$m(x, T, T_0, \tau)(A) = \frac{1}{\tau(T) - \tau(T_0)} \int_{\tau(T_0)}^{\tau(T)} \delta_{X^s(x)}(A)\, ds.$$

We say that a point $x \in M$ is *generic* for a measure $\mu \in \mathcal{M}_1((X^t)_t)$ if the empirical measures $m(x, T)$ converge to μ, in the weak* topology, as $T \to \infty$.

We say that $x \in M$ is a periodic point of the flow $(X^t)_t$ if there exists $t \in \mathbb{R}_+$ so that $X^t(x) = x$. The smallest $t_0 > 0$ satisfying the condition above is called *minimal period* of x; in this case we say that the orbit of x is a *closed orbit* of period t_0. We shall write $\mathrm{Per}((X^t)_t)$ for the set of all closed orbits. It is clear that if x is a periodic point for the flow $(X^t)_t$ with minimal period t_0, then the measure $\gamma(x) = m(x, t_0)$ is invariant for $(X^t)_t$. The set of all measures of this form will be denoted by $\mathcal{M}_1^{co}((X^t)_t)$.

4.2. **Topological entropy, entropy points and expansiveness.** In this subsection we first recall the notion of topological entropy and entropy point for flows. Let $(X^t)_t$ be a continuous flow on a compact metric space (M, d). Given $T, \varepsilon > 0$, two points x, y in M are *(T, ε)-separated* if there exists $0 \le s \le T$ so that $d(X^s(x), X^s(y)) > \varepsilon$. Given $E \subset M$, let $\mathfrak{s}(T, \varepsilon, E)$ denote the maximal cardinality of a (T, ε)-separated set in E. The limit

$$h((X^t)_t, E) = \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log \mathfrak{s}(T, \varepsilon, E) \tag{4.1}$$

is well defined by monotonicity on ε. The *topological entropy* $h_{\text{top}}((X^t)_t)$ of the flow $(X^t)_t$ is defined by (4.1) in the case that $E = M$ (see e.g. [7]). In this setting, we define by $B(x, T, \varepsilon) = \{y \in M : d(X^s(x), X^s(y)) < \varepsilon, \ \forall 0 \leq s \leq T\}$ the *dynamical ball* of length T and size ε centered at x.

We say that $x_0 \in M$ is an *entropy point* for $(X^t)_t$ if the equality $h((X^t)_t, \overline{U}) = h_{\text{top}}((X^t)_t)$ holds for any open neighbourhood U of x_0. Entropy points are those for which the complexity at every local neighborhood reflects the topological complexity of the entire dynamical system. Any finitely generated group acting by a continuous action on a compact metric space M admits an entropy point (see [3]).

Expansive homeomorphisms with specification have positive topological entropy and it coincides with the exponential growth rate of periodic points. Some subtleties arise in the definition of expansiveness for flows, mainly in the presence of singularities. We recall the notion of Komuro expansiveness.

Definition 4.2. Let (M, d) be a compact metric space, $(X^t)_t$ a continuous flow on M, and $\Lambda \subseteq M$ a compact φ-invariant set. We say that the flow $(X^t)_t$ is *Komuro-expansive in* Λ if for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $x, y \in \Lambda$ and $d(X^t(x), X^{h(t)}(y)) < \delta$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ then there is $t_0 \in \mathbb{R}$ such that $X^{h(t_0)}(y) \in X^{[t_0-\varepsilon, t_0+\varepsilon]}(x)$. Here, as usual, $X^{[t_0-\varepsilon, t_0+\varepsilon]}(x) := \{X^t(x) : t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$.

4.3. **Shadowing and gluing orbit properties.** The following definition was introduced in [29] (with the terminology of transitive specification property) and independently in [4] for homeomorphisms and flows.

Definition 4.3. Given an homeomorphism $f \in \text{Homeo}(M)$ on a compact metric space (M, d) we say that f has the *gluing orbit property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \in \mathbb{R}^+$ such that for any points $x_0, x_1, \dots, x_k \in M$ and positive integers $n_0, n_1, \dots, n_k \geq 1$ there are $p_0, p_1, \dots, p_{k-1} \leq K(\varepsilon)$ and $y \in M$ so that

$$d(f^k(y), f^k(x_0)) < \varepsilon \quad \forall 1 \leq k \leq n_0$$

and

$$d(f^{k+\sum_{j=0}^{i-1}(p_j+n_j)}(y), f^k(x_i)) < \varepsilon \quad \forall 1 \leq k \leq n_i$$

for every $1 \leq i \leq k$.

Let $(X^t)_t$ be a continuous flow on a compact metric space (M, d). Given $\delta > 0$ and $T \geq 1$, we say that a sequence $(x_i, t_i)_{i=1\dots j}$ of pairs in $M \times \mathbb{R}^+$ forms a (δ, T)-*pseudo-orbit* for $(X^t)_t$ if $1 \leq t_i \leq T$ and $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for every $i = 1, ..., j-1$. In our time continuous setting the shadowing property should reflect the speed at which different points travel in their trajectories. For that reason we need to consider orbits up to reparametrization. By Rep we denote the set of all increasing homeomorphisms $\tau : \mathbb{R} \to \mathbb{R}$, (*reparametrizations*) satisfying $\tau(0) = 0$. Fixing $\varepsilon > 0$, we define the set

$$\text{Rep}(\varepsilon) = \left\{ \tau \in \text{Rep} : \left| \frac{\tau(t) - \tau(s)}{t - s} - 1 \right| < \varepsilon, \ s, t \in \mathbb{R} \right\}.$$

In rought terms, a reparametrization $\tau \in \text{Rep}$ belongs to $\text{Rep}(\varepsilon)$ whenever it is ε-close to the identity in the sense that the slopes formed by any two points in its graph belong to the interval $(1 - \varepsilon, 1 + \varepsilon)$. This is the case e.g. if τ is C^1-smooth with derivative everywhere in the interval $(1 - \varepsilon, 1 + \varepsilon)$. Given a sequence $(t_i)_{i=0}^j$ we write $t_0 = 0$, $\sigma(n) = t_0 + t_1 + \ldots + t_{n-1}$ and $\sigma(0) = 0$. Given $t \in \mathbb{R}$, let $x_0 \star t$ denote the point

$$x_0 \star t = X^{t-\sigma(i)}(x_i) \quad \text{if} \quad \sigma(i) \leq t < \sigma(i+1).$$

Definition 4.4. We say that the continuous flow $(X^t)_t$ satisfies the *shadowing property* if, for any $\varepsilon > 0$ and $T \geq 1$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that for any (δ, T)-pseudo-orbit $(x_i, t_i)_{i=1}^j$ there exists $\tilde{x} \in M$ and a reparametrization $\tau \in \text{Rep}(\varepsilon)$ such that

$$d(X^{\tau(t)}(\tilde{x}), x_0 \star t) < \varepsilon, \text{ for every } t \in [0, \sigma(j)]. \tag{4.2}$$

For simplicity, we say that the (δ, T)-pseudo-orbit $(x_i, t_i)_{i=1}^j$ is ε-shadowed by $\tilde{x}$ if (4.2) holds. In what follows we recall the notion of reparametrized gluing orbit property, introduced in [4, 2].

Definition 4.5. Let $(X^t)_t$ be a continuous flow on a compact metric space (M,d). We say that $(X^t)_t$ has the *reparametrized gluing property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \in \mathbb{R}^+$ such that for any points $x_0, x_1, \dots, x_k \in M$ and times $t_0, t_1, \dots, t_k \geq 0$ there are $p_0, p_1, \dots, p_{k-1} \leq K(\varepsilon)$, a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ and a point $y \in M$ so that

$$d(X^{\tau(t)}(y)), X^t(x_0)) < \varepsilon \quad \forall t \in [0, t_0]$$

and

$$d(X^{\tau(t+\sum_{j=0}^{i-1} p_j + t_j)}(y), X^t(x_i)) < \varepsilon \quad \forall t \in [0, t_i]$$

for every $1 \leq i \leq k$. If, in addition, the point y can be taken periodic (i.e. $X^{\tau(\sum_{j=0}^{k} p_j + t_j)}(y) = y$ for some $p_k \leq K(\varepsilon)$) we say that $(X^t)_t$ satisfies the *periodic reparametrized gluing orbit property*.

Remark 4.6. By the choice of the class of reparametrizations $\mathrm{Rep}(\varepsilon)$ we have the following property: $\tau(t+p_1) - \tau(t) \leq (1+\varepsilon)p_1 \leq (1+\varepsilon)K(\varepsilon)$, for every $\tau \in \mathrm{Rep}(\varepsilon)$.

One should refer that, similar to what happens with the notion of shadowing and specification, if the gluing orbit property holds for a C^1-open set of diffeomorphism (resp. vector fields) then these generate topologically C^1 Anosov diffeomorphism (resp. flows) (see [29, 4]).

4.4. **Strong transitivity and asymptotic mixing rates.** We introduce some notions of transitivity and recurrence for continuous flows in order to establish a comparison with other topological notions of chaoticity.

Definition 4.7. Let $(X^t)_t$ be a continuous flow on a compact metric space M. We say that:

(1) $(X^t)_t$ is *topologically mixing*, if for any open sets U, V there exists $T > 0$ so that $X^{-t}(U) \cap V \neq \emptyset$ for every $t \geq T$;
(2) $(X^t)_t$ has *positive lower frequency of visits to balls* if for every $\varepsilon > 0$ and for any two balls B_1, B_2 of radius ε
$$\liminf_{t\to+\infty} \frac{1}{t} \mathrm{Leb}(s \in [0,t] : B_1 \cap X^{-s}(B_2) \neq \emptyset) > 0$$
(3) $(X^t)_t$ has *lower asymptotic mixing rate on the family $\mathcal{B}$ of balls* if for any $B(x_1,\varepsilon), B(x_2,\varepsilon) \in \mathcal{B}$ there exists a constant $\tau = \tau(x_2,\varepsilon) > 0$ so that
$$\liminf_{t\to+\infty} \frac{1}{t} \mathrm{Leb}(s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset) \geq \tau(x_2,\varepsilon) > 0;$$
(4) $(X^t)_t$ has *super-linear lower asymptotic mixing rate on the family $\mathcal{B}$ of balls* if for any center x_2 that is a center of a ball in $\mathcal{B}$ there exists $C(x_2) > 0$ so that $\tau(x_2,\varepsilon) \geq C(x_2)\varepsilon$.

If a continuous flow is topologically mixing then it is clear that the limit defined in the left hand side of the expression in Definition 4.7 (2) is equal to one. We recall a criterium established in [2] for proving that a continuous flow has the reparametrized gluing orbit property:

Theorem 4.8. [2, Theorem 1] *Let $(X^t)_t$ be a continuous flow on a compact and connected metric space M satisfying the (periodic) shadowing property and displaying a dense set of periodic orbits. Then $(X^t)_t$ has the (periodic) reparametrized gluing orbit property.*

5. Proofs

5.1. **Proof of Theorem A.** Let $(X^t)_t$ be a continuous flow on the compact and connected Riemannian manifold M with the periodic reparametrized gluing orbit property. For the purpose of analyzing local trajectories of the flow we may assume without loss of generality that the flow evolves on $\mathbb{R}^n$ (this is always the case up to consider some local chart). Since $(X^t)_t$ is the flow obtained by the solutions of the ordinary differential equation $u' = X(u)$ then

$$(X^t)'(x) = X(X^t(x)) \quad \text{and} \quad X^0(x) = x.$$

or, equivalently, $X^t(x) = x + \int_0^t X(X^s(x))\, ds$. Let $L > 0$ be a Lipschitz constant for X.

Proof of (1): Consider arbitrary $\varepsilon > 0$ and $0 < \varepsilon' < \varepsilon$, and let $s(\varepsilon') = (1+\varepsilon')K(\varepsilon')$ be given by the reparametrized gluing orbit property for ε'. Consider the balls $B_i = B(x_i, \varepsilon')$ around $x_i \in M$, for $i = 1, 2$. If there exists $T > 0$ so that $X^t(B_1) \cap B_2 \neq \emptyset$ for every $t \geq T$ then it is clear that

$$\liminf_{t\to+\infty} \frac{1}{t}\mathrm{Leb}\Big(s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\Big) = 1$$

which proves (1) in this case. Otherwise, there exist sequences $(t_i)_i$ and $(e_i)_i$ converging to infinite so that $t_i < e_i < t_{i+1}$, that $X^{t_i}(B_1) \cap B_2 \neq \emptyset$ and $X^{e_i}(B_1) \cap B_2 = \emptyset$ for every $i \geq 1$. Indeed, by the reparametrized gluing orbit property one can choose the sequence $(t_i)_{i\geq 1}$ so that $|t_{i+1} - t_i| < s(\varepsilon')$ for every $i \geq 1$. Moreover, if $y \in B(x, \varepsilon')$ then one can use that

$$\|X^s(y) - x\| \leq \|x - y\| + \|X\|_\infty |s| \tag{5.1}$$

to deduce that $X^s(y) \in B(x, \varepsilon)$ for all $|s| \leq \frac{\varepsilon-\varepsilon'}{\|X\|_\infty}$. Given $t > s(\varepsilon')$ let $n(t,\varepsilon') \in \mathbb{N}$ be uniquely defined by $s(\varepsilon') \cdot n(t,\varepsilon') < t \leq s(\varepsilon') \cdot [n(t,\varepsilon') + 1]$. We conclude that the set

$$\{s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\}$$

contains at least $1 \leq n(t,\varepsilon') \leq \left[\frac{t}{s(\varepsilon')}\right]$ intervals I_i, so that $t_i \in I_i$ and $\mathrm{Leb}(I_i) \geq \frac{2(\varepsilon-\varepsilon')}{\|X\|_\infty}$. This proves that $\mathrm{Leb}\Big(s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\Big) \geq \frac{2(\varepsilon-\varepsilon')}{\|X\|_\infty} n(t,\varepsilon')$ and, consequently,

$$\begin{aligned}
&\frac{1}{t}\mathrm{Leb}\Big(s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\Big) \\
&\quad\geq \frac{1}{[n(t,\varepsilon')+1]\cdot s(\varepsilon')}\mathrm{Leb}\Big(s \in [0\ ,\ n(t,\varepsilon')s(\varepsilon')] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\Big) \\
&\quad\geq \frac{(\varepsilon-\varepsilon')}{s(\varepsilon')\|X\|_\infty}
\end{aligned}$$

for every $t > s(\varepsilon')$. This completes the proof of (1).

Proof of (2): We now prove that the flow has super-linear lower asymptotic mixing rates on the family $\mathcal{B}$ of balls centered at points with closed orbits. Consider arbitrary $\varepsilon > 0$ and points $x_1, x_2 \in M$ with periodic orbits. Assume that x_2 is not a singularity (the case that x_2 is a singularity is simpler). Let $\pi(x_2) > 0$ denote the prime period of x_2 and, for $0 < \varepsilon' < \varepsilon$, let $s(\varepsilon') := (1+\varepsilon')K(\varepsilon')$ be given by the reparametrized gluing orbit property for ε'. Then, for any $t > 0$ there exists $x_{1,t} \in B(x_1, \varepsilon')$ and $0 \leq s_1 = s_1(t) \leq s(\varepsilon')$ so that $X^{s+s_1}(x_{1,t}) \in B(X^s(x_2), \varepsilon')$ for every $s \in [0,t]$. Thus,

$$X^{-s_1-k\,\pi(x_2)}(B_2) \cap B_1 \neq \emptyset \quad \text{for every } 0 \leq k \leq \Big[\frac{t}{\pi(x_2)}\Big].$$

For every $\gamma > 0$, there exists $t_\gamma \gg 1$ so that $\frac{t}{\pi(x_2)} - \left[\frac{t}{\pi(x_2)}\right] \leq 1 \leq \gamma t$ for every $t \geq t_\gamma$ (just take $t_\gamma = [\frac{1}{\gamma}] + 1$). Then, a similar argument to the one used in the proof of (1) implies that (taking $\varepsilon' = \frac{\varepsilon}{2}$)

$$\begin{aligned}
\mathrm{Leb}\Big(s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\Big) &\geq \frac{2(\varepsilon-\varepsilon')}{\|X\|_\infty}\Big[\frac{t}{\pi(x_2)}\Big] \\
&\geq \frac{\varepsilon}{\|X\|_\infty}\Big(\frac{1}{\pi(x_2)} - \gamma\Big)t.
\end{aligned}$$

Since γ was taken arbitrary this proves that

$$\liminf_{t\to\infty} \frac{1}{t}\mathrm{Leb}\Big(\{s \in [0,t] : B(x_1,\varepsilon) \cap X^{-s}(B(x_2,\varepsilon)) \neq \emptyset\}\Big) \geq \frac{\varepsilon}{\pi(x_2)\|X\|_\infty}.$$

Since the right hand side term is linear in ε, this completes the proof of (2).

Proof of (3): Consider arbitrary $\varepsilon > 0$, and let $K(\frac{\varepsilon}{2})$ be given by the periodic reparametrized gluing orbit property for $\frac{\varepsilon}{2}$. For each $T \geq 1$, let $E \subset M$ be a (T,ε)−maximal separated subset. For each $x \in E$ there exists

$y(x) \in B(x, T, \frac{\varepsilon}{2})$, $p(x) \le K(\frac{\varepsilon}{2})$ and $\tau \in \mathrm{Rep}(\frac{\varepsilon}{2})$ such that $X^{\tau(T+p(x))}(y(x)) = y(x)$. Since E is a maximal separated set then $E \ni x \mapsto y(x)$ is injective. Thus,

$$\#P\Big((X^t)_t, (1+\frac{\varepsilon}{2})(T + K(\frac{\varepsilon}{2}))\Big) \ge \#E,$$

and, consequently,

$$\begin{aligned}\limsup_{T\to\infty} &\frac{1}{(1+\frac{\varepsilon}{2})(T+K(\frac{\varepsilon}{2}))} \log \#P((X^t)_t, (1+\frac{\varepsilon}{2})(T+K(\frac{\varepsilon}{2}))) \\ &\ge \frac{1}{1+\frac{\varepsilon}{2}} \limsup_{T\to\infty} \frac{1}{T} \log \#E = \frac{1}{1+\frac{\varepsilon}{2}} \limsup_{T\to\infty} \frac{1}{T} \log \mathfrak{s}(T, \varepsilon, M).\end{aligned}$$

This implies that

$$\limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T) \ge \lim_{\varepsilon\to 0} \limsup_{T} \frac{1}{T} \log \mathfrak{s}(T, \varepsilon, M) = h_{\mathrm{top}}((X^t)_t)$$

as claimed.

Proof of (4) (i): Let $O(p)$ and $O(q)$ be distinct periodic orbits of period smaller or equal to T. Fix $\xi > 0$ and let $\delta_0 = \delta_0(\xi) > 0$ be given by Komuro-expansiveness property. We claim that there exists $t \in \mathbb{R}$ so that $d(X^t(p), X^t(q)) \ge \delta_0$. Otherwise $d(X^t(p), X^t(q)) < \delta_0$ for all $t \in \mathbb{R}$ and, by Komuro-expansiveness (taking the reparametrization $\tau(t) = t$), $p \in X^{t_0}(q)$ for some $t_0 \in \mathbb{R}$ which is a contradiction. This proves that the set of periodic orbits $P((X^t)_t, T)$ is a (T, ε)-separated set for every $0 < \varepsilon < \delta_0$. In particular, periodic orbits of period smaller or equal to T are isolated (hence finite). Thus, $h_{\mathrm{top}}((X^t)_t) \ge \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T)$. This, together with item (3), implies that

$$h_{\mathrm{top}}((X^t)_t) = \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T).$$

Proof of (4) (ii): We claim that every point of M is an entropy point for the flow $(X^t)_t$. Since this is immediate whenever $h_{top}((X^t)_t) = 0$, assume that

$$h := h_{top}((X^t)_t) = \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log \mathfrak{s}(T, \varepsilon, M) > 0.$$

So, for any small $\gamma > 0$ there exists $\varepsilon_\gamma > 0$ so that for every $0 < \varepsilon < \varepsilon_\gamma$.

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathfrak{s}(T, \varepsilon, M) \ge h - \gamma$$

and, consequently, there exists a subsequence of real numbers $(T_k)_{k\ge 1}$ tending to infinity (depending on both γ and ε) in such a way that

$$\mathfrak{s}(T_k, \varepsilon, M) \ge e^{(h-2\gamma)T_k} \tag{5.2}$$

Now, fix $x \in M$ arbitrary and let $U \subset M$ be any open neighborhood of x. Clearly, $h_{top}((X^t)_t, \overline{U}) \le h$. Thus, in order to prove that x in an entropy point we are left to prove the other inequality. Fix $0 < \varepsilon < \varepsilon_\gamma$ so that $B(x, \varepsilon) \subset U$ and let $(T_k)_{k\ge 1}$ be as above. Let $s(\varepsilon) = (1+\varepsilon)K(\varepsilon) > 0$ be given by the reparametrized gluing orbit property.

The naive strategy to create separated orbits in U is to use the gluing orbit property to build finite orbits that depart from U and that shadow long pieces of separated orbits and to obtain essentially $\mathfrak{s}(T_k, \varepsilon, M)$ of such distinct orbits of a fixed size. This idea can be pushed forward in the case of flows with specification because the time needed for a shadowing process to start can be taken constant and independent of the point. In our setting we face two difficulties. The first is that the gluing time is bounded above but usually depends on the orbits involved in the shadowing, and the second one is that since the time of shadowing is not simultaneous one needs to assume some reasonable large set of shadowing orbits to be separated.

Suppose that $(X^t)_t$ is Komuro-expansive and has the periodic reparametrized gluing orbit property. By item 4(i), entropy can be computed using periodic orbits: $h = \lim_{k\to\infty} \frac{1}{T_k} \log \#P((X^t)_t, T_k)$, for some sequence of positive real numbers $(T_k)_k$ tending to infinite. We may assume without loss of generality that $T_{k+1} > 2T_k$ for every $k \geq 1$. Let $t(\varepsilon) > 0$ be given by uniform continuity in such a way that

$$\max_{s\in[0,t(\varepsilon)]} d_{C^0}(X^s, Id) < \frac{\varepsilon}{4} \tag{5.3}$$

and, for every $k \geq 1$, consider the decomposition (depending on k)

$$[0, T_k] = \bigcup_{j=0}^{N(\varepsilon,k)-1} I_j \bigcup I_{N(\varepsilon,k)} \tag{5.4}$$

where $N(\varepsilon, k) = [\frac{T_k}{t(\varepsilon)}]$ denotes the integer part of $\frac{T_k}{t(\varepsilon)}$, $I_j = [jt(\varepsilon), (j+1)t(\varepsilon)[$ for $0 \leq j \leq N(\varepsilon, k) - 1$ and the last interval $I_{N(\varepsilon,k)} = [T_k - N(\varepsilon, k)t(\varepsilon), T_k]$ may be eventually reduced to the empty set. By construction, for every γ we have $\#P((X^t)_t, T_k) \geq e^{(h-2\gamma)T_k}$ for every $k \geq 1$ large.

Fix $0 < \varepsilon \ll \delta_0$ small and $k_0 \geq 1$ large (depending on ε) so that $(1 + \varepsilon)(T_k + 2K(\frac{\varepsilon}{4})) < T_{k+1}$ for every $k \geq k_0$.

The periodic reparametrized gluing orbit property assures that for every periodic point p of period $\pi(p) \leq T_k$ (in other words $O(p) \in P((X^t)_t, T_k)$) there exist $z = z_p \in B(x, \frac{\varepsilon}{4})$, a reparametrization $\tau_p \in \text{Rep}(\varepsilon/4)$ and gluing times $0 < s_p, s'_p \leq K(\frac{\varepsilon}{4})$ so that

$$d(X^{\tau_p(t+s_p)}(z_p), X^t(p)) < \frac{\varepsilon}{4} \text{ for every } t \in [0, \pi(p)] \quad \text{and} \quad X^{\tau_p(\pi(p)+s_p+s'_p)}(z_p) = z_p. \tag{5.5}$$

Such periodic point z_p has period $\pi(z_p) = \tau_p(\pi(p) + s_p + s'_p) \leq (1 + \varepsilon)(T_k + 2K(\frac{\varepsilon}{4})) < T_{k+1}$. Moreover, the Komuro-expansiveness property assures that if p and q are periodic points then there exists $t \in \mathbb{R}$ so that $d(X^t(p), X^t(q)) \geq \delta_0$ (recall item (4)(i)). Recalling the finite decomposition (5.4), the pigeonhole principle guarantees that there exists $0 \leq j \leq N(\varepsilon, k + 1)$ so that

$$\#\Big\{z_p \in B(x, \frac{\varepsilon}{4}) \colon \text{ period of } p \text{ belongs to } I_j\Big\} \geq \frac{1}{N(\varepsilon, k + 1) + 1} e^{(h-2\gamma)T_k}.$$

Claim: *The set $\Big\{z_p \in B(x, \frac{\varepsilon}{4}) \colon$ period of p belongs to $I_j\Big\}$ is $((1 + \varepsilon)(T_k + 2K(\frac{\varepsilon}{4})), \delta_0 - \varepsilon)$-separated in U.*

Proof of the claim. Let $p, q \in P((X^t)_t, T)$ be periodic points and $z_p, z_q \in B(x, \frac{\varepsilon}{4}) \subset U$ be constructed as above (associated to p and q respectively). Suppose that $d(X^t(z_p), X^t(z_q)) < \delta_0 - \varepsilon$ for all $t \in [0, T + s(\frac{\varepsilon}{4})]$. By construction, $d(X^t(z_p), X^t(p)) < \frac{\varepsilon}{4}$ and $d(X^t(z_q), X^t(q)) < \frac{\varepsilon}{4}$ for all $t \in [s(\frac{\varepsilon}{4}), T + s(\frac{\varepsilon}{4})]$. By triangle inequality, $d(X^t(X^{s(\frac{\varepsilon}{4})}p), X^t(X^{s(\frac{\varepsilon}{4})}q)) < \delta_0 - \frac{\varepsilon}{2}$ for all $t \in [0, T]$. As the periods of z_p and p (respectively z_q and q) differ in the maximum $t(\varepsilon)$, it follows from the choice of $t(\varepsilon)$ in (5.3) that we have $d(X^t(p), X^t(q)) < \delta_0 - \varepsilon$ for all $t \in \mathbb{R}$. It contradicts the Komuro-expansiveness property and proves the claim. □

Thus, $h_{\text{top}}(X^t, \overline{U}) \geq \limsup_{T\to\infty} \frac{1}{T} \log \#P((X^t)_t, T) = h_{\text{top}}((X^t)_t)$. This completes the proof of item(4.ii) and finishes the proof of the theorem. □

Remark 5.1. We observe that the conclusion of item 4(ii) holds whenever the periodic reparametrized gluing orbit property assumption is replaced by gluing orbit property. Given $k \geq 1$ large, let $E_k = \{x_i\}_{i=1...r}$ be a (T_k, ε)-maximal separated set with cardinality greater or equal to $e^{(h-2\gamma)T_k}$ (cf. equation (5.2)). The gluing orbit property assures that for every $i = 1, \dots, r$ there exists $z_i \in M$ and a gluing time $0 < s_i \leq K(\frac{\varepsilon}{4})$ so that $d(z_i, x) < \frac{\varepsilon}{4}$ and $d(X^{t+s_i}(z_i), X^t(x_i)) < \frac{\varepsilon}{4}$ for every $t \in [0, T_k]$. As in the previous proof, given the decomposition (5.4), the pigeonhole principle guarantees there exists $0 \leq j \leq N(\varepsilon)$ so that $\#\Big\{1 \leq i \leq r \colon s_i \in I_j\Big\} \geq \frac{1}{N(\varepsilon)+1} e^{(h-2\gamma)T_k}$. It is not hard to check that the set of points $(z_i)_{i\in J}$ is $(T_k + (j + 1)t(\varepsilon), \frac{\varepsilon}{4})$-separated in U, for $J = \{1 \leq i \leq r \colon s_i \in I_j\}$. Thus

$$\mathfrak{s}(T_k, \frac{\varepsilon}{2}, \overline{U}) \geq \frac{1}{N(\varepsilon) + 1} \, e^{(h-2\gamma)T_k} \tag{5.6}$$

and, consequently, $h_{top}((X^t)_t, \overline{U}) \geq h - 2\gamma$. Since $\gamma > 0$ is arbitrary, this argument is enough to guarantee that x is an entropy point in this context.

Remark 5.2. There are several ways to estimate the largest lower bound of the largest escaping time of a point from its ball of a definite radius which may be of independent interest. For instance, in the case of Lipschitz vector fields we can provide estimates where the Lipschitz constant appear naturally. Indeed, instead of (5.1), one could use

$$\|X^t(y) - x\| \le \|X^t(x) - x\| + \|X^t(y) - X^t(x)\|.$$

In the right hand side, the first term $\|X^t(x) - x\|$ is then bounded above by $\|X\|_\infty |t|$, while the second term is bounded by $e^{L|t|}\|x - y\|$ as a consequence of Grownall's inequality. Thus, estimating each of these terms directly and considering the escape time

$$\tau(\varepsilon) := \sup_{x\in M} \inf\{t > 0 : \|X^t(x) - x\| > \varepsilon\} \tag{5.7}$$

one can easily check that if $\|x - y\| \le \varepsilon'$ then $\|X^t(y) - x\| \le \varepsilon$ for every

$$|t| \le \max\Big\{ \sup_{\varepsilon'<\tilde\varepsilon<\varepsilon} \{\frac{\tilde\varepsilon}{\|X\|_\infty} + \frac{1}{L}\log\frac{\varepsilon-\tilde\varepsilon}{\varepsilon'}\},\ \sup_{\varepsilon'<\tilde\varepsilon<\varepsilon} \{\tau(\tilde\varepsilon) + \frac{1}{L}\log\frac{\varepsilon-\tilde\varepsilon}{\varepsilon'}\}\Big\}$$

where $L > 0$ is a Lipschitz constant for the vector field X that generates $(X^t)_t$.

6. The space of invariant measures for flows

The strategy for the proof of Theorem B relies on the extension of the concepts of closeability and linkability introduced in [12] for continuous flows. It is not hard to check that flows with the gluing orbit property are linkable and closable with respect to all periodic orbits. However, an extension of these results for flows where the shadowing is given in terms of reparametrizations is of wider applications and demands a careful analysis.

We need some preliminary definitions. Let $(X^t)_t$ be a continuous flow on the compact and connected metric space (M, d). Given $x \in M$, $\varepsilon > 0$, a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ and an initial time $T_0 \in \mathbb{R}^+$, we define the *reparametrized dynamical ball* of lengh $T - T_0$, with initial time T_0 and size ε centered at x, by

$$B(x, T, T_0, \varepsilon, \tau) = \{y \in M\colon d(X^{\tau(s)}(y), X^s(x)) < \varepsilon,\ \forall\, T_0 \le s \le T\}.$$

The next lemma, which resembles [28, Lemma 3], guarantees that the empirical measures associated to points that remain in reparametrized dynamical balls remain close. The difficulties here are that the two pieces of orbits are close but each of these in a suitable velocity. Since the information is required to hold at small scales we assume $\varepsilon_0 = \frac{1}{2}$ and $0 < \varepsilon < \varepsilon_0$.

Lemma 6.1. *Let $(X^t)_t$ be a continuous flow on M, $x \in M$, $0 < \varepsilon \le \varepsilon_0$, $\tau \in Rep(\varepsilon)$ be a reparametrization and let $p, q \in \mathbb{R}^+$ be such that $\tau(p) \le \tau(q) \le (1 + \varepsilon)\tau(p)$. Then, for every $y \in B(x, p, 0, \varepsilon, \tau)$ we have that $\rho(m(y, q, 0, \tau), m(x, p)) \le 6\varepsilon$.*

Proof. Take any Borel set $A \subset M$. For every $y \in B(x, p, 0, \varepsilon, \tau)$ we have

$$\begin{aligned} m(y, q, 0, \tau)(A) &= \frac{1}{\tau(q)} \int_0^{\tau(q)} \delta_{X^s(y)}(A)\, ds \\ &\le \frac{1}{\tau(p)} \int_0^{\tau(p)} \delta_{X^s(y)}(A)\, ds + \frac{1}{\tau(p)} \int_{\tau(p)}^{\tau(q)} \delta_{X^s(y)}(A)\, ds \\ &\le \frac{1}{p} \int_0^{p} \delta_{X^s(x)}(A^\varepsilon)\, ds + 5\varepsilon + \frac{1}{\tau(p)} \int_{\tau(p)}^{\tau(q)} \delta_{X^s(y)}(A)\, ds. \end{aligned} \tag{6.1}$$

Here we used that $X^{\tau(s)}(y) \in A$ implies $X^s(x) \in A^\varepsilon$, for all $0 \le s \le p$, and that the reparametrization τ is increasing, hence Lebesgue almost everywhere differentiable. Indeed, since $\tau \in \mathrm{Rep}(\varepsilon)$, Rademacher's theorem guarantees that the derivative τ' to lie in the interval $[1 - \varepsilon, 1 + \varepsilon]$. Therefore

$$\frac{1}{\tau(p)} \int_0^{\tau(p)} \delta_{X^s(y)}(A)\, ds = \frac{1}{\tau(p)} \int_0^{p} \tau'(s)\, \delta_{X^{\tau(s)}(y)}(A)\, ds.$$

Then, this is enough to deduce the estimate

$$\begin{aligned}
&\Big|\frac{1}{\tau(p)}\int_0^p \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds - \frac{1}{p}\int_0^p \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| \\
&\quad\le \Big|\frac{1}{\tau(p)}-\frac{1}{p}\Big|\,\Big|\int_0^p \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds\Big| \\
&\quad+ \Big|\frac{1}{p}\int_0^p \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds - \frac{1}{p}\int_0^p \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| \\
&\quad\le (1+\varepsilon)\,\frac{|\tau(p)-p|}{\tau(p)} + 2\varepsilon \\
&\quad\le \Big(\frac{1+\varepsilon}{1-\varepsilon}+2\Big)\varepsilon \le 5\varepsilon.
\end{aligned}$$

In order to prove the lemma, now observe that the third term in the right-hand side of (6.1) can be bounded above as follows

$$\frac{1}{\tau(p)}\int_{\tau(p)}^{\tau(q)} \delta_{X^s(y)}(A)\,ds \le \frac{\tau(q)-\tau(p)}{\tau(p)} \le \varepsilon.$$

Altogether, we conclude that $m(y,q,0,\tau)(A) \le m(x,p)(A^\varepsilon) + 6\varepsilon$. Hence, $\rho(m(y,q,0,\tau), m(x,p)) \le 6\varepsilon$, which proves the lemma. □

The following notions are adapted from similar concepts in [12].

Definition 6.1. A point $x \in M$ is *closeable* if for every $\varepsilon > 0$ and $T > 0$ there exist positive real numbers $p = p(x,\varepsilon,T)$ and $q = q(x,\varepsilon,T)$ and a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ such that there is a point $y \in B(x,p,0,\varepsilon,\tau) \cap \mathrm{Per}((X^t)_t)$ satisfying $X^{\tau(q)}(y) = y$ and $T \le \tau(p) \le \tau(q) \le (1+\varepsilon)\tau(p)$.

Definition 6.2. The set $\mathrm{Per}((X^t)_t)$ is *linkable* if for every $x_0, x_1 \in \mathrm{Per}((X^t)_t)$, $\varepsilon > 0$ and $\lambda \in [0,1]$ there exist a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$, times $t_0, t_1, q_0, q_1 \in \mathbb{R}^+$ and $y \in \mathrm{Per}((X^t)_t)$ satisfying the following conditions:

(1) $X^{\tau(q_1)}(y) = y$ and $y \in B(x_0,t_0,0,\varepsilon,\tau) \cap B(x_1, q_0+t_1, q_0, \varepsilon, \tau)$;
(2) $\tau(t_0) \le \tau(q_0) \le (1+\varepsilon)\tau(t_0)$;
(3) $\tau(q_0+t_1) - \tau(q_0) \le \tau(q_1) - \tau(q_0) \le (1+\varepsilon)[\tau(q_0+t_1) - \tau(q_0)]$;
(4) $\lambda - \varepsilon \le \frac{\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)} \le \lambda + \varepsilon$.

The previous notion means, roughly, that any convex combination of Dirac masses on closed orbits can be approximated by an empirical measure over a suitable closed orbit. The following stronger notion requires such convex combinations to be given after a fixed amount of time.

Definition 6.3. The set $\mathrm{Per}((X^t)_t)$ is *strongly linkable* provided that for every $x_0, x_1 \in \mathrm{Per}((X^t)_t)$ and every $\varepsilon > 0$ there exists $T = T(x_0,x_1,\varepsilon)$ such that if $t_0, t_1 > 0$ satisfy $X^{t_i}(x_i) = x_i$ for $i = 0,1$, $\tau(t_0) \ge T$ and $t_1 \ge \frac{T}{1-\varepsilon}$, then there are a periodic point $y \in \mathrm{Per}((X^t)_t)$, a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ and real numbers $0 < q_0 \le q_1$ such that $X^{\tau(q_1)}(y) = y$ and

(1) $\tau(t_0) \le \tau(q_0) \le (1+\varepsilon)\tau(t_0)$ and $y \in B(x_0,t_0,0,\varepsilon,\tau)$
(2) $\tau(q_0+t_1) - \tau(q_0) \le \tau(q_1) - \tau(q_0) \le (1+\varepsilon)[\tau(q_0+t_1) - \tau(q_0)]$ and $y \in B(x_1, q_0+t_1, q_0, \varepsilon, \tau)$.

Remark 6.4. The later condition is stronger than the one of Definition 6.2. Indeed, for any $\delta > 0$ and $\tilde\tau \in \mathrm{Rep}(\delta)$

$$\frac{1-\delta}{1+\delta}\,\frac{t_0}{t_0+t_1} \le \frac{\tilde\tau(t_0)}{\tilde\tau(t_0)+\tilde\tau(q_0+t_1)-\tilde\tau(q_0)} \le \frac{1+\delta}{1-\delta}\,\frac{t_0}{t_0+t_1}.$$

Given $\varepsilon > 0$ and $\lambda \in (0,1)$, if $\delta > 0$ satisfies $\frac{1+\delta}{1-\delta} < 2$ then

$$\lambda - \frac{\varepsilon}{2} \le \frac{1-\delta}{1+\delta}\,\lambda \le \lambda \le \frac{1+\delta}{1-\delta}\,\lambda \le \lambda + \frac{\varepsilon}{2}.$$

Thus, in order to prove that an estimate as in Definition 6.2(4) it is enough to prove that one can choose t_0, t_1 large so that $X^{t_i}(x_i) = x_i$ for $i = 0, 1$ and

$$\lambda - \frac{\varepsilon}{4} \le \frac{t_0}{t_0 + t_1} \le \lambda + \frac{\varepsilon}{4},$$

which is simple to check.

As an immediate consequence of the previous lemma we deduce that empirical measures of closeable points are approximated by invariant measures over the approximating closed orbits.

Corollary 6.1. *Let $x \in M$ be closeable. Given $T \in \mathbb{R}^+$ and $\varepsilon > 0$ there exist $y \in M$, $\tau \in Rep(\varepsilon)$ and $p, q \in \mathbb{R}^+$ so that $X^{\tau(q)}(y) = y$, that $T \le \tau(p) \le \tau(q) \le (1+\varepsilon)\tau(p)$ and $\rho(\gamma(y)), m(x,p)) < 6\varepsilon$, where $\gamma(y) = m(y, q, 0, \tau)$.*

In what follows we shall prove that the reparametrized gluing orbit property implies both the closeability and linkability properties.

Proposition 6.1. *If a continuous flow $(X^t)_t$ on M satisfies the periodic reparametrized gluing orbit property then:*

(1) *every point is closeable;*
(2) *$Per((X^t)_t)$ is strongly linkable.*

Proof. Fix $x \in M$ arbitrary. Take any $T \in \mathbb{R}^+$ and $\varepsilon > 0$, and let $K(\varepsilon) > 0$ be provided by the reparametrized gluing orbit property. Let $t_0 \in \mathbb{R}^+$ (depending on ε and T) be such that, for all $\tilde\tau \in \mathrm{Rep}(\varepsilon)$,

(i) $\tilde\tau(t_0) \ge T$,
(ii) $(1+\varepsilon)\,K(\varepsilon) \le \varepsilon\,\tilde\tau(t_0)$.

Given $x_0 = x$ and t_0 as above, by the reparametrized gluing orbit property there exists a time $q := t_0 + p_0$, where $p_0 \le K(\varepsilon)$, a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ and a periodic point $y \in M$ such that $y \in B(x, t_0, 0, \varepsilon, \tau)$ and $X^{\tau(q)}(y) = y$. In order to prove that x is closeable it is enough to estimate the proportion of time spent during the shadowing as follows

$$T \le \tau(t_0) < \tau(t_0 + p_0) \le \tau(t_0) + (1+\varepsilon)p_0 \le \tau(t_0) + (1+\varepsilon)K(\varepsilon) \le (1+\varepsilon)\tau(t_0),$$

thus completing the proof of item (1) in the proposition.

It remains to prove that the set $\mathrm{Per}((X^t)_t)$ is strongly linkable. Let $x_0, x_1 \in \mathrm{Per}((X^t)_t)$. Given $\varepsilon > 0$ let $K = K(\varepsilon)$ be given by the reparametrized gluing orbit property and let $T = T(\varepsilon) \in \mathbb{R}^+$ be defined by $T(\varepsilon) := \frac{(1+\varepsilon)K(\varepsilon)}{\varepsilon}$. Take any $t_0, t_1 \in \mathbb{R}^+$ periods of x_0 and x_1, respectively, in such a way that

(1) $\tilde\tau(t_0) \ge T(\varepsilon)$;
(2) $t_1 \ge \frac{T(\varepsilon)}{1-\varepsilon} = \frac{(1+\varepsilon)K(\varepsilon)}{\varepsilon(1-\varepsilon)}$;
(3) $X^{t_j}(x_j) = x_j$ for $j = 0, 1$,

for all $\tilde\tau \in \mathrm{Rep}(\varepsilon)$. By the reparametrized gluing orbit property, there exists $\tau \in \mathrm{Rep}(\varepsilon)$, $p_0, p_1 \le K(\varepsilon)$ and a periodic point $y \in M$ such that $y \in B(x_0, t_0, 0, \varepsilon, \tau) \cap B(x_1, q_0 + t_1, q_0, \varepsilon, \tau)$ and $X^{\tau(q_1)}(y) = y$, where $q_0 := t_0 + p_0$ and $q_1 := t_0 + p_0 + t_1 + p_1$. Moreover, by the choice of $t_0, t_1 > 0$, we conclude

$$\begin{aligned} T \le \tau(t_0) < \tau(t_0 + p_0) &\le \tau(t_0) + (1+\varepsilon)p_0 \le \tau(t_0) + (1+\varepsilon)K(\varepsilon) \\ &\le \tau(t_0) + \varepsilon\, T(\varepsilon) \le \tau(t_0) + \varepsilon\,\tau(t_0) = (1+\varepsilon)\tau(t_0) \end{aligned}$$

and, similarly,

$$\begin{aligned} \tau(t_0 + p_0 + t_1) - \tau(t_0 + p_0) &\le \tau(t_0 + p_0 + t_1 + p_1) - \tau(t_0 + p_0) \\ &= \tau(t_0 + p_0 + t_1 + p_1) - \tau(t_0 + p_0 + t_1) + \tau(t_0 + p_0 + t_1) - \tau(t_0 + p_0) \\ &\le (1+\varepsilon)K(\varepsilon) + \tau(t_0 + p_0 + t_1) - \tau(t_0 + p_0) \\ &\le \varepsilon(1-\varepsilon)t_1 + \tau(t_0 + p_0 + t_1) - \tau(t_0 + p_0) \\ &\le (1+\varepsilon)[\tau(p_0 + t_0 + t_1) - \tau(t_0 + p_0)]. \end{aligned}$$

This proves (2) and completes the proof of the proposition. □

In what follows, given a locally convex space X and $A \subset X$, define

$$\text{conv}(A) := \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \mu_j : \lambda_j \in [0,1], \sum_{j=1}^{n} \lambda_j = 1, \mu_j \in A \right\}$$

(resp. $\overline{conv}(A)$) the smallest convex set (resp. the smallest closed convex set) containing A and let $\overline{A}$ denote the closure of A.

Theorem 6.5. *Let $(X^t)_t$ be a continuous flow on M. If every point is closeable then the set $\mathcal{M}_1^{co}((X^t)_t)$ is dense in $\mathcal{M}_1^e((X^t)_t)$. Moreover, if Per$((X^t)_t)$ is linkable then $\overline{\mathcal{M}_1^{co}((X^t)_t)} = \overline{conv}(\mathcal{M}_1^{co}((X^t)_t))$.*

Proof. Let $(X^t)_t$ be a continuous flow on a compact and connected metric space M. Assume first that every point of M is closeable and take $\mu \in \mathcal{M}_1^e((X^t)_t)$ and $\varepsilon > 0$ arbitrary small. Let $x \in M$ be a generic point for μ and let $T > 0$ be such that $\rho(\mu, m(x,t)) < \varepsilon/7$ for every $t \geq T$. Since every point is closeable, taking $p = T$ and $\varepsilon/7$, there exist $\tau \in \text{Rep}(\varepsilon/7)$, $p, q \in \mathbb{R}^+$ with $T \leq \tau(p) \leq \tau(q) \leq (1+\varepsilon/7)\tau(p)$ and a periodic point $y \in B(x,p,0,\varepsilon/7,\tau)$ such that $X^{\tau(q)}(y) = y$. Corollary 6.1 implies that $\rho(m(y,q,0,\tau), m(x,p)) \leq 6\varepsilon/7$ and, consequently,

$$\rho(\gamma(y),\mu) = \rho(m(y,q,0,\tau),\mu) \leq \rho(m(y,q,0,\tau), m(x,p)) + \rho(m(x,p),\mu) \leq \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrary this proves that μ is accumulated by invariant probability measures on closed orbits. This proves that $\mathcal{M}_1^{co}((X^t)_t)$ is dense in $\mathcal{M}_1^e((X^t)_t)$ and finishes the proof of the first part of the theorem.

Now, assume that Per$((X^t)_t)$ is linkable. We will use the following lemma (see [12, Lemma 5.14]).

Lemma 6.2. *Let $L \subset K \subset \mathcal{M}_1(M)$. If L is dense in $\{\lambda\mu_1 + (1-\lambda)\mu_2 : \lambda \in [0,1],\ \mu_1,\mu_2 \in K\}$ then L is dense in the convex combination $conv(K)$ of measures in K. Hence $\overline{L} = \overline{(conv(K))} = \overline{conv}(K)$.*

By the later, in order to deduce that $\overline{\mathcal{M}_1^{co}((X^t)_t)} = \overline{conv}(\mathcal{M}_1^{co}((X^t)_t))$ it is enough to prove that for any periodic measures $\gamma(x_0), \gamma(x_1) \in \mathcal{M}_1^{co}((X^t)_t)$ supported on orbits of $x_0, x_1 \in \text{Per}((X^t)_t)$, respectively, $\lambda \in [0,1]$ and $\varepsilon > 0$ there exists $y \in M$ such that $X^{\tau(q_1)}(y) = y$ and

$$\rho(m(y,q_1,0,\tau), \lambda\gamma(x_0) + (1-\lambda)\gamma(x_1)) \leq 8\varepsilon. \tag{6.2}$$

Fix $\varepsilon > 0$, $\lambda \in [0,1]$ and $x_0, x_1 \in \text{Per}((X^t)_t)$. Assume, without loss of generality, that $0 < \varepsilon < \frac{1}{2}$ is small. Since Per$((X^t)_t)$ is linkable there exist a reparametrization $\tau \in \text{Rep}(\varepsilon)$, times $t_0, t_1, q_0, q_1 \in \mathbb{R}^+$ and $y \in \text{Per}((X^t)_t)$ satisfying:

(1) $X^{t_i}(x_i) = x_i$ for $i = 0,1$
(2) $X^{\tau(q_1)}(y) = y$ and $y \in B(x_0,t_0,0,\varepsilon,\tau) \cap B(x_1, q_0+t_1, q_0, \varepsilon, \tau)$;
(3) $\tau(t_0) \leq \tau(q_0) \leq (1+\varepsilon)\,\tau(t_0)$;
(4) $\tau(q_0+t_1) - \tau(q_0) \leq \tau(q_1) - \tau(q_0) \leq (1+\varepsilon)[\tau(q_0+t_1) - \tau(q_0)]$;
(5) $\lambda - \varepsilon \leq \frac{\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)} \leq \lambda + \varepsilon$.

We claim that the periodic measure $\gamma(y)$ satisfies (6.2). First observe that

$$\rho(m(y,q_0,0,\tau), \gamma(x_0)) < 6\varepsilon$$

as a direct consequence of Lemma 6.1 (taking $p = t_0$ and $q = q_0$). We also need the following estimate.

Claim: $\rho(m(y,q_1,q_0,\tau)), \gamma(x_1)) \leq 6\varepsilon$.

Proof of the claim. The computations are similar to the ones in the proof of Lemma 6.1 using the estimates given by the linkability property. Indeed, using Rademacher's theorem we have

$$\frac{1}{\tau(q_0+t_1)-\tau(q_0)} \int_{\tau(q_0)}^{\tau(q_0+t_1)} \delta_{X^s(y)}\,ds = \frac{1}{\tau(q_0+t_1)-\tau(q_0)} \int_{q_0}^{q_0+t_1} \tau'(s)\,\delta_{X^{\tau(s)}(y)}\,ds$$

and so it is not hard to check that

$$\begin{aligned}\Big|m(y,q_1,q_0,\tau)(A) - \frac{1}{t_1}\int_0^{t_1} \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| &= \Big|\frac{1}{\tau(q_1)-\tau(q_0)}\int_{\tau(q_0)}^{\tau(q_1)} \delta_{X^s(y)}(A)\,ds - \frac{1}{t_1}\int_0^{t_1} \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| \\ &\le \Big|\frac{1}{\tau(q_0+t_1)-\tau(q_0)}\int_{q_0}^{q_0+t_1} \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds - \frac{1}{t_1}\int_0^{t_1} \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| \\ &\quad + \Big|\frac{1}{\tau(q_0+t_1)-\tau(q_0)}\int_{\tau(q_0+t_1)}^{\tau(q_1)} \delta_{X^s(y)}(A)\,ds\Big|\end{aligned}$$

and, consequently,

$$\begin{aligned}\Big|m(y,q_1,q_0,\tau)(A) - \frac{1}{t_1}\int_0^{t_1} \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| &\le \Big|\frac{1}{\tau(q_0+t_1)-\tau(q_0)} - \frac{1}{t_1}\Big|\,\Big|\int_{q_0}^{q_0+t_1} \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds\Big| \\ &\quad + \Big|\frac{1}{t_1}\int_{q_0}^{q_0+t_1} \tau'(s)\,\delta_{X^{\tau(s)}(y)}(A)\,ds - \frac{1}{t_1}\int_0^{t_1} \delta_{X^s(x)}(A^\varepsilon)\,ds\Big| \\ &\quad + \Big|\frac{1}{\tau(q_0+t_1)-\tau(q_0)}\int_{\tau(q_0+t_1)}^{\tau(q_1)} \delta_{X^s(y)}(A)\,ds\Big| \\ &\le (1+\varepsilon)\,\frac{|\tau(q_0+t_1)-\tau(q_0)-t_1|}{|\tau(q_0+t_1)-\tau(q_0)|} + 2\varepsilon + \varepsilon \\ &\le \frac{1+\varepsilon}{1-\varepsilon}\,\varepsilon + 2\varepsilon + \varepsilon \\ &\le \Big(\frac{1+\varepsilon}{1-\varepsilon} + 3\Big)\varepsilon \le 6\varepsilon.\end{aligned}$$

This proves the claim. □

We are now in a position to prove (6.2). Observe that

$$\frac{\tau(q_0)}{\tau(q_1)}\, m(y,q_0,0,\tau) = \frac{1}{\tau(q_1)}\int_0^{\tau(q_0)} \delta_{X^s(y)}\,ds$$

and

$$\frac{\tau(q_1)-\tau(q_0)}{\tau(q_1)}\, m(y,q_1,q_0,\tau) = \frac{1}{\tau(q_1)}\int_{\tau(q_0)}^{\tau(q_1)} \delta_{X^s(y)}\,ds\,.$$

Thus, writting

$$m(y,q_1,0,\tau) = \frac{1}{\tau(q_1)}\int_0^{\tau(q_1)} \delta_{X^s(y)}\,ds = \frac{1}{\tau(q_1)}\int_0^{\tau(q_0)} \delta_{X^s(y)}\,ds + \frac{1}{\tau(q_1)}\int_{\tau(q_0)}^{\tau(q_1)} \delta_{X^s(y)}\,ds,$$

we can conclude (using [12, Lemma 2.1(iii)])

$$\rho\Big(m(y,q_1,0,\tau), \frac{\tau(q_0)}{\tau(q_1)}\,\gamma(x_0) + \frac{\tau(q_1)-\tau(q_0)}{\tau(q_1)}\,\gamma(x_1)\Big) \le 6\varepsilon.$$

Using properties (3) and (4) above, we also have

$$\frac{\tau(t_0)}{(1+\varepsilon)(\tau(t_0)+\tau(q_0+t_1)-\tau(q_0))} \le \frac{\tau(q_0)}{\tau(q_1)} \le \frac{(1+\varepsilon)\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)},$$

and thus

$$\Big|\frac{\tau(q_0)}{\tau(q_1)} - \frac{\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)}\Big| \le \varepsilon.$$

By triangular inequality we obtain

$$\rho\Big(m(y,q_1,0,\tau), \frac{\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)}\,\gamma(x_0) + \frac{\tau(q_0+t_1)-\tau(q_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)}\,\gamma(x_1)\Big) \le 7\varepsilon.$$

Now observe that property (5) can be rewritten as

$$\left|\frac{\tau(t_0)}{\tau(t_0)+\tau(q_0+t_1)-\tau(q_0)}-\lambda\right| \le \varepsilon.$$

This implies that

$$\rho(m(y,q_1,0,\tau),\lambda\,\gamma(x_0)+(1-\lambda)\,\gamma(x_1)) \le 8\varepsilon.$$

Since $\varepsilon > 0$ was taken arbitrary this proves that $\gamma(y)$ is close to the convex combination $\lambda\,\gamma(x_0)+(1-\lambda)\,\gamma(x_1)$ and finishes the proof of the lemma. □

Proof of Theorem B. As an immediate consequence of Theorem 6.5 we obtain that periodic measures are dense in $\mathcal{M}_1((X^t)_t)$. Since the set of extreme points of $\mathcal{M}_1((X^t)_t)$ is always a G_δ set, the set of ergodic measures is residual in $\mathcal{M}_1((X^t)_t)$. □

Proof of Corollary 1. By Theorem 4.8, the denseness of periodic orbits together with the periodic shadowing property guarantee the flow $(X^t)_t$ to satisfy the periodic reparametrized gluing orbit property. Then Theorem B implies that periodic measures are dense in the space of invariant measures.

The converse is simpler. Assume that periodic measures are dense in $\mathcal{M}_1((X^t)_t)$, and let Λ denote the union of the supports of the ergodic measures. Given any $x \in \Lambda$ let μ be an ergodic measure so that $x \in \operatorname{supp}(\mu)$. Let U be an arbitrary open neighborhood of x for which, in particular, $\mu(U) > 0$. By assumption, there exists a sequence of points $(p_n)_n$ with closed orbits in such a way that $\gamma(p_n) \to \mu$ in the weak* topology. In consequence, $\limsup_{n\to\infty} \gamma(p_n)(U) \ge \mu(U) > 0$. This implies that there are points with closed orbit in the set U. Since U was taken arbitrary one concludes that x is accumulated by periodic points, finishing the proof of the corollary. □

7. Coboundaries, irregular sets and periodic orbits

This section is devoted to the proof of Corollary 2.

Proof of Corollary 2. We prove the implications separately.

(1) $\Rightarrow$ (2)*:* It is a simple consequence of the fact that the periodic measures are dense in the set of all invariant measures (cf. Theorem B).

(2) $\Rightarrow$ (1)*:* Immediate.

(3) $\Rightarrow$ (1)*:* If the Birkhoff averages at x do not converge then the sequence of empirical probability measures $(\frac{1}{t}\int_0^t \delta_{X^s(x)} ds)_{t\ge 0}$ have necessarily two accumulation points that are invariant probability measures $\mu_1 \neq \mu_2$ and $\int \phi\, d\mu_1 < \int \phi\, d\mu_2$.

(1) $\Rightarrow$ (3)*:* Let $\mu_1 \neq \mu_2$ be two $(X^t)_t$-invariant and ergodic probability measures so that $\int \phi\, d\mu_1 < \int \phi\, d\mu_2$. Take $0 < \varepsilon < \frac14(\int \phi\, d\mu_2 - \int \phi\, d\mu_1)$. For any $s = 1, 2$ pick $z_s \in M$ in the ergodic basin of attraction of μ_s. For any $i \ge 1$ take $x_i := z_{i (\text{mod } 1)}$. The strategy of the proof is to construct a point for which the Birkhoff averages oscillate between the values $\int \phi\, d\mu_1$ and $\int \phi\, d\mu_2$. To do so, let $\delta > 0$ be so that $|\phi(x) - \phi(y)| \le \frac{\varepsilon}{2}$ whenever $d(x, y) \le \delta$ (obtained by uniform continuity of the observable ϕ), and let $K(\delta) > 0$ be given by the reparametrized gluing orbit property. Consider the sequence of positive times $(T_k)_k$ so that $T_k \gg T_{k-1}$ for every $k \ge 1$ (the choice of such sequence will be clear in the construction). For any $k \ge 1$ there are $z_k \in M$, times $0 < s_0, \dots, s_{k-1} \le K(\delta)$ and a reparametrization $\tau \in \operatorname{Rep}(\delta)$ so that

$$d(\, X^{\tau(t)}(z_k), X^t(x_i)\,) < \delta \quad \text{for every} \sum_{0\le j<i} (T_j + s_j) \le t \le T_i + \sum_{0\le j<i} (T_j + s_j) \tag{7.1}$$

and every $1 \le i \le k$. If $L_i := \sum_{0\le j<i} (T_j + s_j)$, the previous expression implies that

$$\begin{aligned}&\Big|\frac{1}{t-L_i}\int_{L_i}^t \phi(X^{\tau(s)}(z_k))\, ds - \frac{1}{t-L_i}\int_{L_i}^t \phi(X^s(x_i))\, ds\Big| \\ &\le \frac{1}{t-L_i}\int_{L_i}^t \Big|\phi(X^{\tau(s)}(z_k)) - \phi(X^s(x_i))\Big|\, ds \le \frac{\varepsilon}{2}\end{aligned}$$

for every $1 \le i \le k$ and $L_i \le t \le T_i + L_i$. Therefore, by the choice of $(T_k)_k$,

$$\begin{aligned}
\Big| \int_0^{L_i+T_i} \phi(X^{\tau(s)}(z_k))\, ds - \int_0^{L_i+T_i} \phi(X^s(x_i))\, ds \Big| \\
&\le \Big| \int_0^{L_i} \phi(X^{\tau(s)}(z_k))\, ds - \int_0^{L_i} \phi(X^s(x_i))\, ds \Big| \\
&+ \Big| \int_{L_i}^{L_i+T_i} \phi(X^{\tau(s)}(z_k))\, ds - \int_{L_i}^{L_i+T_i} \phi(X^s(x_i))\, ds \Big| \\
&\le 2L_i \|\phi\|_\infty + \int_{L_i}^{L_i+T_i} \Big| \phi(X^{\tau(s)}(z_k)) - \phi(X^s(x_1)) \Big|\, ds
\end{aligned}$$

and so

$$\frac{1}{L_i + T_i} \Big| \int_0^{L_i+T_i} \phi(X^{\tau(s)}(z_k))\, ds - \int_0^{L_i+T_i} \phi(X^s(x_i))\, ds \Big| \le \frac{2L_i}{L_i + T_i} \|\phi\|_\infty + \frac{T_i}{L_i + T_i} \frac{\varepsilon}{2} < \varepsilon \tag{7.2}$$

(here we used that one can take $L_i \ll T_i$ simply by $T_{i+1} \gg T_i$). This is sufficient to get that

$$\begin{aligned}
\frac{1}{L_i + T_i} \int_0^{L_i+T_i} \phi(X^{\tau(s)}(z_k))\, ds < \int \phi\, d\mu_1 + \frac{3}{2}\varepsilon < \int \phi\, d\mu_2 - \frac{3}{2}\varepsilon \\
\le \frac{1}{L_j + T_j} \int_0^{L_j+T_j} \phi(X^{\tau(s)}(z_k))\, ds
\end{aligned} \tag{7.3}$$

for all (large) i odd and j even in the interval $[1, k]$. Now, observe that the estimates (7.1) to (7.3) hold for all points $z_k \in M$ independently from the reparametrizations $\tilde{\tau} \in \mathrm{Rep}(\delta)$ considered. Since the sequence $(z_k)_k$ lies on the compact set $\overline{B(x_1, \delta)}$ accumulation points exist and the Birkhoff averages for ϕ on any of these points do not converge. This completes the proof of Corollary 2.

□

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References

[1] E. Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics, 1, American Mathematical Society, Providence, RI, 1993.

[2] M. Bessa, M. J. Torres and P. Varandas, *On the periodic orbits, shadowing and strong transitivity of continuous flows*, In preparation 2016.

[3] A. Biś, *An analogue of the variational principle for group and pseudogroup actions.,* Ann. Inst. Fourier. Grenoble, 63,3 (2013), 839–863.

[4] T. Bomfim and P. Varandas, *Gluing orbit property and large deviations principles for flows*, Preprint 2015.

[5] M. Boshernitzan, G. Galperin, T. Krüger and S. Troubetzkoy. *Periodic billiard orbits are dense in rational polygons*, Trans. Amer. Math. Soc., 350:9, (1998) 3523–3535.

[6] R. Bowen. *Entropy for group endomorphisms and homogeneous spaces.* Trans. Amer. Math. Soc., **153** (1971), 401–414.

[7] R. Bowen and D. Ruelle. *The ergodic theory of Axiom A flows. Invent. Math.*, 29:181–202, 1975.

[8] V. Climenhaga and D. Thompson. *Unique equilibrium states for flows and homeomorphisms with non-uniform structure.* Preprint 2015.

[9] Y. Coudène and B. Shapira. *Generic measures for geodesic flows on nonpositively curved manifolds*, Journal de l'Ecole Polytechnique Vol. 1 (2014), p. 387–408.

[10] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces.*, Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.

[11] B. Fayad. *Polynomial decay of correlations for a class of smooth flows on the two torus.* Bull. Soc. Math. France, 129 (2001), no. 4, p. 487–503.
[12] K. Gelfert and D. Kwietniak, *On density of ergodic measures and generic points,* Ergod. Th. & Dynam. Sys. (to appear).
[13] E. Gutkin, *Billiards in polygons: survey of recent results ,* J. Statist. Phys., Volume 83:1–2, (1996), 7–26.
[14] N. Haydn and D. Ruelle *Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification*, Commun. Math. Phys. 148, 155–167 (1992).
[15] S. Kerckhoff, H. Masur, and J. Smillie, *Ergodicity of billiard flows and quadratic differentials,* Annals of Math. 124 (1986), 293–311.
[16] M. Kulczycki, D. Kwietniak and P. Oprocha, *On almost specification and average shadowing properties,* Fund. Math., 224 (2014), 241–278.
[17] N. Haydn and D. Ruelle *Equivalence of Gibbs and equilibrium states for homeomorphisms satisfying expansiveness and specification*, Commun. Math. Phys. 148, 155–167 (1992).
[18] M. Hirayama, *Periodic probability measures are dense in the set of invariant measures*, Disc. Cont. Dyn. Sys. 9 (2003), 1185–1192.
[19] C. Liang, G. Liu and W. Sun, Approximation properties on invariant measure and Oseledec splitting in non-uniformly hyperbolic systems. Trans. Amer. Math. Soc. 361 (2009), no. 3, 1543–1579.
[20] K. Oliveira and X.Tian. *Non-uniform hyperbolicity and non-uniform specification.* Trans. Amer. Math. Soc., **365** (2013), 4371–4392.
[21] R. Pavlov *On intrinsic ergodicity and weakenings of the specification property.* Adv. Math., **295**, (2016), 250–270.
[22] C.-E. Pfister and W. Sullivan. *Large deviations estimates for dynamical systems without the specification property: application to the beta-shifts.* Nonlinearity, **18** (2005), 237–261.
[23] Y. Prohorov, Convergence of random processes and limit theorems in probability theory. Teor. Veroyatnost. i Primenen. 1 (1956), 177–238.
[24] K. Sigmund, *Generic properties of invariant measures for Axiom A-diffeomorphisms,* Invent. Math. 11 (1970),99–109.
[25] K. Sigmund, *On the space of invariant measures for hyperbolic flows,* Amer. Journal of Math. 94, No. 1 (1972), 31–37.
[26] N. Sumi, P. Varandas and K.Yamamoto, *Partial hyperbolicity and specification.* Proc. Amer. Math. Soc. 144:3 (2016) 1161–1170.
[27] N. Sumi, P. Varandas and K. Yamamoto, *Specification and partial hyperbolicity for flows*, Dyn. Sys. 30:4, (2015) 501–524.
[28] W. Sun, X. Tian and E. Vargas, *Non-uniformly hyperbolic flows and shadowing*, J. Differential Equations 261 (2016) 218–235.
[29] W. Sun and X. Tian *Diffeomorphisms with various C^1-stable properties.* Acta Mathematica Scientia 2012, 32B(2):552–558.
[30] D. Thompson, *The irregular set for maps with the specification property has full topological pressure,* Dyn. Syst. 25 (2010), no. 1, 25-51.
[31] P. Varandas. *Non-uniform specification and large deviations for weak Gibbs measures.* J. Statist. Phys., **146** (2012), 330–358.
[32] A. Zemlyakov and A. Katok, *Topological transitivity of billiards in polygons,* Math. Notes 18 (1975) 760–764.

Thiago Bomfim, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil

E-mail address: tbnunes@ufba.br

Maria Joana Torres, CMAT and Departamento de Matemática e Aplicações, Universidade do Minho, Campus de Gualtar, 4700-057 Braga, Portugal

E-mail address: jtorres@math.uminho.pt

Paulo Varandas, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil

E-mail address: paulo.varandas@ufba.br