A CRITERION FOR THE TRIVIALITY OF THE CENTRALIZER FOR VECTOR FIELDS AND APPLICATIONS

WESCLEY BONOMO AND PAULO VARANDAS

Abstract. In this paper we establish a criterion for the triviality of the C^1-centralizer for vector fields and flows. In particular we deduce the triviality of the centralizer at homoclinic classes of C^r vector fields ($r \geq 1$). Furthermore, we show that the set of flows whose C^1-centralizer is trivial include: (i) C^1-generic volume preserving flows, (ii) C^2-generic Hamiltonian flows on a generic and full Lebesgue measure set of energy levels, and (iii) C^1-open set of non-hyperbolic vector fields (that admit a Lorenz attractor).

1. Introduction and statement of the main results

1.1. **Introduction.** In this article we study the centralizer of flows and vector fields. The centralizer of a flow is the set of flows that commute with the original one. In [31], Smale proposed a problem on the triviality of the centralizer in dynamical systems, asking if 'typical' dynamics (meaning open and dense, or Baire generic) would have trivial centralizers. In the case of discrete time dynamical systems there have been substantial contributions towards both an affirmative solution to the problem raised by Smale and also to applications that exploit the relation of the centralizer with differentiability of conjugacies, the embedding of diffeomorphisms as time-1 maps of flows, the rigidity of measurable $\mathbb{Z}^d$-actions on tori or the characterization of conjugacies for structurally stable diffeomorphisms, just to mention a few (see e.g. [7, 9, 10, 11, 15, 21, 23, 27, 28, 33] and references therein).

The centralizer of continuous time dynamics has been less studied. Indeed, in the continuous time setting the problem of the centralizer has been considered for Anosov flows [12], C^∞-Axiom A flows with the strong transversality condition [30], and for Komuro-expansive flows, singular-hyperbolic attractors and expansive $\mathbb{R}^d$-actions [8, 20]. In most cases, the ingredients in the proofs of the previous results involve either a strong form of hyperbolicity and/or C^∞-smoothness of the dynamics in order to use linearization of the dynamics in a neighborhood of a periodic or singular point.

In order to describe a larger class of vector fields (and flows) we establish a criterion for the triviality of the C^1-centralizer of vector fields based on the denseness of hyperbolic periodic orbits (Theorem A). In particular the triviality of the centralizer can be deduced whenever there exists a dense set of hyperbolic periodic orbits, a condition that is satisfied by all homoclinic classes of C^r-vector

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fields in view of Birkhoff-Smale's theorem ($r \geq 1$) (Corollary 1). We use this fact to show that C^1-generic volume preserving flows have a trivial centralizer (Corollary 2), obtaining the counterpart of [7] for continuous time dynamics, and that Lorenz attractors have trivial centralizer (cf. Corollary 3). In the case of Hamiltonian dynamics the relation between symmetries and elements of the centralizer of a Hamiltonian flow is more evident. Indeed, it follows from Noether's theorem that there exists a bijection between conserved quantities, modulo constants, and the set of infinitesimal symmetries (see Subsection 1.2.3 for more details). First results on commuting Hamiltonians were obtained in [3], which described almost-periodic commuting Hamiltonian vector fields. Moreover, as the Hamiltonian is always a first integral for the Hamiltonian vector field it is natural to ask whether typical Hamiltonians have other independent first integrals. Arnold-Liouville's integrability theorem assures that the existence of n independent commuting integrals for a C^2-Hamiltonian H implies on the integrability of the Hamiltonian vector field and that the restriction of the flow to each invariant torus consists of a translation flow (see e.g. [2]). However, C^2-generic Hamiltonians are transitive on each connected component of generic energy levels ([5]) and, consequently, continuous first integrals are constant on the corresponding level sets. We use this fact here to prove that the centralizer of C^2-generic Hamiltonians on compact symplectic manifolds is trivial on generic level sets (we refer the reader to Corollary 4 for the precise statement).

The problem of the centralizer for flows and vector fields differs substantially from the discrete-time setting. For instance, the statement of the criterion for triviality established here for vector fields is false even for Anosov diffeomorphisms (see Remark 1.1). In rough terms, the first part of the strategy to the proof of Theorem A is to show that the closure of the set of hyperbolic critical elements is preserved by all flows in the C^1-centralizer of a flow $(X_t)_t$. The second part of the argument is to use transitivity to show that every element in the centralizer of $(X_t)_t$ is a linear (actually constant) reparametrization of $(X_t)_t$. One should mention that if one was interested to describe a Baire generic subset of dynamical systems the transitivity assumption could be removed. Indeed, as proposed by R. Thom [32], for every $1 \leq r \leq \infty$, C^r-generic vector fields do not admit non-trivial first integrals (see e.g. [17]).

This paper is organized as follows. Subsection 1.2 is devoted to definitions and preliminary results on hyperbolicity, centralizers and Hamiltonians. In Subsection 1.3 we state our main results. The proofs of the main results appear in Sections 2 and 3. Finally, in Section 4 we provide some examples and applications and Section 5 is devoted to some final comments on C^0-centralizers of continuous flows and possible directions of research.

1.2. **Preliminaries.** In this subsection we recall some necessary definitions and set some notation.

1.2.1. *Non-wandering set, periodic orbits and hyperbolicity.* Let M be a compact, connected Riemannian manifold. Given a C^1-vector field $X \in \mathfrak{X}^1(M)$, we often denote by $(X_t)_{t\in\mathbb{R}}$ the C^1-flow generated by X and let $\Omega(X)$ denote its non-wandering set. Given a compact $(X_t)_{t\in\mathbb{R}}$-invariant set $\Lambda \subset M$, we say that Λ is a *hyperbolic set* for $(X_t)_{t\in\mathbb{R}}$ if there exists a DX_t-invariant splitting $T_\Lambda M = E^s \oplus E^0 \oplus E^u$ so that: (a) E^0 is one dimensional and generated by the vector field

X, and (b) there are constants $C > 0$ and $\lambda \in (0,1)$ so that

$$\|DX_t(x) \mid_{E^s_x} \| \le C\lambda^t \quad \text{and} \quad \|(DX_t(x) \mid_{E^u_x})^{-1}\| \le C\lambda^t$$

for every $x \in \Lambda$ and $t \ge 0$. The flow $(X_t)_{t\in\mathbb{R}}$ is *Anosov* if $\Lambda = M$ is a hyperbolic set. A point $p \in M$ is: (i) a *singularity* if $X_t(p) = p$ for all $t \in \mathbb{R}$, and (ii) *periodic* if there exists $t > 0$ such that $X_t(p) = p$ and $\pi(p) := \inf\{t > 0 : X_t(p) = p\} > 0$. We denote by $Sing((X_t)_t)$ (or $Sing(X)$) the set of singularities, by $\mathrm{Per}((X_t)_t)$ (or $\mathrm{Per}(X)$) the set of periodic orbits and by $Crit((X_t)_t) := Sing((X_t)_t) \cup \mathrm{Per}((X_t)_t)$ the set of all critical elements for the flow $(X_t)_t$. Every non-singular point is called *regular.* A flow $(X_t)_t$ is called *Axiom A* if $\Omega(X)$ is a uniformly hyperbolic set and $\overline{\mathrm{Crit}((X_t)_t)} = \Omega(X)$. A point periodic $p \in M$ is *hyperbolic* if its orbit $\gamma_p := \{X_t(p) : t \in \mathbb{R}\}$ is a hyperbolic set. A singularity $\sigma \in M$ is *hyperbolic* if $DX(\sigma)$ has no purely imaginary eigenvalues. Denote by $W^s(\gamma_p)$ (resp. $W^u(\gamma_p)$) the usual stable (resp. unstable) manifolds of the hyperbolic periodic orbit γ_p. The stable and unstable manifolds for a singularity σ are denoted similarly by $W^s(\sigma)$ and $W^u(\sigma)$, respectivelly. The *homoclinic class* associated to a hyperbolic critical element z is the compact invariant set $H(z) = \overline{W^s(\gamma_z) \pitchfork W^u(\gamma_z)}$. Finally, we say that a flow $(X_t)_t$ is *transitive* if there exists $x \in M$ so that $(X_t(x))_{t\in\mathbb{R}_+}$ is dense in M. We refer to [14, 22] for more details on uniform hyperbolicity and homoclinic classes.

1.2.2. *Commuting flows and centralizers.* Given $r \ge 0$, let $\mathcal{F}^r(M)$ denote the space of C^r-flows on M. We say that the flows $(X_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$ *commute* if $Y_s \circ X_t = X_t \circ Y_s$ for all $s, t \in \mathbb{R}$. Given a flow $(X_t)_{t\in\mathbb{R}} \in \mathcal{F}^r(M)$, the *centralizer of* X is the set of C^r-flows that commute with X, that is,

$$\mathcal{Z}^r((X_t)_t) = \{(Y_s)_{s\in\mathbb{R}} \in \mathcal{F}^r(M) \colon Y_s \circ X_t = X_t \circ Y_s, \forall\, s, t \in \mathbb{R}\}. \tag{1.1}$$

It is clear from the definition that all flows obtained as smooth reparametrizations of $(X_t)_t$ belong to $\mathcal{Z}^r((X_t)_t)$. For that reason, the centralizer of the time-one map a flow is never a discrete subgroup of the space of diffeomorphisms. Given $r \ge 0$, we say a flow $(X_t)_t \in \mathcal{F}^r(M)$ has *quasi-trivial centralizer* if for any $Y \in \mathcal{Z}^r(X)$ there exists a continuous function $h : M \to \mathbb{R}$ so that

(i) (orbit invariance) $h(x) = h(X_t(x))$ for every $(t,\, x) \in \mathbb{R} \times M$, and
(ii) $Y_t(x) = X_{h(x)t}(x)$ for every $(t,\, x) \in \mathbb{R} \times M$.

In the case that the reparametrizations h are necessarily constant then we say the centralizer is *trivial.* The previous notion has a dual formulation in terms of vector fields. Given $r \ge 1$ and $X \in \mathfrak{X}^r(M)$, one can define the *centralizer of the vector field* X by

$$\mathcal{Z}^r(X) = \{Y \in \mathfrak{X}^r(M) \colon [X, Y] = L_Y X = 0\}, \tag{1.2}$$

where $[X, Y]$ denotes the usual comutator of the vector fields X and Y, and $L_Y X$ denotes the Lie derivative of the vector field X along Y. We say that the vector field $X \in \mathfrak{X}^r(M)$ has *quasi-trivial centralizer* if for any $Y \in \mathcal{Z}^r(X)$ there exists a continuous $h : M \to \mathbb{R}$ so that $Y = h \cdot X$ and $X(h) = 0$. As before, the centralizer is *trivial* if any such h is necessarily constant. Observe that any reparametrization $h : M \to \mathbb{R}$ satisfying $X(h) = 0$ is constant along the orbits of the flow and, hence, is a first integral for the flow. If $(X_t)_t$ is transitive then any first integral is necessarily constant. For that reason, any transitive flow with a quasi-trivial centralizer has trivial centralizer. Moreover, as C^r-generic vector fields do not admit non-trivial first integrals [17], the set of C^1-flows with a quasi-trivial and non-trivial centralizer is meager.

1.2.3. *Hamiltonian vector fields and flows.* Let (M^{2n}, ω) be a compact symplectic Riemannian manifold. Given $r \geq 1$ and a Hamiltonian $H \in C^{r+1}(M, \mathbb{R})$, the Hamiltonian vector field $X_H \in \mathfrak{X}^r(M)$ is defined by $\omega(X_H(x), v) = DH(x)\, v$ for every $x \in M$ and $v \in T_xM$. In the case that $r \geq 2$ the Hamiltonian vector field X_H is at least C^1 and we denote by $(\varphi_t^H)_t$ the Hamiltonian flow generated by X_H. We say that $K \in C^{r+1}(M, \mathbb{R})$ is a *first integral* (or an infinitesimal symmetry) for the Hamiltonian $H \in C^{r+1}(M, \mathbb{R})$ if $K(\varphi_t^H(x)) = K(x)$ for every $x \in M$ and $t \in \mathbb{R}$. A first integral K for a Hamiltonian H is characterized by $\{K, G\} = 0$, where the Poisson bracket $\{\cdot, \cdot\} : C^{r+1}(M, \mathbb{R}) \times C^{r+1}(M, \mathbb{R}) \to C^r(M, \mathbb{R})$, defined by

$$\{H, K\}(x) = \frac{d}{dt} H(\varphi_t^K) \mid_{t=0}, \tag{1.3}$$

measures the derivative of H along the orbits of $(\varphi_t^K)_t$. Recall $\{H, K\} = -\{K, H\}$ and that the Poisson and Lie brackets for Hamiltonians are related by $[X_K,\, X_H] = X_{\{H,\, K\}}$ (see e.g. [16]). Hence, the Poisson bracket determines commutativity of Hamiltonians: $H, K \in C^r(M, \mathbb{R})$ commute if and only if the Poisson bracket $\{H, K\}$ is locally constant (see e.g. [2, Section 40]). Nevertheless, two Hamiltonian flows may commute and not preserve level sets (see Example 4.2). On the one hand, Noether's Theorem (see e.g. [14] or [16, Theorem 18.27]) establishes a bijective correspondence between the set of vector fields $Y \in \mathfrak{X}^r(M)$ such that $H(Y_t(x)) = H(x)$ and $Y_t^*\omega = \omega$ for any $t \in \mathbb{R}$ and $x \in M$ (also known as conserved quantities) and the set of first integrals for X_H, modulo addition by constants. On the other hand, K is a first integral of X_H iff $\{K,\, H\} = 0$ (see e.g. [18, Theorem 2]).

These facts motivate the following definition. Given $r \geq 1$, a Hamiltonian $H \in C^{r+1}(M, \mathbb{R})$ with associated vector field $X_H \in \mathfrak{X}^r(M)$, the *Hamiltonian C^r-centralizer* of X_H is

$$\mathcal{Z}^r_\omega(X_H) = \{X_K \in \mathfrak{X}^r_\omega(M) :\, \{K,\, H\} = 0\}. \tag{1.4}$$

Then, we say that the centralizer of a Hamiltonian flow $(\varphi_t^H)_t$ is: (i) quasi-trivial if there exists a continuous map h such that $X_H(x) = h(x) X_K(x)$ for every $x \in M$, and (ii) trivial on the connected component $\mathcal{E}_{H,e} \subset H^{-1}(e)$ if for every $X_K \in \mathcal{Z}^r_\omega(X_H)$ there exists $c \in \mathbb{R}$ such that $X_K(x) = cX_H(x)$ for every $x \in \mathcal{E}_{H,e}$.

1.3. **Statement of the main results.** First we will state the criterion for the triviality of the centralizer of vector fields and flows.

Theorem A. *Let M be a compact Riemannian manifold and and let $(X_t)_t$ be the flow generated by $X \in \mathfrak{X}^r(M)$, for some integer $r \geq 1$. Assume that $\Lambda \subset M$ is a compact, $(X_t)_t$-invariant and transitive subset. If the set of hyperbolic periodic orbits of $(X_t)_t$ is dense in Λ then $\mathcal{Z}^1(X \mid_\Lambda)$ is trivial: if $Y \in \mathcal{Z}^1(X)$ then the flow $(Y_t)_t$ preserves Λ and there exists $c \in \mathbb{R}$ such that $Y \mid_\Lambda = cX \mid_\Lambda$.*

We note that the assumptions of Theorem A are satisfied by homoclinic classes. Indeed, an homoclinic classes Λ is transitive and it follows from Birkhoff-Smale's theorem that hyperbolic periodic orbits are dense. Thus

Corolary 1. *Assume that $r \geq 1$, $X \in \mathfrak{X}^r(M)$, $p \in M$ is a hyperbolic critical element for X and $\Lambda = H(p)$ is a homoclinic class for X. Then $\mathcal{Z}^1(X \mid_\Lambda)$ is trivial.*

This corollary, which applies e.g. to hyperbolic basic pieces of C^r-flows, will be useful to study the triviality of the centralizer for non-hyperbolic homoclinic classes, including volume preserving and Hamiltonian vector fields, and vector fields that exhibit Lorenz attractors (see Section 1.2 for definitions).

Remark 1.1. *Theorem A and Corollary 1 do not admit a counterpart for discrete time dynamics. Indeed, there are linear Anosov automorphisms on $\mathbb{T}^n$ (hence transitive, with a dense set of periodic orbits, all of them hyperbolic) that do not have trivial centralizer (see e.g. [24]). Nevertheless, any Anosov flow (e.g. suspensions of an Anosov diffeomorphisms) satisfies the hypothesis of Theorem A and, consequently, has trivial centralizer.*

In order to state some consequences of Theorem A and Corollary 1 we recall that a subset R of a topological space E is called *Baire residual* if it contains a countable intersection of open and dense sets, and it is called *meager* if it is the complementary set of a Baire residual subset. It follows from Baire category theorem that if E is a Baire space then all residual subsets are dense in E. Given $r \geq 1$, let $\mathfrak{X}^r_\mu(N)$ denote the space of C^r-volume preserving vector fields on N.

Corolary 2. *Let $r \geq 1$ and let N be a compact and connected Riemannian manifold. There exists a C^r-residual subset $\mathcal{R} \subset \mathfrak{X}^r(N)$ such that if $X \in \mathcal{R}$, $Y \in \mathcal{Z}^1(X)$ and $\Lambda \subset \overline{Per(X)}$ is a transitive invariant set then there exists $c \in \mathbb{R}$ so that $Y = cX$ on Λ. Moreover, C^1-generic vector fields in $\mathfrak{X}^1_\mu(N)$ have trivial C^1-centralizer.*

The second assertion in Corollary 2 is the counterpart of the results in [7] for C^1-volume preserving vector fields. Now we draw our attention to the existence of C^1-open sets of vector fields with trivial centralizer. Kato and Morimoto [13] used the notion of topological stability to prove that all C^1-Anosov flows have quasi-trivial centralizer (hence transitive Anosov flows have trivial centralizer). In [30], Sad used a linearization technique to prove that the centralizer is trivial for a C^∞-open and dense set of C^∞-Axiom A flows with the strong transversality condition. Theorem A can also be used to exhibit new C^1-open sets of vector fields that have non-uniformly hyperbolic attractors restricted to which the centralizer is trivial. These are open sets of vector fields that exhibit Lorenz attractors (we refer the reader to [1] for definitions and a large account on Lorenz attractors). More precisely we obtain the following:

Corolary 3. *Let $\mathcal{U} \subset \mathfrak{X}^1(\mathbb{R}^3)$ be a C^1-open set of vector fields and let $V \subset \mathbb{R}^3$ be an open elipsoid containing the origin such that every $X \in \mathcal{U}$ exhibits a Lorenz attractor $\Lambda_X = \bigcap_{t\geq 0} \overline{X_t(V)}$. Then $\mathcal{Z}^1(X \mid_{\Lambda_X})$ is trivial for every $X \in \mathcal{U}$: for any $Y \in \mathcal{Z}^1(X)$ the flow generated by Y preserves the $(X_t)_t$-invariant set Λ_X and there exists $c \in \mathbb{R}$ such that $Y \mid_{\Lambda_X} = cX \mid_{\Lambda_X}$.*

The previous result is complementary [8, Theorem A] where the authors proved that a $\mathcal{C}^1$-open and $\mathcal{C}^\infty$-dense subset of vector fields $\mathfrak{X}^\infty(M^3)$ that exhibit Lorenz attractors have trivial centralizer on their topological basin of attraction.

In what follows we discuss some applications of our result in the case of Hamiltonian vector fields. In this setting, the triviality of the centralizer is not immediate from the denseness of periodic orbits whose periods are isolated. Indeed, given a Hamiltonian $H \in C^2(M,\mathbb{R})$, any $K \in C^2(M,\mathbb{R})$ such that $X_K \in \mathcal{Z}^1_\omega(X_H)$ is a first integral for the flow $(\varphi^H_t)_t$ (cf. Subsection 1.2.3). However is not clear *a priori* that K preserves the level sets $H^{-1}(e)$ and, even if this is the case, the first integrals H and K could be independent. Here we prove that the centralizer of C^2-generic Hamiltonians have trivial centralizer. More precisely:

Corolary 4. *Let (M^{2n}, ω) be a compact symplectic Riemannian manifold. If $n \geq 2$ then there exists a residual subset $\mathcal{R} \subset C^2(M, \mathbb{R})$ such that the following holds: for every $H \in \mathcal{R}$ there exists a residual and full Lebesgue measure subset $\mathfrak{R}_H \subset H(M)$ of energies such that if $e \in \mathfrak{R}_H$ then the centralizer of Hamiltonian flow $(\varphi_t^H)_t$ is trivial on each connected component $\mathcal{E}_{H,e} \subset H^{-1}(e)$.*

As Anosov energy levels persist by C^1-perturbations of the vector field and volume preserving Anosov flows are transitive, the following is an immediate consequence from Theorem A:

Corolary 5. *Let (M^{2n}, ω) a compact symplectic Riemannian manifold with $n \geq 2$, let $H_0 \in C^r(M, \mathbb{R})$ ($r \geq 2$) and let $e_0 \in \mathbb{R}$ be such that the Hamiltonian flow $(\varphi_t^{H_0})_t$ restricted to the connected component $\mathcal{E}_{H_0,e_0} \subset H_0^{-1}(e_0)$ is Anosov. There is a C^r-open neighborhood $\mathcal{U}$ of H_0 and a open neighborhood $V \subset \mathbb{R}$ of e_0 such that for every $(H, e) \in \mathcal{U} \times V$ the flow $(\varphi_t^H)_t$ on has trivial centralizer on the connected component $\mathcal{E}_{H,e} \subset \tilde{H}^{-1}(\tilde{e})$ obtained as analytic continuation of $\mathcal{E}_{H_0,e_0}$.*

One should note that in the case of surfaces one cannot avoid the presence of elliptic islands even in low topologies and, in particular, the triviality of the centralizer cannot be expected to hold in general (cf. Example 4.3).

2. Proof of Theorem A

Fix an integer $r \geq 1$ and $X \in \mathfrak{X}^r(N)$, and take $Y \in \mathcal{Z}^1(X)$. Assume that the vector fields X and Y generate the flows $(X_t)_t$ and $(Y_s)_s$, respectivelly. By the commutative relation $X_t \circ Y_s = Y_s \circ X_t$ for all t, $s \in \mathbb{R}$ it follows that the diffeomorphism Y_s is a conjugation between the flow $(X_t)_t$ with itself, for every $s \in \mathbb{R}$. Hence, if $p \in N$ is a periodic point of period $\pi(p) > 0$ for $(X_t)_t$ (γ_p denotes its orbit) and $s \in \mathbb{R}$ is arbitrary then

$$X_{\pi(p)}(Y_s(p)) = Y_s(X_{\pi(p)}(p)) = Y_s(p). \tag{2.1}$$

We make use of the following simple lemma.

Lemma 2.1. *Let N be a compact Riemannian manifold and assume that the C^1 flow $(Y_s)_s$ on N belongs to the C^1-centralizer of the C^1-flow $(X_t)_t$. Then, for every $s \in \mathbb{R}$ and every hyperbolic critical element $p \in Crit((X_t)_t)$ one has $Y_s(\gamma_p) = \gamma_p$.*

Proof. Let $p \in \mathrm{Per}((X_t)_t)$ be a hyperbolic periodic point of prime period $\pi(p) > 0$ and take $s \neq 0$. Differentiating $X_{\pi(p)} \circ Y_s = Y_s \circ X_{\pi(p)}$ we obtain

$$DX_{\pi(p)}(Y_s(p))\, DY_s(p) = DY_s(X_{\pi(p)}(p))\, DX_{\pi(p)}(p) = DY_s(p)\, DX_{\pi(p)}(p)$$

which shows, since Y_s is a diffeomorphism, that $DX_{\pi(p)}(p)$ and $DX_{\pi(p)}(Y_s(p))$ are linearly conjugate. This proves that $(Y_s(p))_{s\in\mathbb{R}}$ is a family of hyperbolic periodic points of period $\pi(p)$ for the flow $(X_t)_t$. Since hyperbolic periodic orbits of the same period are isolated we conclude that $Y_s(\gamma_p) = \gamma_p$ for every $s \in \mathbb{R}$, which proves the lemma. ∎

The previous lemma assures that every $Y \in \mathcal{Z}^1(M)$ preserves the set of hyperbolic periodic orbits in Λ (and so it preserves Λ).

Now, as vector fields X and Y are collinear on γ_p, for all periodic points $p \in \Lambda$, we use that the periodic orbits of $(X_t)_t$ are dense in N to derive that the vector fields X and Y are collinear on $\Lambda \setminus Sing(X)$. More precisely:

Proposition 2.2. *Let $r \geq 1$ and let $X \in \mathfrak{X}^r(N)$ generate a flow $(X_t)_t$ so that $\Lambda \subset N$ is a compact, $(X_t)_t$-invariant and transitive subset with a dense subset of hyperbolic periodic orbits. For any $Y \in \mathcal{Z}^1(X)$ the flow $(Y_t)_t$ preserves Λ and there exists a map $h : \Lambda \setminus Sing(X) \to \mathbb{R}$ such that $Y(x) = h(x) \cdot X(x)$ for all $x \in \Lambda \setminus Sing(X)$. Moreover, the following properties hold:*

(a) *h is uniquely defined,*
(b) *h is constant along regular orbits of X, and*
(c) *h is continuous.*

Proof. Fix $Y \in \mathcal{Z}^1(X)$ arbitrary. We know that Λ is preserved by the flow $(Y_t)_t$. First, suppose by contradiction that exists $x_0 \in N \setminus Sing(X)$ such that the vectors $X(x_0)$ and $Y(x_0)$ are linearly independent. Consider the continuous map $r : \Lambda \setminus Sing(X) \to T_xN$ given by

$$r(x) = Y(x) - \frac{\langle Y(x),\, X(x)\rangle_x}{\langle X(x),\, X(x)\rangle_x} X(x),$$

where $\langle \cdot\,, \cdot\rangle_x$ denotes the Riemannian metric of N at T_xN. As $r(x_0) \neq 0$, by continuity there exists an open neighborhood $V_{x_0} \subset \Lambda$ of x_0 such that $r(x) \neq 0$ for all $x \in V_{x_0}$. This cannot occur because $r(x) = 0$ for all $x \in \Lambda \cap Per((X_t)_t)$ and $Per((X_t)_t)$ is dense in Λ. Therefore we conclude that r is identically zero on $\Lambda \setminus Sing(X)$ or, in other words, the vector fields X and Y are collinear on $\Lambda \setminus Sing(X)$. In consequence, there exists a map $h : N \setminus Sing(X) \to \mathbb{R}$ such that $Y(x) = h(x) \cdot X(x)$ for all $x \in \Lambda \setminus Sing(X)$.

We proceed to prove properties (a), (b) and (c). The conclusion of (a) is immediate. Indeed, if there are $h_1, h_2 : \Lambda \setminus Sing(X) \to \mathbb{R}$ such that $h_1(x)\, X(x) = Y(x) = h_2(x)\, X(x)$, then $(h_1(x) - h_2(x))\, X(x) = 0$ for all regular points $x \in \Lambda \setminus Sing(X)$. Thus $h_1(x) = h_2(x)$ for all $x \in \Lambda \setminus Sing(X)$ and h is uniquely defined.

Let us prove now that h is constant along regular orbits of X. Given a regular point p for X, we have that

$$Y(X_t(p)) = h(X_t(p))X(X_t(p)) \qquad \text{for every } t \in \mathbb{R}. \tag{2.2}$$

Then, differentiating the relation $Y_s \circ X_t = X_t \circ Y_s$ with respect to s (at $s = 0$) we conclude that

$$\begin{aligned} Y(X_t(p)) &= DX_t(p)Y(p) = DX_t(p)[h(p)X(p)] \\ &= h(p)DX_t(p)X(p) = h(p)X(X_t(p)) \end{aligned} \tag{2.3}$$

for every $t \in \mathbb{R}$. Since $X_t(p)$ is a regular point for X, equations (2.2) and (2.3) show that the reparametrization h is constant along the orbits of $(X_t)_t$. In other words, $h(x) = h(X_t(x))$ for all $x \in \Lambda \setminus Sing(X)$ and $t \in \mathbb{R}$, which proves (b).

We are left to prove the continuity of the reparametrization h. First we observe that $Y_t(x) = X_{h(x)t}(x)$ for all $x \in \Lambda \setminus Sing(X)$. This is a simple consequence of the uniqueness of solution of the ordinary differential equation $u' = Y(u)$ and the fact that h is constant along the orbits of $(X_t)_t$.

Assume by contradiction that h is not continuous. Then there exists $\delta_0 > 0$, and a sequence $(x_n)_{n\in\mathbb{N}}$ convergent to x in $\Lambda \setminus Sing(X)$ such that $\inf_{n\in\mathbb{N}} |h(x_n) - h(x)| > 0$. First observe that, as the vector fields X and Y are continuous, $X(x) =$

$\lim\limits_{n\to\infty} X(x_n) \neq 0$ and $|h(x_n)| = \|Y(x_n)\|/\|X(x_n)\|$, the sequence $(h(x_n))_n$ of real numbers is bounded. Moreover, since x is a regular point, the tubular flowbox theorem assures that there exists $\varepsilon_x > 0$, an open neighborhood $V_x \subset M$ of x, and a C^r-diffeomorphism $h : V_x \to (-\varepsilon_x, \varepsilon_x) \times B(\vec{0}, \varepsilon) \subset \mathbb{R} \times \mathbb{R}^{n-1}$ so that $h_* X(y) = (1, 0, \ldots, 0)$ for every $y \in V_x$. Choose any sequence $(s_n)_n$ of real numbers so that $(s_n,\, x_n) \to (s,\, x) \in (\mathbb{R} \setminus \{0\}) \times (\Lambda \setminus Sing(X))$, that $|h(x) \cdot s| < \frac{\varepsilon_x}{2}$ and $|h(x_n) \cdot s_n - h(x) \cdot s| \leq \frac{\varepsilon_x}{2}$ for all $n \in \mathbb{N}$. By construction, $\delta_0 := \inf_{n\in\mathbb{N}} |h(x_n) \cdot s_n - h(x) \cdot s| > 0$. Since the points $X_{h(x_n)\cdot s_n}(x)$ and $X_{h(x)\cdot s}(x)$ lie in the same piece of orbit in the tubular flowbox chart then there exists $\delta_1 > 0$ (depending on δ_0) so that $d(X_{h(x_n)\cdot s_n}(x),\, X_{h(x)\cdot s}(x)) \geq \delta_1 > 0$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\delta_1 &\leq d(X_{h(x_n)\cdot s_n}(x),\, X_{h(x)\cdot s}(x)) \\
&\leq d(X_{h(x_n)\cdot s_n}(x),\, X_{h(x_n)\cdot s_n}(x_n)) + d(X_{h(x_n)\cdot s_n}(x_n),\, X_{h(x)\cdot s}(x)) \\
&= d(X_{h(x_n)\cdot s_n}(x),\, X_{h(x_n)\cdot s_n}(x_n)) + d(Y_{s_n}(x_n),\, Y_s(x))
\end{aligned}$$

for all $n \in \mathbb{N}$. The second term in the right hand side above is clearly convergent to zero as n tends to infinity by the continuous dependence on initial conditions of the flow $(Y_s)_s$. Since $(h(x_n))_n$ is bounded, choosing a convergent subsequence $(h(x_{n_k}))_k$ we conclude that

$$0 < \delta_1 \leq d(X_{h(x_{n_k})\cdot s_{n_k}}(x),\, X_{h(x_{n_k})\cdot s_{n_k}}(x_{n_k})) + d(Y_{s_{n_k}}(x_{n_k}),\, Y_s(x)) \xrightarrow{k\to\infty} 0$$

leading to a contradiction. This proves item (c) and completes the proof of the proposition. $\square$

We are now in a position to complete the proof of Theorem A. The previous argument shows that there exists a continuous map $h : \Lambda \setminus Sing(X) \to \mathbb{R}$ that is constant along orbits of $(X_t)_t$ such that $Y(x) = h(x) \cdot X(x)$ for all $x \in \Lambda \setminus Sing(X)$. Using that the flow $(X_t)_t$ is transitive on Λ, there exists $x_0 \in \Lambda \setminus Sing(X)$ such that $\overline{\{X_t(x_0) : t \in \mathbb{R}_+\}} = \Lambda$. As h is constant along the orbits of $(X_t)_t$ we conclude that $h(x) = h(x_0)$ for every $x \in \Lambda \setminus Sing(X)$. Take $c = h(x_0)$. The later implies that the vector fields Y and cX coincide in an open and dense subset of Λ. Thus $Y = cX$, which proves the triviality of the centralizer.

3. Applications: volume preserving flows, Lorenz attractors and Hamiltonian flows

3.1. **Proof of Corollary 2.** Given $r \geq 1$, let $\mathcal{R} \subset \mathfrak{X}^r(N)$ denote the C^r-residual subset formed by Kupka-Smale vector fields (see e.g. [22]). In particular, all critical elements of vector fields $X \in \mathcal{R}$ are hyperbolic (hence the ones with the same period are isolated). Thus, if $\Lambda \subset \overline{Per(X)}$ is a transitive invariant set for the flow generated by $X \in \mathcal{R}$, the argument used in the first part of the proof of Theorem A assures that all the periodic orbits in $\Lambda \cap \mathrm{Per}(X)$ are preserved by every $Y \in \mathcal{Z}^1(X)$. In particular, the flow generated by Y preserves the set Λ, and the restriction of the vector field to the set Λ has trivial centralizer, that is, there exists $c \in \mathbb{R}$ so that $Y = cX$ on Λ. This proves the first assertion of the corollary.

In the case of volume preserving vector fields one has that $\Omega(X) = N$ for every $X \in \mathfrak{X}^1_\mu(N)$, by Poincaré recurrence theorem. Moreover, there are C^1-generic sets $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \subset \mathfrak{X}^1_\mu(N)$ so that every $X \in \mathcal{R}_1$ generates a topologically mixing (hence transitive) flow [4], that periodic orbits of every vector field $X \in \mathcal{R}_2$ are dense in N [25, Theorem 11.1], and that hyperbolic periodic orbits for vector fields $X \in \mathcal{R}_3$

are dense in the non-wandering set (cf. [25], [26] and [19, Proposition 3.1]). These facts, together with Theorem A prove the second assertion in the corollary and complete its proof.

3.2. **Proof of Corollary 3.** Let $\mathcal{U} \subset \mathfrak{X}^1(\mathbb{R}^3)$ be a C^1-open set of vector fields and an open elipsoid $V \subset \mathbb{R}^3$ containing the origin such that every $X \in \mathcal{U}$ exhibits a geometric Lorenz attractor $\Lambda_X = \bigcap_{t\geq 0} \overline{X_t(V)}$. For every $X \in \mathcal{U}$ there exists a periodic point $p_X \in \Lambda_X$ so that the Lorenz attractor Λ_X coincides with the homoclinic class $H(p_X) := \overline{W^s(p_X) \pitchfork W^u(p_X)}$ (cf. [1, Proposition 3.17]). In particular, the restriction of the flow to the attractor is transitive and, by Birkhoff-Smale's theorem, admits a dense set of hyperbolic periodic orbits. The corollary is then a direct consequence of Theorem A.

3.3. **Proof of Corollary 4.** Assume that $n \geq 2$. First we will verify that vector fields of C^2-generic Hamiltonians and connected components of generic energy levels satisfy the following: the restriction of the Hamiltonian flow to such connected components is transitive and admits a dense set of hyperbolic periodic orbits. Our starting point is the following result, which assures that transitive level sets for C^2-generic Hamiltonians are few from both topological and measure theoretical senses.

Theorem 3.1. [5, Theorems 2 and 3] *There is a residual set $\mathcal{R}_0$ in $C^2(M,\mathbb{R})$ such that for any $H \in \mathcal{R}$ there exists a Baire residual, full Lebesgue measure set $\mathfrak{R}_H$ of regular energy levels in $H(M) \subset \mathbb{R}$ such that for every $e \in \mathfrak{R}_H$ the restriction of the flow $(\varphi_t^H)_t$ to every connected component $\mathcal{E}_{H,e}$ of the level set $H^{-1}(e)$ is transitive.*

In order to prove the corollary we are left to show that hyperbolic periodic orbits are dense in typical level sets of C^2-generic Hamiltonians. Although this argument is contained in [5] we include it here for completeness reasons. Robinson [26] proved that there exists a C^2 residual subset $\mathcal{R}_1 \subset C^2(M,\mathbb{R})$ of Hamiltonians such that for every $H \in \mathcal{R}_1$ all closed orbits are either hyperbolic or elliptic. Note that, by Birkhoff fixed point theorem, the hyperbolic periodic points are dense on M (cf. [19] Proposition 3.1, Corollary 3.2 and §6). Moreover, by Theorem 11.5 in [25], Newhouse's result can be strengthened in a way that there exists a C^2-generic subset $\mathcal{R}_2 \subset C^2(M,\mathbb{R})$ such that the generic connected components of energy levels for $H \in \mathcal{R}_2$ contains a dense set of hyperbolic periodic orbits. The density of hyperbolic periodic orbits and the fact that the ones displaying the same period are isolated guarantees, as in Theorem A, that for every Hamiltonian $H \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2$, every $X_K \in \mathcal{Z}^1_\omega(X_H)$ and every connected component $\mathcal{E}_{H,e} \subset H^{-1}(e)$ associated to an energy $e \in \mathfrak{R}_H$, there exists $h : \mathcal{E}_{H,e} \to \mathbb{R}$ continuous, invariant along regular orbits of X_H, such that $X_K(x) = h(x) \cdot X_H(x)$ for all $x \in \mathcal{E}_{H,e}$. Transitivity at the hypersurface $\mathcal{E}_{H,e}$ assures that the first integral h is constant on $\mathcal{E}_{H,e}$, which complete the proof of the Corollary 4.

4. Examples

In this section we give some examples that illustrate our results. The following simple example, of constant vector fields, is specially interesting when one is interesting in applying the tubular flowbox theorem.

Example 4.1. *Let $X \in \mathfrak{X}^r(\mathbb{R}^n)$ the constant field $X(x) = (1, 0, \cdots, 0)$. If $Y \in \mathcal{Z}^r(X)$ is written in coordinates by $Y(x) = (Y_1(x), \cdots, Y_n(x))$, the vector fields X and Y commute then, in local coordinates,*

$$
\begin{aligned}
0 = [X, Y] &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \right] \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_1} \frac{\partial}{\partial x_i} = \left(\frac{\partial Y_1}{\partial x_1}, \frac{\partial Y_2}{\partial x_1}, \cdots, \frac{\partial Y_n}{\partial x_1} \right).
\end{aligned}
$$

In other words, X and Y commute if and only if $\frac{\partial Y_i}{\partial x_1} = 0$ for all $1 \leq i \leq n$ or, equivalently, the vector field Y does not depend on the x_1-coordinate, hence it is constant along the orbits of the constant vector field. If M is a Riemannian

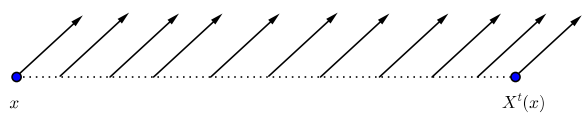

FIGURE 1. Local rigidity on flowbox charts caused by commutativity

manifold of dimension n, $Z \in \mathfrak{X}^r(M)$, $r \geq 1$, and $p \in M$ is a regular point for Z, the tubular flowbox theorem (cf. [22]) assures the existence of a neighborhood V_p of x in M and a C^r-diffeomorphism $h : V_p \to h(V_p) \subset \mathbb{R}^n$ so that $h_* Z = X \mid_{h(V_p)}$. If the vector field $\tilde{Z} \in \mathfrak{X}^r(M)$ commutes with Z commutes on V_p then $0 = [Z, \tilde{Z}] = [h_* Z, h_* \tilde{Z}] = [X, h_* \tilde{Z}]$. Here we used that Lie brackets are invariant by induced vector fields (see e.g. [16, Corollary 8.31]). The latter implies that the vector field $h_* \tilde{Z}$ does not depend on the x_1-coordinate in the local coordinates of $h(V_p) \subset \mathbb{R}^n$.

Now we note that commuting Hamiltonian vector fields may not preserve level sets, which justifies the definition of the Hamiltonian centralizer.

Example 4.2. *Consider the space $\mathbb{R}^2$ endowed with the usual symplectic form $\omega = dx \wedge dy$. Then, it is not hard to check that for any C^{r+1} Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ the Hamiltonian vector field $X_H \in \mathfrak{X}^r(M)$ can be written as $X_H(x, y) = \left(-\frac{\partial H(x,y)}{\partial y}, \frac{\partial H(x,y)}{\partial x} \right)$, for every $(x, y) \in \mathbb{R}^2$. It is not hard to check that the Hamiltonian flows $(\varphi_t^H)_t$, $(\varphi_t^K)_t$ generated by $H, K : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x$ and $K(x, y) = y$, commute. However, the flow $(\varphi_t^K)_t$ does not preserve level sets of $(\varphi_t^H)_t$. In fact, the corresponding Poisson bracket is given by*

$$
\{H, K\} = \frac{\partial H}{\partial x} \frac{\partial K}{\partial y} - \frac{\partial H}{\partial y} \frac{\partial K}{\partial x} = 1 \neq 0.
$$

Example 4.3. *Consider the linear vector field in $\mathbb{R}^2$ given by $X(x, y) = (-y, x)$, that generates the flow $(X_t)_{t \in \mathbb{R}}$ such that X_t is the rotation of angle t in $\mathbb{R}^2$. If $Z(x, y) = (ax + by, cx + dy)$ is a linear vector field in $\mathbb{R}^2$ that commutes with X*

then

$$\begin{aligned} 0 = [X,\, Z] &= \Big(X_1 \frac{\partial Z_1}{\partial x} - Z_1 \frac{\partial X_1}{\partial x} + X_2 \frac{\partial Z_1}{\partial y} - Z_2 \frac{\partial X_1}{\partial y}, \\ &\qquad X_1 \frac{\partial Z_2}{\partial x} - Z_1 \frac{\partial X_2}{\partial x} + X_2 \frac{\partial Z_2}{\partial y} - Z_2 \frac{\partial X_2}{\partial y} \Big) \\ &= \big((b+c)x + (d-a)y,\ \ (d-a)x - (b+c)y \big), \end{aligned}$$

that is, $a = d$ and $c = -b$. Thus, the vector field X commutes with the family of linear vector fields

$$Z_{a,b}(x,\, y) = (ax + by,\, -bx + ay) \tag{4.1}$$

for every $(x,y) \in \mathbb{R}^2$ $(a, b \in \mathbb{R})$. The centralizer of the linear vector field X contains also nonlinear vector fields. For instance, it is not hard to check that the vector field $Y \in \mathfrak{X}^\infty(\mathbb{R}^2)$ given by $Y(x,\, y) = \big(y + x(1-x^2-y^2),\ -x + y(1-x^2-y^2)\big)$ is such that $[X,\, Y] = 0$. This shows that the one parameter group of rotations on $\mathbb{R}^2$,

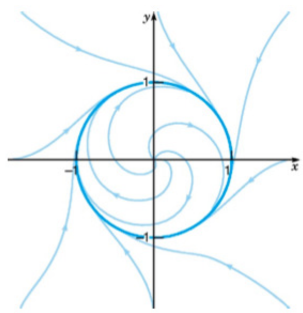

Figure 2. Representation of some solutions of the flow generated by the non-linear vector field $Y(x,\, y) = \big(y + x(1-x^2-y^2),\ -x + y(1-x^2-y^2)\big)$, $(x,y) \in \mathbb{R}^2$.

which correspond to the solutions of the vector field X, are radial symmetries for the solutions of the flow generated by $Y \in \mathcal{Z}^r(X)$ (see Figure 2).

In the next example we consider the Hamiltonian centralizer of a linear center in $\mathbb{R}^2$.

Example 4.4. *Consider the linear vector field $X \in \mathfrak{X}^1(\mathbb{R}^2)$ given in the previous example. It is a Hamiltonian vector field: $X = X_H$ for $H \in C^2(\mathbb{R}^2, \mathbb{R})$ given by $H(x,y) = \frac{1}{2}(x^2 + y^2)$. If $Y \in \mathcal{Z}^1_\omega(X)$, there exists $K \in C^2(\mathbb{R}^2,\, \mathbb{R})$ such that $Y := Y_K = (-\frac{\partial K}{\partial y}, \frac{\partial K}{\partial x})$ and*

$$0 = \{H,\, K\} = \frac{\partial H}{\partial x}\frac{\partial K}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial K}{\partial x} = x\,\frac{\partial K}{\partial y} - y\,\frac{\partial K}{\partial x} = \Big\langle (x,\, y),\ \Big(\frac{\partial K}{\partial y},\, -\frac{\partial K}{\partial x}\Big) \Big\rangle.$$

Hence, the vectors $(x,\, y)$ and $\left(\frac{\partial K}{\partial y},\, -\frac{\partial K}{\partial x}\right)$ are orthogonal. As the vector field $X_H(x,y) = (-y, x)$ is also orthogonal to (x,y) then the vector fields X_H and X_K are collinear and, consequently, there exists $\kappa : \mathbb{R}_+ \to \mathbb{R}$ continuous such that $X_K(x,y) = \kappa(\|(x,y)\|) \cdot X_H(x,y)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. In particular, since the orbits of the flow $(\varphi^H_t)_t$ are circles centered at the origin, the orbit of each point

$(x, y) \in \mathbb{R}^2 \backslash \{(0,0)\}$ by the flow $(\varphi_t^K)_t$ is either a singularity, in which case all points in the circle γ centered at the origin passing through (x, y) are also singularities, or is periodic and the orbit coincides the circle γ. Moreover, $(0,0)$ is a singularity for the vector field X_K. Hence, the Hamiltonian centralizer of X_H is smaller than its centralizer in the space of all vector fields described in Example 4.3 (since there are non-volume preserving flows in the family (4.1)). Nevertheless, the centralizer $\mathcal{Z}^1(X_H)$ is not trivial as, for instance, the Hamiltonian $K(x, y) = \frac{1}{2}(x^2 + y^2)^2$ satisfies $X_K(x, y) = \kappa(\|(x, y)\|)X_H(x, y)$, where $\kappa(z) = 4z^2$ tends to zero as $z \to 0^+$, is continuous but it is not constant.

5. Final comments

In this final section we give further comments on the assumptions of the Theorem A and the C^0-centralizer of continuous flows, and discuss on some open questions. First we note that while the proof of Theorem A is mainly topological, the C^1-smoothness of the vector field (and the hyperbolicity of a dense subset of periodic orbits) is used to assure that the invariant set Λ is preserved by elements in the centralizer, and that hyperbolic periodic orbits are also preserved (recall Lemma 2.1 and the argument following it at page 6). For general vector fields, invariant sets may not be preserved by elements of the centralizer. For instance, any annulus $\mathbb{D}$ centered at the origin is preserved by the flow $(X_t)_t$ in Example 4.3 while the flow generated by the vector field $Y(x, y) = (x, y)$ commutes with $(X_t)_t$ but does not preserve $\mathbb{D}$.

We will discuss further on the C^0-centralizer of flows. Given a C^0-flow $(X_t)_t$ on a compact Riemannian manifold N, denote by $X(x) = \frac{d}{dt}X_t(x) \mid_{t=0}$, $x \in N$, the vector field $X \in \mathfrak{X}^0(N)$ tangent to the flow. With some abuse of notation, we say that $\mathcal{Z}^0(X)$ *is trivial* if for every continuous flow $(Y_t)_t$ that commutes with $(X_t)_t$, the associated C^0-vector field Y satisfies $Y = cX$ for some $c \in \mathbb{R}$.

Given the commuting C^0-flows $(X_t)_t$ and $(Y_t)_t$, the homeomorphism Y_s is a conjugation between the flow $(X_t)_t$ with itself, for every $s \in \mathbb{R}$. In particular, the periodic points of a fixed period for $(X_t)_t$ are preserved by Y_s (recall equation (2.1)): $(Y_s(p))_{s\in\mathbb{R}}$ defines a smooth curve formed by periodic points for $(Y_t)_t$, all of these with constant period $\pi(p)$. If the periodic orbits of $(X_t)_t$ with the same period are isolated, we conclude that $Y_s(\gamma_p) = \gamma_p$ for all $s \in \mathbb{R}$. Using *ipsis literis* the arguments in the proof of Theorem A we also have the following:

Theorem A' *Let N be a compact Riemannian manifold and let $(X_t)_t$ be a continuous flow generated by a C^0-vector field X such that*

1. *$(X_t)_t$ is transitive,*
2. *$Per((X_t)_t)$ is dense in N, and*
3. *periodic orbits of $(X_t)_t$ with the same period are isolated,*

then $\mathcal{Z}^0(X)$ is trivial.

In this paper we provided a positive solution to the problem proposed by Smale in the context of C^1-volume preserving vector fields. In [6], the authors introduced a notion of unbounded distortion to prove that C^1-generic diffeomorphisms have trivial centralizer. It is an interesting open question wether that method can be pushed to the context of C^1-generic vector fields.

Finally, it is natural to expect our results to extent to the setting of transitive and locally free $\mathbb{R}^d$-actions with a dense set of closed orbits. Although some open classes of $\mathbb{R}^d$-actions can be obtained as suspensions of $\mathbb{Z}^d$-actions (see e.g. [8] and references therein), to the best of our knowledge, the existence of closed orbits is itself a wide open problem in this context with just a contribution in the case of $\mathbb{R}^2$ actions on 3-manifolds [29].

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W. Bonomo, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil
E-mail address: wescleybonomo@yahoo.com.br

Paulo Varandas, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil
E-mail address: paulo.varandas@ufba.br
URL: http://www.pgmat.ufba.br/varandas