# OPEN SETS OF AXIOM A FLOWS WITH EXPONENTIALLY MIXING ATTRACTORS

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Abstract. For any dimension $d \geq 3$ we construct $\mathcal{C}^1$-open subsets of the space of $\mathcal{C}^3$ vector fields such that the flow associated to each vector field is Axiom A and exhibits a non-trivial attractor which mixes exponentially with respect to the unique SRB measure.

## 1. Introduction

The Axiom A flows of Smale [24] have been extensively studied in the last four decades and are now relatively well understood. One important remaining question concerns the rate of mixing for such flows. Let $M$ be a Riemannian manifold (dimension $d \geq 3$) and let $\Lambda$ be a basic set for an Axiom A flow $X^t : M \to M$. This means that $\Lambda \subset M$ is an invariant, closed, topologically transitive, locally maximal hyperbolic set. A basic set $\Lambda$ is called an *attractor* if there exists a neighbourhood $U$ of $\Lambda$, and $t_0 > 0$, such that $\Lambda = \cap_{t \in \mathbb{R}_+} X^t U$. A basic set is *non-trivial* if it is neither an equilibrium nor a periodic solution. In this article we focus on the SRB measure; this is the invariant probability measure which is characterised by having absolutely continuous conditional measures on unstable manifolds. It is known that for every attractor of an Axiom A flow there exists a unique SRB measure. In this setting it is known that the unique SRB measure is also the unique physical measure and is also the Gibbs measure associated to the potential chosen as minus the logarithm of the unstable Jacobian [26]. We say that an invariant probability measure $\mu$ is mixing if the correlation function $\rho_{\phi,\psi}(t) := \int_\Lambda \phi \circ X^t \cdot \psi \, d\mu - \int_\Lambda \phi \, d\mu \cdot \int_\Lambda \psi \, d\mu$ tends to zero for $t \to +\infty$, for all bounded measurable observables $\phi, \psi : U \to \mathbb{R}$. We say that it mixes exponentially if, for any fixed Hölder exponent $\alpha \in (0,1)$, there exist $\gamma, C > 0$ such that, for all $\phi$, $\psi$ which are $\alpha$-Hölder on $U$, $|\rho_{\phi,\psi}(t)| \leq C \left\|\phi\right\|_{\mathcal{C}^\alpha} \left\|\psi\right\|_{\mathcal{C}^\alpha} e^{-\gamma t}$ for all $t \geq 0$.

The conjecture that all mixing Axiom A flows, with respect to Gibbs measures for Hölder continuous potentials, mix exponentially was proven false by considering suspension semiflows with piecewise constant return times [23] and Pollicott [19] constructed examples with arbitrarily slow mixing rates. Building on work by Chernov [8], Dolgopyat [9] demonstrated that $\mathcal{C}^{2+\varepsilon}$ transitive Anosov flows with $\mathcal{C}^1$ stable and unstable foliations which are jointly nonintegrable mix exponentially

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with respect to the SRB measure and Hölder continuous observables. Unfortunately this good regularity of the invariant foliations is not typical [13]. Dolgopyat also showed [10] that rapid mixing (superpolynomial) is typical, in a measure theoretic sense of prevalence, for Axiom A flows with respect to any equilibrium state associated to a Hölder potential. Building on these ideas Field, Melbourne and Török [12] showed that there exist $\mathcal{C}^2$-open, $\mathcal{C}^r$-dense sets of $\mathcal{C}^r$-Axiom A flows ($r \geq 2$) for which each non-trivial basic set is rapid mixing.

It is tempting to think that exponential mixing is a robust property, i.e., if an Axiom A flow mixes exponentially then all sufficiently close Axiom A flows also mix exponentially. This remains an open problem, even limited to the case of Anosov flows. Dolgopyat [11] studied suspension semiflows over topologically mixing subshifts of finite type with respect to Gibbs measures and showed that an open and dense subset (in the Hölder topology) mix exponentially. He conjectured [11, Conjecture 1] that the set of exponentially mixing flows contains a $\mathcal{C}^r$-open, $\mathcal{C}^r$-dense subset of the set of all Axiom A flows. Unfortunately the previous mentioned result does not help in proving this conjecture since the Hölder topology is pathological for these purposes (for details see the second remark after Theorem 1.1 in [11]). Liverani [16] was able to bypass the regularity of the stable and unstable invariant foliations and show exponential mixing for $\mathcal{C}^4$ contact Anosov flows using the contact structure, an observation that suggests that the regularity of the invariant foliations is not essential. Unfortunately, contact flows are a thin subset of Anosov flows (or Axiom A flows), in particular there do not exist open subsets of Axiom A flows which preserve a contact structure. Exponential rates of mixing were proved for $\mathcal{C}^2$ uniformly expanding surface suspension semiflows by Baladi and Vallée [4] under the assumption that the return time function is cohomologous to a piecewise constant function. This was extended to arbitrary dimension by Avila, Gouëzel, and Yoccoz [3]. In a recent preprint, Araújo and Melbourne [1] extended [4] relaxing $C^2$ to $C^{1+\alpha}$. These results for non-invertible flows can be applied to invertible systems (e.g., [3]) when the stable foliation is of sufficient regularity. For a more complete history of the question of mixing rates of hyperbolic flows we refer to the reader the introductions of [16, 12].

The purpose of this paper is to construct open sets of Axiom A flows which mix exponentially with respect to the SRB measure of its attractor, thus making the step from open sets of symbolic flows to open sets of Axiom A flows at the cost of assuming that the stable foliation is $\mathcal{C}^2$-smooth. The existence of such open sets of Axiom A flows was expected [11]. The first and third author previously constructed [2] open sets of three-dimensional singular hyperbolic flows (geometric Lorenz attractors) which mix exponentially with respect to the unique SRB measure. It was unclear if the singularity actually aids the mixing and allows for the robust exponential mixing. In this article we show that we do not need the singularity but actually we can just take advantage of the volume contraction of the flow (and consequently a domination condition) in order to carry out a similar construction.

## 2. Results & Outline of the Proof

Given an Axiom A flow $X^t : M \to M$ associated to a vector field $X$ we consider the $\mathcal{C}^r$ distance on the space of $\mathcal{C}^r$-vector fields $\mathfrak{X}^r(M)$, that induces a natural distance on the space of flows. The following two theorems are the main results of this article.

**Theorem A.** *Given any Riemannian manifold $M$ of dimension $d \geq 3$ there exists a $\mathcal{C}^1$-open subset of $\mathcal{C}^3$-vector fields $\mathcal{U} \subset \mathfrak{X}^3(M)$ such that for each $X \in \mathcal{U}$ the associated flow is Axiom A and exhibits a non-trivial attractor which mixes exponentially with respect to the unique SRB measure.*

As far as the authors are aware, this is the first result concerning the existence of robustly exponentially mixing Axiom A flows. The strategy for the construction of the open sets in the above theorem is similar to the one developed in [2] for singular flows. Theorem A is a consequence (details in Section 4) of the following more fundamental result.

**Theorem B.** *Suppose that $X^t : M \to M$ is a $\mathcal{C}^2$ Axiom A flow, $\Lambda$ is an attractor, and that the stable foliation is $\mathcal{C}^2$. If the stable and unstable foliations are not jointly integrable then the flow mixes exponentially with respect to the unique SRB measure for $\Lambda$.*

The proof of the above is described in Section 3 and involves quotienting along stable manifolds of a well-chosen Poincaré section to reduce to the case of a suspension semiflow over an expanding Markov map. We can then apply the result of [3] which implies exponential mixing for the semiflow unless the return time function is cohomologous to a piecewise constant function. This is then related to the exponential mixing for the original flow and the joint integrability of the stable and unstable foliations. It is expected that [1] to allow the $\mathcal{C}^2$ requirement for the stable foliation to be weakened to $\mathcal{C}^{1+\alpha}$. We have no reason to believe that the requirement of good regularity (better than $\mathcal{C}^1$) of the stable foliation is essential to the above theorem, however the present methods rely heavily on this fact. Note that although we require this good regularity of the stable foliation we have no requirements on the regularity of the unstable foliation. We observe that the required good regularity of the stable foliation can hold robustly, in contrast to Dolgopyat's original argument [9] which required $\mathcal{C}^1$ regularity for both the stable and unstable foliations.

The following questions remain: Are all exponentially mixing Axiom A flows also robustly exponentially mixing? And the above mentioned conjecture, for any $r > 1$, does the set of exponentially mixing Axiom A flows contain an $\mathcal{C}^r$-open and dense subset of the set of all Axiom A flows? It would appear that both these questions are of a higher order of difficulty.

The joint nonintegrability of stable and unstable foliations (as required in Theorem B) can be seen in several different ways. The stable and unstable foliations of an Axiom A flow are always transversal, consequently, if they are jointly integrable, this provides a codimension one invariant foliation which is transversal to the flow direction. Conversely, if there exists a codimension one invariant foliation which is transversal to the flow direction, then this foliation must be subfoliated by both the stable and unstable foliations which must therefore be jointly integrable. In this case it is known [12, Proposition 3.3] that the flow is (bounded-to-one) semiconjugate to a locally constant suspension over a subshift of finite type. Such flow need not mix, or may mix slower than exponentially.

The additional ingredient in order to use Theorem B to prove Theorem A is a result concerning the regularity of foliations. Let $\|\cdot\|$ denote the Riemannian norm on the tangent space of $M$. As before $X^t : M \to M$ is an Axiom A flow and $\Lambda$ is a non-trivial basic set. Since the flow is Axiom A the tangent bundle restricted to $\Lambda$ can be written as the sum of three $DX^t$-invariant continuous subbundles, $T_\Lambda M = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$ where $\mathbb{E}^c$ is the one-dimensional bundle tangent to the flow and there exists $C, \lambda > 0$ such that $\| DX^t|_{\mathbb{E}^s} \| \leq Ce^{-\lambda t}$, and $\| DX^{-t}|_{\mathbb{E}^u} \| \leq Ce^{-\lambda t}$, for all $t \geq 0$.

**Theorem 1.** *Suppose that $\Lambda$ is an attractor for the $\mathcal{C}^3$ Axiom A flow $X^t : M \to M$. Further suppose that*

$$\sup_{x\in\Lambda} \left\| DX_x^{t_0}\big|_{\mathbb{E}^s} \right\| \cdot \|DX_x^{t_0}\|^2 < 1, \tag{1}$$

*for some $t_0 > 0$. Then the stable foliation of $X^t$ is $\mathcal{C}^2$.*

This is a well-known consequence of the arguments described by Hirsch, Pugh, and Shub [15] and also in the proof of [14, Theorem 6.1]. We refer the reader to Section 6 in [21] for a discussion on three definitions for $C^k$-smoothnesss of foliations. We observe that in our context, where $k > 1$, the results from [14, 15] provide the strongest of these three notions, where the smoothness of the stable foliation is obtained when it is regarded as a section into a certain Grassmannian (cf. [21]). This is enough to guarantee that local sections obtained by collecting local stable manifolds through a local unstable manifold and stable holonomies are indeed $C^2$ smooth (cf. construction of Poincaré sections and return maps in Section 3).

This question of regularity is a subtle issue. For Anosov flows, one cannot in general expect the stable foliation (or equivalently the unstable foliation) to have better regularity than Hölder [13]. If the invariant foliation of interest has codimension one, then $\mathcal{C}^1$ regularity can be obtained; see e.g. [18, Appendix 1]. However the stable foliation of a hyperbolic flow can never have codimension one since in the splitting $T_\Lambda M = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$ the complementary direction to $\mathbb{E}^s$ is $\mathbb{E}^c \oplus \mathbb{E}^u$.

We will construct open sets of Axiom A flows which satisfy the assumptions of Theorem 1. Note that the domination condition (1) implies that the flows we consider are volume contracting and so the attractor has zero volume and consequently the flow is necessarily not Anosov. Since the domination condition is an open condition it implies that there exist open sets of Axiom A flows which possess a $\mathcal{C}^2$ stable foliation. Since the non-joint integrability of stable and unstable foliations is an open and dense condition (see e.g. [12, Remark 1.10] and references therein), to prove that robust exponential mixing does exist we apply Theorem B. Details of this argument are given in Section 4.

## 3. Axiom A attractors with $C^2$-stable foliation.

The purpose of this section is to prove Theorem B. For the duration of this section we fix $X \in \mathfrak{X}^2(M)$ where $M$ is $d$-dimensional and assume that $X^t : M \to M$ is a $\mathcal{C}^2$ Axiom A flow and $\Lambda \subset M$ is an attractor (in particular topologically transitive) with a $\mathcal{C}^2$ stable foliation.

Given $x \in \Lambda$ we denote by $W^s_\varepsilon(x) = \{y \in M : d(X^t(y), X^t(x)) \leq \varepsilon, \ \forall t \geq 0 \text{ and } d(X^t(y), X^t(x)) \to 0 \text{ as } t \to +\infty\}$ the *local (strong) stable manifold* of $x$ which is forward invariant, that is, $X^t(W^s_\varepsilon(x)) \subset W^s_\varepsilon(X^t(x))$ for every $x \in \Lambda$ and $t \geq 0$. Consider also the *local centre-stable manifold* of $x$ defined as $W^{cs}_\varepsilon(x) = \cup_{|t|\leq\varepsilon} X^t(W^s_\varepsilon(x))$. The local (strong) unstable and centre-unstable manifolds, $W^u_\varepsilon(x)$ and $W^{cu}_\varepsilon(x)$ respectively, correspond to the local strong stable and local centre-stable manifolds for the flow $(X^{-t})_t$. It is known [5] that the attractor $\Lambda$ has a local product structure. This means that there exists an open neighbourhood $\mathcal{J}$ of the diagonal of $M \times M$ and $\varepsilon > 0$ such that, for all $(x, y) \in \mathcal{J}_\Lambda := \mathcal{J} \cap (\Lambda \times \Lambda)$, then $W^{cu}_\varepsilon(x) \cap W^s_\varepsilon(y) \neq \emptyset$ and $W^u_\varepsilon(x) \cap W^{cs}_\varepsilon(y) \neq \emptyset$ each consist of a single point and this intersection point belongs to $\Lambda$. In this case it makes sense to consider the continuous maps $[\cdot,\cdot]_s : \mathcal{J}_\Lambda \to \Lambda$ and $[\cdot,\cdot]_u : \mathcal{J}_\Lambda \to \Lambda$ defined by

$$W^{cu}_\varepsilon(x) \cap W^s_\varepsilon(y) = \{[x,y]_s\}, \quad W^u_\varepsilon(x) \cap W^{cs}_\varepsilon(y) = \{[x,y]_u\}.$$

Following [12, §4.1] we note that the set $\mathcal{J}$, and $\varepsilon > 0$ may be chosen fixed for some $\mathcal{C}^1$-open set $\mathcal{U} \subset \mathfrak{X}^1(M)$ containing any given initial Axiom A flow.

**Definition 2** ([5])**.** *A differentiable closed $(d-1)$-dimensional disk $\mathcal{S} \subset M$, transverse to the flow direction is called a* local cross-section*. A set $\mathcal{R} \subset \Lambda \cap \mathcal{S}$ is called a* rectangle *if $W^{cs}_\varepsilon(x) \cap W^{cu}_\varepsilon(y) \cap \mathcal{R}$ consists of exactly one point for all $x, y \in \mathcal{R}$.*

Let $\mathcal{R}^*_i$ denote the interior of $\mathcal{R}_i$ as a subset of the metric space $\mathcal{S}_i \cap \Lambda$.

**Definition 3** ([5])**.** *A finite set of rectangles $\mathbf{R} = \{\mathcal{R}_i\}_i$ is called a* proper family *(of size $\varepsilon$) if*

(1) $\Lambda = \cup_{t\in[-\varepsilon,0]} X^t (\cup_i \mathcal{R}_i) =: X^{[-\varepsilon,0]} (\cup_i \mathcal{R}_i)$,

*and there exist local sections $\{\mathcal{S}_i\}_i$ of diameter less than $\varepsilon$ such that*

(2) $\mathcal{R}_i \subset \operatorname{int}(\mathcal{S}_i)$ *and* $\mathcal{R}_i = \overline{\mathcal{R}^*_i}$,

(3) *For $i \neq j$, at least one of the sets $\mathcal{S}_i \cap X^{[0,\varepsilon]}\mathcal{S}_j$ and $\mathcal{S}_j \cap X^{[0,\varepsilon]}\mathcal{S}_i$ is empty.*

Given a proper family as above, let $\Gamma := \cup_i \mathcal{R}_i$, and denote by $P$ the Poincaré return map to $\Gamma$ associated to the flow $X^t$, and by $\tau$ the return time. Although $P$ and $\tau$ are not continuous on $\Gamma$, they are continuous on

$$\Gamma' := \left\{x \in \Gamma : P^k(x) \in \cup_i \mathcal{R}^*_i \text{ for all } k \in \mathbb{Z}\right\}.$$

**Definition 4** ([5])**.** *A proper family $\mathbf{R} = \{\mathcal{R}_i\}_i$ is a* Markov family *if*

(1) $x \in U(\mathcal{R}_i, \mathcal{R}_j) := \overline{\{w \in \Gamma' : w \in \mathcal{R}_i, P(w) \in \mathcal{R}_j\}}$ *implies* $\mathcal{S}_i \cap W^{cs}_\varepsilon(x) \subset U(\mathcal{R}_i, \mathcal{R}_j)$,

(2) $y \in V(\mathcal{R}_k, \mathcal{R}_i) := \overline{\{w \in \Gamma' : P^{-1}(w) \in \mathcal{R}_k, w \in \mathcal{R}_i\}}$ *implies* $\mathcal{S}_i \cap W^{cu}_\varepsilon(y) \subset V(\mathcal{R}_k, \mathcal{R}_i)$.

It follows from [5, Theorem 2.4] that for any Markov family $\mathbf{R} = \{\mathcal{R}_i\}$ the flow is (bounded-to-one) semiconjugate to a suspension flow, with a bounded roof function also bounded away from zero, over the subshift of finite type $\sigma_{\mathbf{R}} : \Sigma_{\mathbf{R}} \to \Sigma_{\mathbf{R}}$ where $\Sigma_{\mathbf{R}} = \{(a_i)_i \in \mathbb{Z} : A_{a_i a_j} = 1 \ \forall i, j\}$ and $A_{a_i a_j} = 1$ if and only if there exists $x \in \Gamma' \cap \mathcal{R}_i$ and $P(x) \in \mathcal{R}_j$.

The first step in the proof of Theorem B is to carefully choose local cross-sections for the flow near the attracting basic set $\Lambda$. Since $\Lambda$ is an hyperbolic attractor then the local unstable manifold of each point of the attractor is contained within the attractor; see e.g. [6]. Hence for any $x \in \Lambda$ and small enough $\varepsilon > 0$ we have $W^u_\varepsilon(x) \subset \Lambda$ and the $(d-1)$-submanifold generated by the union of local stable manifolds through points of $W^u_\varepsilon(x)$,

$$\mathcal{S}_\varepsilon(x) = \bigcup_{y \in W^u_\varepsilon(x)} W^s_\varepsilon(y), \tag{2}$$

is a local cross-section containing $x$. Moreover, since the stable foliation of $X^t : \Lambda \to \Lambda$ is $\mathcal{C}^2$ and $W^u_\varepsilon(x)$ is a $\mathcal{C}^2$-disk in $W^u(x)$, then $\mathcal{S}_\varepsilon(x)$ is a codimension one $\mathcal{C}^2$-hypersurface foliated by $\mathcal{C}^2$ local stable manifolds. In addition, there exists a natural projection $\pi_{\varepsilon,x} : S_\varepsilon(x) \to W^u_\varepsilon(x)$ through the stable leaves which is of class $C^2$, by construction. In general there is no reason to expect that these local cross-sections will be foliated by local unstable manifolds even though it contains one local unstable manifold at the centre.

**Lemma 5.** *Let $X^t : M \to M$ be an Axiom A flow and $\Lambda \subset M$ an attractor. There exists a finite number of $\mathcal{C}^2$ local cross-sections $\mathcal{S} = \{\mathcal{S}_i\}_i \subset U$ such that the sets $\mathcal{R}_i = \mathcal{S}_i \cap \Lambda$ are rectangles and $\mathbf{R} = \{\mathcal{R}_i\}_i$ is a Markov family for $X^t$. Moreover, each hypersurface $\mathcal{S}_i$ is subfoliated by strong stable leaves and for each rectangle $\mathcal{R}_i$ there exists $x_i \in \mathcal{R}_i$ and a $\mathcal{C}^2$-disk $\Delta_i \subset W^u_\varepsilon(x_i) \subset \Lambda$ such that $\mathcal{S}_i = \cup_{y\in\Delta_i}\gamma^s(y)$ where $\gamma^s(y)$ is an open subset of $W^s_\varepsilon(y)$ that contains $y$. In addition, the projection $\pi_i : \mathcal{S}_i \to \Delta_i$ along stable manifolds is $C^2$ smooth.*

*Proof.* We have shown (2) that through each point $x$ of $\Lambda$ there passes a $C^2$ codimension one disk transversal to the flow, whose diameter can be made arbitrarily small, formed by the union of stable leaves through the points of $W^u_\varepsilon(x)$, and these disks have a $C^2$ smooth projection along the stable leaves onto the unstable disk $W^u_\varepsilon(x)$. This is the starting point of the proof of [5, Theorem 2.5], detailed in [5, Section 7], to show that the $X^t : \Lambda \to \Lambda$ admits a Markov family as in the statement of lemma, consisting of a finite number of rectangles of arbitrarily small size $\varepsilon$ contained in the interior of cross-sectional disks $\mathcal{S} = \{\mathcal{S}_i\}_i \subset U$, each one endowed with an unstable disk $\Delta_i \subset \mathcal{S}_i$ and a projection $\pi_i$, as in the statement.

The extra properties of these disks are consequences of the initial construction of smooth local cross-sections to the flow taking advantage of the fact that $\Lambda$ is an attractor with a $C^2$ smooth stable foliation.[1] □

Let $\mathcal{S}$ be as given by Lemma 5. Now we consider the flow on $U \supset \Lambda$ as a suspension flow, return map $P : \mathcal{S} \to \mathcal{S}$ and return time $\tau : \mathcal{S} \to [\underline{\tau}, \bar{\tau}]$ for some fixed $0 < \underline{\tau} < \bar{\tau} < \infty$. Let $\mathcal{F}_s$ denote the foliation of $U$ by local stable manifolds. Let $\Delta = \mathcal{S}/\mathcal{F}_s$ (the quotient of $\mathcal{S}$ with respect to the local stable manifolds). A concrete realization of this quotient is given by $\Delta = \cup_i \Delta_i$. A key point in this construction is that the return time $\tau$ is constant along the local stable manifolds and $C^2$-smooth. Quotienting along these manifolds we obtain a suspension semiflow over an expanding map $f : \Delta \to \Delta$. The fact that, by Lemma 5, the local cross-section $\mathcal{S} = \{\mathcal{S}_i\}_i$ is foliated by local stable manifolds is essential. We write $\widetilde{\mathbf{m}}$ for the normalised restriction of the Riemannian volume to the family $\Delta$ of $C^2$ disks $\{\Delta_i\}_i$. Since $\mathbf{R}$ is a Markov family, this ensures the Markov structure of $f : \Delta \to \Delta$ and hence, for each $\Delta_i$, there exists a partition $\{\Delta_{i,j}\}_j$ of a full $\widetilde{\mathbf{m}}$-measure subset of $\Delta_i$ such that $f : \Delta_{i,j} \to \Delta_j$ is a bijection. The $\mathcal{C}^2$-smooth regularity of the local cross-sections $\mathcal{S}_i$ and the flow $X^t$ is enough to guarantee that the return map $P : \mathcal{S} \to \mathcal{S}$ is also $\mathcal{C}^2$ on each component. Because the projection $\pi$ is also $\mathcal{C}^2$-smooth, we conclude that $f : \Delta_{i,j} \to \Delta_j$ is a $\mathcal{C}^2$ diffeomorphism for each $i$, $j$. For future convenience, let $\pi : \mathcal{S} \to \Delta$ denote the collection of the projections $\pi_i$, so that $f \circ \pi = \pi \circ P$. Since the return time function is constant along stable leaves, we also denote the return time function on $\Delta$ by $\tau : \Delta \to \mathbb{R}_+$.

Subsequently we wish to make the connection to the flows studied in [3]. It is therefore convenient to work with a full branch expanding map whereas the quotient return map $f : \cup_{i,j}\Delta_{i,j} \to \Delta$ is a transitive Markov expanding map but might not

[1]As pointed out to us by Mark Pollicott, a similar idea to the above was used by Ruelle [22], modifying the construction of Bowen [5] to produce local sections with improved regularity.

be full branch. We consider an induced map to guarantee the full branch property. Let $F$ denote the first return map to some element $\Delta_0$ of the Markov partition of $f$ (the choice of $\Delta_0$ is arbitrary and we could choose $\Delta_0$ from a refinement of the partition and proceed identically). This induced system is a full branch Markov map $F = f^R : \cup_\ell \Delta_0^{(\ell)} \to \Delta_0$ where $\{\Delta_0^{(\ell)}\}_\ell$ is a countable partition (the $\Delta_0^{(\ell)}$ are open sets) of a full measure subset of $\Delta_0$ and the first return time function $R : \Delta_0 \to \mathbb{N}$ is constant on each $\Delta_0^{(\ell)}$. We define the induced return map $\widehat{F}$ over $\widehat{\Delta} := \pi^{-1}\Delta_0$ by $\widehat{F}(x) = P^{R(\pi(x))}(x)$ and the induced return time $r := \sum_{j=0}^{R\circ\pi-1} \tau \circ P^j$. Let $\mathbf{m}$ denote the normalised restriction of the Riemannian volume to $\Delta_0$. For future convenience let $\widehat{\Delta}_0^{(\ell)} := \pi^{-1}\Delta_0^{(\ell)}$ for each $\ell$. Observe that $\widehat{F}$ and $r$ are the return map and return time respectively of the flow $X^t$ to the section $\widehat{\Delta} \subset U$ since $F$ was chosen as the first return of $f$ to $\Delta_0$. Let $\mathcal{S}_r = \{(x, u) : x \in \widehat{\Delta}, 0 \le u < r(x)\}$ be the phase space of the suspension semiflow[2] $F_t : \mathcal{S}_r \to \mathcal{S}_r$ which is defined by

$$F_t(x, u) = \begin{cases} (x, u + t) & \text{whenever } u + t < r(x) \\ (\widehat{F}(x), 0) & \text{whenever } 0 \le u + t - r(x) < r(\widehat{F}(x)). \end{cases} \tag{3}$$

The flow is defined for all $t \ge 0$ assuming that the semigroup property $F_{t+s} = F_t \circ F_s$ holds. The suspension flow is conjugated to the original flow $X^t : U \to U$ for some neighbourhood $U$ of the attractor by

$$\Phi : \mathcal{S}_r \to U; \quad (x, u) \mapsto X^u(x). \tag{4}$$

Note that $\Phi$ is invertible since $\widehat{F}$ is the first return to $\widehat{\Delta}_0$. Furthermore $\Phi$ directly inherits the good regularity of $X^t$.

In summary, up until this point, this section has been devoted to choosing the local section $\widehat{\Delta} \subset U$ and so representing $X^t$ as a suspension with return map $\widehat{F}$ and return time $r$. The special feature of this choice is that $r$ is constant on the stable leaves in this local section, it preserves the required regularity and is of the correct form to apply the results of [3] as we will see shortly. The previous choice of local section $\mathcal{S}$ produced a transitive Markov return map $P$ and bounded return time $\tau$. Unfortunately the quotiented return map $f$ need not be full branch. On the other hand $F$ is full branch Markov (countable partition) but now the return time could be unbounded.

**Lemma 6.** *Let $F : \cup_\ell \Delta_0^{(\ell)} \to \Delta_0$ be as defined above. The following hold:*

1. *For each $\ell$, $F : \Delta_0^{(\ell)} \to \Delta_0$ is a $\mathcal{C}^1$ diffeomorphism;*
2. *There exists $\lambda \in (0, 1)$ such that $\|DF^{-1}(x)\| \le \lambda$ for all $x \in \cup_\ell \Delta_0^{(\ell)}$;*
3. *Let $J$ denote the inverse of the Jacobian of $F$ with respect to $\mathbf{m}$. The function $\log J$ is $\mathcal{C}^1$ and $\sup_h \|D((\log J) \circ h)\|_{\mathcal{C}^0} < \infty$, where the supremum is taken over every inverse branch $h$ of $F$.*

The statement of the above lemma is precisely the definition of a $\mathcal{C}^2$ uniformly-expanding full-branch Markov map [3, Definition 2.2]. The only difference is that the reference is more general, the domain is there required just to be what they term a "John domain" [3, Definition 2.1], it is immediate that $(\Delta_0, \mathbf{m})$ satisfies the

[2] Note that the skew product $\widehat{F}$ is invertible on its image but it is not onto. It is in this sense that the corresponding suspension is a semiflow and not a flow and corresponds to the fact that the attractor we are considering has zero volume. This is the same use of terminology as [3].

requirements since $\Delta_0$ is a $\mathcal{C}^2$ disk and $\mathbf{m}$ is the restriction of a smooth measure to $\Delta_0$.

*Proof of Lemma 6.* The required regularity of $F$ and $\log J$ are satisfied since $F$ is $\mathcal{C}^2$. The uniform hyperbolicity of the flow means that there exists $C > 0$, $\tilde{\lambda} \in (0, 1)$ such that $\|Df^{-n}(x)\| \leq C\tilde{\lambda}^n$ for all $x \in \Delta_0$, $n \in \mathbb{N}$. The definition of the induced return map was based on the choice of some element of the Markov partition which we denoted by $\Delta_0$. By choosing first a refinement of the Markov partition and then choosing $\Delta_0$ from the refined partition we may guarantee that the inducing time $R$ is as large as we require. Since $\|DF^{-1}(x)\| \leq C\tilde{\lambda}^{R(x)}$ the uniform expansion estimate of item (2) follows for $\lambda = C \sup_x \tilde{\lambda}^{R(x)} \in (0, 1)$. Finally, since $\|D(\log Jf)\|_{\mathcal{C}^0}$ is bounded

$$\begin{aligned}\|D((\log JF) \circ h)\|_{\mathcal{C}^0} &= \Big\|D(\sum_{j=0}^{R-1} \log Jf \circ f^j \circ h)\Big\|_{\mathcal{C}^0} \\ &\leq \sum_{j=0}^{R-1} \Big\|D(\log Jf)(f^j \circ h) D(f^j \circ h)\Big\|_{\mathcal{C}^0} \\ &\leq C \sum_{j=0}^{R-1} \Big\|D(f^j \circ h)\Big\|_{\mathcal{C}^0} \leq C \sum_{j\geq 0}^{R-1} \lambda^{R-j} \leq C \sum_{k\geq 0}^{\infty} \lambda^k.\end{aligned} \tag{5}$$

is uniformly bounded for all inverse branches $h$ of $F$, proving item (3). □

Since $F$ is a uniformly expanding full-branch Markov map there exists a unique invariant probability measure which is absolutely continuous with respect to $\mathbf{m}$. We denote this by $\nu$ and its density by $\varphi = \frac{d\nu}{d\mathbf{m}}$. Moreover we know that $\varphi \in \mathcal{C}^1(\Delta_0, \mathbb{R}^+)$ and is bounded away from zero.

Let $d_u$, $d_s$ denote the dimension of $\mathbb{E}^u$, $\mathbb{E}^s$ respectively. Taking advantage of the smoothness of the stable foliation and the smoothness of the section we may assume that $\widehat{\Delta} = \Delta_0 \times \Omega$ where $\Delta_0$ is a $d_u$-dimensional ball, $\Omega$ is a $d_s$-dimensional ball (in particular compact) and that $\widehat{F} : (x, z) \mapsto (Fx, G(x, z))$ where $F$ is as before the uniformly expanding $\mathcal{C}^2$ Markov map and $G$ is $\mathcal{C}^2$ and contracting in the second coordinate. This puts us in the setting of [7].

Given $v : \widehat{\Delta}_0 \to \mathbb{R}$, define $v_+, v_- : \Delta_0 \to \mathbb{R}$ by setting $v_+(x) = \sup_z v(x, z)$, $v_-(x) = \inf_z v(x, z)$. Using that $\nu$ is an $F$-invariant probability measure and the contraction in the stable direction we know that the limits

$$\lim_{n\to\infty} \int_{\Delta_0} (v \circ \widehat{F}^n)_+ \, d\nu \quad \text{and} \quad \lim_{n\to\infty} \int_{\Delta_0} (v \circ \widehat{F}^n)_- \, d\nu \tag{6}$$

exist and coincide for all $v$ continuous. Denote the common limit by $\eta(v)$. This defines an $\widehat{F}$-invariant probability measure $\eta$ on $\widehat{\Delta_0}$ and $\pi_*\eta = \nu$ (see, for example, [7, Proposition 1]).

Let $\mathcal{L}$ denote the transfer operator given by $\int_\Delta g \circ F \cdot v \, d\nu = \int_\Delta g \cdot \mathcal{L}v \, d\nu$. By [7, Propostion 2], for any $v \in \mathcal{C}^0(\widehat{\Delta}, \mathbb{R})$, the limit

$$\eta_x(v) := \lim_{n\to\infty} (\mathcal{L}^n v_n)(x), \quad v_n(x) := v \circ \widehat{F}^n(x, 0), \tag{7}$$

exists for all $x \in \Delta$ and defines a probability measure supported on $\pi^{-1}(x)$ (without loss of generality we may assume that 0 denotes some element of $\Omega$). Moreover

$x \mapsto \eta_x(v)$ is continuous and

$$\eta(v) = \int_\Delta \eta_x(v)\ d\nu(x). \tag{8}$$

Despite the fact that $\eta$ is singular along stable manifolds, we can take advantage of the regularity of the skew product form of $\widehat{F}$ (due to the regularity of the stable foliation) and the uniform hyperbolicity in order to prove rather good regularity for the decomposition of $\eta$ into $\{\eta_x\}_{x\in\Delta_0}$. By [7, Propostion 9], this decomposition is $\mathcal{C}^1$ in the sense that there exists $C > 0$ such that, for any open set $\omega \subset \Delta$ and for any $v \in \mathcal{C}^1(\widehat{\Delta}, \mathbb{R})$, the function $x \mapsto \bar{v}(x) := \eta_x(v)$ is $\mathcal{C}^1$ and

$$\sup_{x\in\omega} \|D\bar{v}(x)\| \le C \sup_{(x,z)\in\pi^{-1}\omega} |v(x,z)| + C \sup_{(x,z)\in\pi^{-1}\omega} \|Dv(x,z)\|. \tag{9}$$

Since $\eta$ is an $\widehat{F}$-invariant probability measure $\widehat{\eta} := \frac{1}{\eta(r)}\eta \times \mathrm{Leb}$ is an $F_t$-invariant probability measure. Since this measure has absolutely continuous conditional measure on unstable manifolds due to the connection to the absolutely continuous $\nu$, we know that $\mu = \Phi_*\widehat{\eta}$ is the unique SRB measure for $X^t : U \to U$.

*Remark* 7. In the following we will crucially use the result of [3] concerning exponentially mixing hyperbolic suspension semiflows and consequently it is essential that $\nu$ is absolutely continuous with respect to Riemannian volume on $\Delta_0$. This is the reason we obtain a result for the SRB measure of the Axiom A attractor and not for any other Gibbs measure.

Recall that the uniform hyperbolicity of the original flow implies that there exists $\kappa \in (0, 1)$ and $C > 0$ such that, for all $w_1, w_2 \in \mathcal{S}_i$ in the same local stable leaf, i.e. $\pi(w_1) = \pi(w_2)$, we have $d(\widehat{F}^n w_1, \widehat{F}^n w_2) \le C\kappa^n d(w_1, w_2)$ for all $n \in \mathbb{N}$. As previously, in the proof of Lemma 6, by choosing first a refinement of the Markov partition and then choosing $\Delta_0$ from the refined partition we may guarantee that the inducing time $R$ is a large as we require so that there exists contraction at each iteration by $\hat{F}$ or, in other words, we may take $C = 1$. For that choice of $\Delta_0$ we have the following result.

**Lemma 8.** *Let $\widehat{F} : \cup_\ell \widehat{\Delta}_0^{(\ell)} \to \widehat{\Delta}_0$, $F : \cup_\ell \Delta_0^{(\ell)} \to \Delta_0$ and $F$-invariant probability measure $\nu$ be as defined above.*

1. *There exists a continuous map $\pi : \widehat{\Delta}_0 \to \Delta_0$ such that $F \circ \pi = \pi \circ \widehat{F}$ whenever both members of the equality are defined;*
2. *There exists an $\widehat{F}$-invariant probability measure $\eta$ giving full mass to the domain of definition of $\widehat{F}$;*
3. *There exists a family of probability measures $\{\eta_x\}_{x\in\Delta_0}$ on $\widehat{\Delta}_0$ which is a disintegration of $\eta$ over $\nu$, that is, $x \mapsto \eta_x$ is measurable, $\eta_x$ is supported on $\pi^{-1}(x)$ and, for each measurable subset $A$ of $\widehat{\Delta}_0$, we have $\eta(A) = \int \eta_x(A)\, d\nu(x)$. Moreover, this disintegration is smooth: we can find a constant $C > 0$ such that, for any open subset $\omega \subset \cup_\ell \Delta_0^{(\ell)}$ and for each $u \in \mathcal{C}^1(\pi^{-1}(\omega))$, the function $\tilde{u} : \omega \to \mathbb{R}, x \mapsto \tilde{u}(x) := \int u(y)\, d\eta_x(y)$ belongs to $\mathcal{C}^1(\omega)$ and satisfies*

$$\sup_{x\in\omega} \|D\tilde{u}(x)\| \le C \sup_{(x,z)\in\pi^{-1}\omega} |v(x,z)| + C \sup_{y\in\pi^{-1}(\omega)} \|Du(y)\|.$$

(4) *There exists* $\kappa \in (0,1)$ *such that, for all* $w_1, w_2$ *such that* $\pi(w_1) = \pi(w_2)$, *we have* $d(\widehat{F}w_1, \widehat{F}w_2) \le \kappa d(w_1, w_2)$.

The statement of the above proposition corresponds precisely with [3, Definition 2.5] and says, in their terminology, that $\widehat{F}$ is a *hyperbolic skew-product* over $F$.

*Proof of Lemma 8.* Item (1) is clear. Item (2) follows from the definition by the limits (6). Item (3) follows from the definition (7) and the estimates (9). Property (4) is a consequence of the choice of $\Delta_0$. □

**Lemma 9.** *Let* $F : \cup_\ell \Delta_0^{(\ell)} \to \Delta_0$ *and* $r : \cup_\ell \Delta_0^{(\ell)} \to \mathbb{R}_+$ *be the full branch Markov map with countable partition and the induced return time as defined before.*

(1) *There exists* $r_0 > 0$ *such that* $r$ *is bounded from below by* $r_0$*;*
(2) *There exists* $K > 0$ *such that* $|D(r \circ h)| \le K$ *for every inverse branch* $h$ *of* $F$.

*Proof.* We note that $R(x)\underline{\tau} \le r(x) \le R(x)\bar{\tau}$ and so both items are consequences of the boundedness of $\tau$ (for item (2) we just follow the same estimates (5) in the proof of Lemma 6). □

**Lemma 10.** *Let* $r : \cup_\ell \Delta_0^{(\ell)} \to \mathbb{R}_+$ *be defined as before. There exists* $\sigma_0 > 0$ *such that* $\int e^{\sigma_0 r}\, d\mathbf{m} < \infty$.

*Proof.* Since $R$ was defined as the first return time of the uniformly expanding map $f$ to $\Delta_0$ we know[3] that there exists $\alpha > 0$, $C > 0$ such that $\mathbf{m}(\{x \in \Delta_0 : R(x) \ge n\}) \le Ce^{-\alpha n}$ for all $n \in \mathbb{N}$. As $\tau$ is uniformly bounded and $r = \sum_{j=0}^{R-1} \tau \circ f^j$ the estimate of the lemma follows. □

Consider the following cohomology property known as the *uniform non integrability* characteristic of the flow [9]:

(UNI): There does *not* exist any $\mathcal{C}^1$ function $\gamma : \Delta_0 \to \mathbb{R}$ such that $r - \gamma \circ F + \gamma$ is constant on each $\Delta_0^{(\ell)}$.

The above property is also described as "$r$ not being cohomologous to a locally constant function".

**Lemma 11.** *Suppose that* $\widehat{F} : \cup_\ell \widehat{\Delta}_0^{(\ell)} \to \widehat{\Delta}_0$, $r : \cup_\ell \Delta_0^{(\ell)} \to \mathbb{R}_+$, *and* $(F)_t$ *the suspension semiflow on* $\mathcal{S}_r$ *preserving the measure* $\widehat{\eta}$ *are defined as before. Further suppose that assumption (UNI) holds. Then there exist* $C > 0$, $\delta > 0$ *such that, for all* $\phi, \psi \in \mathcal{C}^1(\mathcal{S}_r)$, *and for all* $t \ge 0$,

$$\left| \int \phi \cdot \psi \circ F_t \; d\widehat{\eta} - \int \phi \; d\widehat{\eta} \cdot \int \psi \; d\widehat{\eta} \right| \le C \, \|\phi\|_{\mathcal{C}^1} \, \|\psi\|_{\mathcal{C}^1} \, e^{-\delta t}.$$

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[3]Recall that $f : \cup_k \Delta_k \to \Delta$ is a Markov expanding map with a finite partition (writing $\{\Delta_k\}_k$ instead of $\{\Delta_{i,j}\}_{i,j}$ for conciseness) and that $F$ was chosen as the first return map of $f$ to the partition element $\Delta_0$ (return time denoted by $R$). By transitivity and the finiteness of the original partition there exists $\beta > 0$, $n_0 \in \mathbb{N}$, $Q_k \subset \Delta_k$ such that $f^{n_0}(x) \in \Delta_0$ for all $x \in Q_k$ and $\mathbf{m}(Q_k) \ge \beta \mathbf{m}(\Delta_k)$. Let $A_n := \{x : R(x) \ge n\}$ and note that $A_n$ is the disjoint union of elements of the $n^{\text{th}}$-level refinement of the partition. Using bounded distortion ($D > 0$)

$$\frac{\mathbf{m}(A_n \cap f^{-n} Q_k)}{\mathbf{m}(A_n \cap f^{-n} \Delta_k)} \ge D^{-1} \frac{\mathbf{m}(Q_k)}{\mathbf{m}(\Delta_k)}.$$

This means that $\mathbf{m}(A_n \cap f^{-n} Q_k) \ge D^{-1} \beta \mathbf{m}(A_n)$, i.e., a fixed proportion of the points which have not yet returned to $\Delta_0$ will return to $\Delta_0$ within $n_0$ iterates.

*Proof.* In order to prove the lemma we make the connection to the systems studied in [3]. Lemma 6 corresponds to [3, Definition 2.2]; Lemma 9 combined with assumption (UNI) corresponds to [3, Definition 2.3]; Lemma 10 corresponds to [3, Definition 2.4]; and finally Lemma 8 corresponds to [3, Definition 2.5]. This implies that the assumptions of [3, Theorem 2.7] are satisfied and consequently that the suspension semiflow $F_t$ mixes exponentially for $\mathcal{C}^1$ observables. □

This ensures that the original flow $X^t : U \to U$ also mixes exponentially for $\mathcal{C}^1$ observables: if $\phi, \psi : U \to \mathbb{R}$ are $\mathcal{C}^1$, then $\phi \circ \Phi, \psi \circ \Phi$ are $\mathcal{C}^1$ observables on $\mathcal{S}_r$ ($\Phi$ is the conjugacy (4)) and so

$$\int \phi \cdot (\psi \circ X^t)\, d\mu = \int (\phi \circ \Phi) \cdot (\psi \circ X^t \circ \Phi)\, d\widehat{\eta} = \int (\phi \circ \Phi) \cdot (\psi \circ \Phi) \circ F_t\, d\widehat{\eta}$$

goes exponentially fast to zero. By an approximation argument [9, Proof of Corollary 1] this implies exponential mixing for Hölder observables.

A more direct proof of Theorem A would be to establish the open and denseness of the (UNI) assumption with respect to a given choice of cross-sections. However, since Theorem B in terms of non joint-integrability of stable and unstable foliations is of independent interest, we choose now to make the connection between (UNI) and non joint-integrability.

**Lemma 12.** *Suppose that $\widehat{F} : \cup_\ell \widehat{\Delta}_0^{(\ell)} \to \widehat{\Delta}_0$ and $r : \cup_\ell \Delta_0^{(\ell)} \to \mathbb{R}$ are defined as above. Property (UNI) fails if and only if the stable and unstable foliations of the underlying Axiom A flow $X^t : U \to U$ are jointly integrable.*

*Proof.* First suppose that (UNI) fails and so there exists a $\mathcal{C}^1$ function $\gamma : \cup_\ell \Delta_0^{(\ell)} \to \mathbb{R}$ such that $r - \gamma \circ F + \gamma$ is constant on each $\Delta_0^{(\ell)}$. For notational convenience extend $\gamma$ to $\hat{\Delta}_0^{(\ell)}$ as the function which is constant along local stable manifolds. This means that $r - \gamma \circ \hat{F} + \gamma$ is constant on each $\hat{\Delta}_0^{(\ell)}$. For each $\ell$ fix some $x_\ell \in \overline{\hat{\Delta}_0^{(\ell)}}$ such that $\gamma(x) \le \gamma(x_\ell)$ for all $x \in \hat{\Delta}_0^{(\ell)}$ (recall that $\gamma : \Delta_0 \to \mathbb{R}$ is a $C^1$ function). Choosing, if required, a refinement of the Markov partition we guarantee that $\gamma(x_\ell) - \gamma(x) \le \inf r$. Since $\gamma$ is a $\mathcal{C}^1$ function consider (for each $\ell$) the $(d_u + d_s)$-dimensional hypersurface

$$\mathcal{D}_\ell := \left\{ (x, \gamma(x_\ell) - \gamma(x)) : x \in \hat{\Delta}_0^{(\ell)} \right\} \subset \mathcal{S}_r. \tag{10}$$

We claim that the return time function to this family of local cross-sections $\cup_\ell \mathcal{D}_\ell$ is locally constant. For some $\ell, \ell'$ consider the set $(x, a) \in \mathcal{D}_\ell$ such that $\hat{F}(x) \in \hat{\Delta}_0^{(\ell')}$ For $x$ such that $0 < t - r(x) < r(Fx)$

$$F_t((x, \gamma(x_\ell) - \gamma(x))) = (\hat{F}x, t + \gamma(x_\ell) - \gamma(x) - r(x))$$

Then the first $t > 0$ so that $F_t((x, \gamma(x_\ell) - \gamma(x))) \in \cup_{\ell'} \mathcal{D}_{\ell'}$ is given as a solution of $t + \gamma(x_\ell) - \gamma(x) - r(x) = \gamma(Fx_{\ell'}) - \gamma(Fx)$ or, equivalently,

$$t = [\gamma(Fx_{\ell'}) - \gamma(x_\ell)] + [r(x) - \gamma(Fx) + \gamma(x)],$$

which is locally constant since it does not depend on $x \in \hat{\Delta}_0^{(\ell)} \cap \hat{F}^{-1} \hat{\Delta}_0^{(\ell')}$. This proves that

$$\left\{ (x, a + \gamma(x_\ell) - \gamma(x) : x \in \hat{\Delta}_0^{(\ell)}, a \in \mathbb{R}^+ \right\} \tag{11}$$

defines a codimension-one invariant $\mathcal{C}^1$-foliation transversal to the flow direction, which implies that the stable and unstable foliations are jointly integrable. This shows that if (UNI) fails, then we obtain joint integrability. The other direction of the statement is well known [12]. □

## 4. Robust exponential mixing

The purpose of this section is to prove Theorem A. We make use of Theorem B by constructing open sets of Axiom A flows that have attractors whose stable and unstable foliations are not jointly integrable in a robust way. Let $M$ be a Riemannian manifold of dimension $d \geq 3$. It is sufficient to prove that there exists an open set of vector fields supported on the $d$-dimensional hypercube $D^d := (0,1)^d$ satisfying the conclusion of Theorem A.

**Lemma 13.** *For any $d \geq 3$ there exists a vector field $X \in \mathfrak{X}^3(\mathbb{R}^d)$ such that $X^t : \mathbb{R}^d \to \mathbb{R}^d$ exhibits an Axiom A attractor contained within $(0,1)^d$. Moreover the domination condition* (1) *holds.*

*Proof.* The *Plykin attractor* [20, §8.9] is a smooth diffeomorphism of a bounded subset of $\mathbb{R}^2$ which exhibits an Axiom A attractor. By [17, §2] we may use the existence of an Axiom A diffeomorphism on $D^2$ to construct an Axiom A flow exhibiting an attractor in a bounded subset of $D^2 \times S^1$. This can be embedded in $\mathbb{R}^3$ as a solid torus. It is a simple matter to ensure that the contraction in the stable direction is much stronger that the expansion in the unstable direction so that the domination condition (1) holds. An attractor of higher dimensions is achieved by a similar construction but with additional uniformly contracting directions added. □

The above construction is far from being the only possibility. To construct a four dimensional flow another obvious choice is to start with the *Smale Solenoid Attractor*. This is an Axiom A diffeomorphism in $\mathbb{R}^n$ which exhibits an attractor $\Lambda$ with one-dimensional unstable foliation and $(n-1)$-dimensional stable foliation. Constructing a flow from this as per the above proof gives an $(n+1)$-dimensional Axiom A flow exhibiting an attractor. For a wealth of possibilities in higher dimension we may take advantage of the work of Williams [25] on expanding attractors. By this the expanding part of the the system is determined by a symbolic system of an $n$-solenoid, this means that the expanding part of the system may be any dimension desired. Then he shows that the stable directions may be added and the whole system embedded as a vector field on $\mathbb{R}^d$.

Since the domination condition (1) is open we obtain the following as an immediate consequence of Lemma 13 and of the regularity given by Theorem 1: There exists a $C^1$-open subset $\mathcal{U} \subset \mathfrak{X}^2(M)$ of Axiom A flows exhibiting an attractor $\Lambda$ with $\mathcal{C}^2$ stable foliation. Then, using Theorem B, for any $X \in \mathcal{U}$ the corresponding attractor for the Axiom A flow $X^t : M \to M$ admits a $C^2$ stable foliation and, consequently the unique SRB measure mixes exponentially as long as the stable foliation and unstable foliation are not jointly integrable. Since it is known since Brin (see [12] and references within) that the non integrability of the stable and unstable foliations is $\mathcal{C}^r$-open and dense this completes the proof of Theorem A.

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